Solving the Minimum-Fuel Low-Thrust Geostationary Station Keeping Problem via the Switching Systems Theory

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Abstract

The optimal station keeping control problem of a geostationary satellite equipped with electric thrusters is recast as a switched system commutation times optimisation problem, considering that a system with a bang-bang control profile is composed by several subsystems, one for which all the controls are off and the other ones one for which one of the controls is on. In order to optimise the commutation times, the optimal firing sequence has to be known in advance. This sequence is provided by a two-step decomposition technique and the proposed method can be interpreted as a third step. Simulation results on a realistic example validate the benefit of this third optimisation step on the control sequence fuel consumption.

1. Introduction

Spacecraft orbiting the Earth on Geostationary Earth Orbits (GEO) undergo orbital disturbing forces, resulting in a natural drift outside their operational station keeping (SK) windows (a rectangular box of a given geographical longitude and latitude range). It is therefore mandatory to design an accurate SK control strategy, in order not to let the spacecraft operating conditions deteriorate.

The usual spacecraft propulsion system is composed of chemical thrusters (see Soop[12] or Sidi[11]). However, the idea of using electric propulsion has been proposed in the sixties (see for instance the references Barret[2] and Hunziker[10]), followed by some theoretical developments in the eighties (as for example the development conducted by Anzel[1] and Eckstein[6]). Despite the difficult operational constraints arising from the use of the electric propulsion (disjunctive thrusts, minimum elapsed time between two thrusts, on-off profile of the thrusters), this kind of propulsion is a viable alternative to the commonly used chemical one thanks to its high specific impulse. Indeed, this particular features naturally implies fuel consumption savings enabling to increase the spacecraft longevity.

The minimum-fuel station keeping problem may be mathematically expressed as an optimal control problem (OCP). Two types of methods may be used to solve optimal SK control problems. The collocation based direct methods rely on a discretisation of the state and the control vectors in order to create a non linear programming problem (see Betts[4] or Hull[9]) whose solution is a discretized approximation of the optimal solution. Indirect methods are based on the Pontryagin Maximum Principle (PMP). Despite the difficulty of handling the operational constraints of the electric propulsion within the PMP framework, Gazzino et al.[8] set up a decomposition method to solve the optimal station keeping control problem taking the operational constraints into account. However, due to the great number of tuning parameters and to the complexity of the problem, the obtained local solutions appear to be highly sensitive to the initialisation process, thus resulting in some convergence issues in some cases or in a possible optimality deficiency.

If the operational constraints are respected, the spacecraft trajectory is composed of thrusting arcs separated by coasting arcs. Thus, the control profile switches from a time interval where all thrusters are off to a time interval for which one thruster is on, and vice-versa. Therefore, the system can be viewed as a switched system composed by

one subsystem per thruster and one subsystem to describe the the case where all thrusters are off. Xu and Antsaklis[13] proposes a method in order to compute the optimal switching times of a switched systems thanks to a time change of coordinates. The idea of this paper is to use the method of Xu and Antsaklis[13] to optimise the switching times for the optimal station keeping control problem. As this method requires to know in advance the optimal firing sequence of the thrusters, the two-step decomposition technique of Gazzino et al.[8] is first used as initial guess. The proposed optimisation step can hence be considered as a third step of the decomposition method. The efficiency of the proposed technique for improving the results of Gazzino et al.[8] is illustrated on a realistic numerical example.

2. Problem statement

2.1 Geostationary Spacecraft Dynamics

The motion of a spacecraft orbiting the Earth on a Geostationary Earth Orbit (GEO) can be described with the equinoctial orbital elements as defined in Battin:[3]

$$x_{eoe} = [a \; e_x \; e_y \; i_x \; i_y \; \ell_{M\Theta}]^t \in \mathbb{R}^6, \tag{1}$$

where a is the semi-major axis, (e_x, e_y) the eccentricity vector, (i_x, i_y) the inclination vector, $\ell_{M\Theta} = \omega + \Omega + M - \Theta$ is the mean longitude where Ω is the right ascension of the ascending node, ω is the perigee's argument, M is the mean anomaly and $\Theta(t)$ is the right ascension of the Greenwich meridian. The dynamics of the spacecraft may be represented by the following non linear state-space model:

$$\frac{dx_{eoe}}{dt} = f_L(x_{eoe}, t) + f_G(x_{eoe}, t)u. \tag{2}$$

$f_L \in \mathbb{R}^6$ is the Lagrange contribution part of the external disturbing forces. For a GEO spacecraft, these disturbing forces are the non-spherical part of the Earth gravitational potential that mainly affects the mean longitude $\ell_{M\Theta}$, the solar radiation pressure (SRP) that mainly affects the eccentricity vector (e_x, e_y) and the gravitational attraction of the sun and the moon that mainly affects the inclination vector (i_x, i_y). These forces are described by the CNES ORANGE model (cf. Campan et Brousse[5]). $f_G \in \mathbb{R}^{6 \times 3}$ is the Gauss contribution part for the disturbing forces that do not derive from a potential.

$u = \begin{bmatrix} u_R & u_T & u_N \end{bmatrix}^t \in \mathbb{R}^3$ is the control vector expressed in the local orbital RTN frame (also written RSW) defined in Soop[12] by:

- N is the unit vector along the kinetic momentum;
- R is the unit vector along the direction Earth's center - satellite;
- T completes the orthogonal direct basis.

In order to deal with the station keeping problem, the relative state of the satellite with respect to the station keeping state is defined as:

$$x_{sk} = \begin{bmatrix} a_{sk} & 0 & 0 & 0 & 0 & \ell_{M\Theta_{sk}} \end{bmatrix}^t \tag{3}$$

where a_{sk} is the synchronous semi-major axis and $\ell_{M\Theta_{sk}}$ is the station mean longitude.

The relative dynamics equations are developed by linearization of Equation (2) about the station keeping point (3). By denoting $x = x_{eoe} - x_{sk}$ the relative state model for the SK problem reads:

$$\frac{dx}{dt} = A(t)x + D(t) + B(t)u, \tag{4}$$

where the matrices $A \in \mathbb{R}^{6 \times 6}$, $B \in \mathbb{R}^{6 \times 3}$, $C \in \mathbb{R}^{3 \times 6}$ and $D \in \mathbb{R}^6$ are defined as follows:

$$A(t) = \frac{\partial}{\partial x_{eoe}} (f_L(x_{eoe}(t), t)) \Big|_{x_{eoe} = x_{sk}}, \tag{5}$$

$$B(t) = f_G(x_{sk}, t), \tag{6}$$

$$D(t) = f_L(x_{sk}, t), \tag{7}$$

The geographical coordinates of the satellite:

$$y_{eoe} = T(x_{eoe}, t)x_{eoe}, \tag{8}$$

are of interest because the station keeping problem consists in constraining them in the vicinity of the station position $y_{sk} = \begin{bmatrix} r_{sk} & 0 & \lambda_{sk} \end{bmatrix}^t$ where r_{sk} is the synchronous radius and λ_{sk} is the station keeping geographical longitude. The relative geographical position with respect to the station-keeping position is denoted by:

$$y = y_{eoe} - y_{sk} = T(x_{sk}, t)x = C(t)x, \tag{9}$$

and is obtained by linearising Equation (8).

After the linearisation, the station keeping requirement on the latitude and the longitude of the spacecraft is expressed by constraining the relative geographical position with respect to the centre of the SK window:

$$|[0\ 1\ 0]C(t)x(t)| \leqslant \delta \text{ and } |[0\ 0\ 1]C(t)x(t)| \ \forall t \in [0, T], \tag{10}$$

with δ being the half width of the SK window in the latitude and longitude directions. A trajectory satisfying the SK constraints (10) is called SK-feasible.

2.2 Electric Thrusting System

The considered satellite is equipped with four electric thrusters mounted on the anti-nadir face, each of them having an orientation defined by a cant angle θ and a slew angle α. These angles define the North-East, the South-East, the North-West and the South-West directions of thrust. The satellite dynamics can be expressed considering the four thrusts provided by the four engines as control variables. The control $u(t)$ expressed in the local orbital frame is a linear combination of the four thrusts such that $u = \Gamma F$, where $\Gamma = [\Gamma_1 \mid \Gamma_2 \mid \Gamma_3 \mid \Gamma_4] \in \mathbb{R}^{3 \times 4}$ and $F = [F_1\ F_2\ F_3\ F_4]^t \in [0, F_{max}]^4$. The thrust direction matrices $\Gamma_j \in \mathbb{R}^3$ are defined such that:

$$\Gamma_j = \frac{1}{m}\begin{bmatrix} -\sin\theta_j\cos\alpha_j & -\sin\theta_j\sin\alpha_j & -\cos\theta_j \end{bmatrix}^t, \tag{11}$$

where the cant and slew angles θ_j and α_j respectively are defined exactly as in

For the sake of simplicity, the thrust vector is normalized by the maximum level of thrust F_{max}. It is thus possible to write $F = F_{max}\tilde{F}$ with $\tilde{F} \in [0, 1]^4$. As the thrust are on-off, the thrust profile is modelled as a rectangular signal that is parametrized by the date $t_{i,j}$ corresponding to the middle instant of the thrust and by its half width duration denoted $\Delta t_{i,j}$:

$$F_i(t) = F_{max}\sum_{j=1}^{P_i} \text{RectangleFunction}(t, t_{i,j}, \Delta t_{i,j}). \tag{12}$$

where P_i is the number of thrusts of thruster i.

Some technological operational constraints that are inherently related to the actuation system have to be taken into account:

(a) thrusters cannot have simultaneous thrusts;

(b) a thrust must last at least $T_l : 2\Delta t_{i,j} \geqslant T_l$;

(c) two successive thrusts of a given thruster must be separated of an interval of latency equal to T_s;

(d) two thrusts of two different thrusters must be separated by an interval of latency equal to T_d.

The constraint for the time latency between the thrust k of thruster i and the thrust l of thruster j is thus mathematically expressed as:

$$|t_{i,k} - t_{j,l}| - (\Delta t_{i,k} + \Delta t_{j,l}) \geqslant K_{i,j}, \tag{13}$$

for $k = 1 \ldots P_i$ and $l = 1 \ldots P_j$, where $K_{i,j} = T_s$ if $i = j$ (constraint (c)) and $K_{i,j} = T_d$ otherwise (constraint (d)). In addition, some other convenient constraints are enforced in order to make sure that the thrusts will not begin before 0 and end after T:

$$t_{i,j} - \Delta t_{i,j} \geqslant 0 \text{ and } t_{i,j} + \Delta t_{i,j} \leqslant T. \tag{14}$$

A trajectory satisfying the operational constraints for the electric propulsion system is called operationally feasible.

2.3 Statement of the Station Keeping Optimal Control Problem

The main goal of the station keeping problem is to maintain the longitude and the latitude of the satellite in a box defined by its size δ by acting on the orbital parameter via the 4 thrusters while optimising the fuel consumption. The associated optimal control problem is in general defined over a fixed horizon for the computation of optimal open loop control laws. In this context, optimality means that a minimum fuel-solution is looked for to extend the operational life time of the satellite. Therefore, performing minimum-fuel station keeping is minimizing the performance index:

$$J = 2F_{max} \sum_{\text{thruster } i=1}^{4} \|\Gamma_i\|_1 \sum_{j=1}^{P_i} \Delta t_{i,j}, \tag{15}$$

Considering all the constraints described above, the minimum-fuel SK problem to solve may be summarized as the following optimal control problem:

Problem 1 Find the sequence of dates $\{t_{i,j}\}$ and durations $\{\Delta t_{i,j}\}$, for $i = 1 \ldots 4$, $j = 1 \ldots P_i$ (with P_i fixed) solutions of the minimization problem:

$$\min_{t_{i,j}, \Delta t_{i,j}} J = \sum_{i=1}^{4} \sum_{j=1}^{P_i} \Delta t_{i,j},$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = A(t)\, x(t) + D(t) + \bar{B}(t)\Gamma F_{max}\, \tilde{F}(t), \\ x(0) = 0, \\ |[0\ 1\ 0]C(t)x(t)| \leqslant \delta,\ |[0\ 0\ 1]C(t)x(t)| \leqslant \delta, \\ 2\Delta t_{i,j} \geqslant T_l,\ t_{i,j} - \Delta t_{i,j} \geqslant 0,\ t_{i,j} + \Delta t_{i,j} \leqslant T, \\ |t_{i,k} - t_{j,l}| - (\Delta t_{i,k} + \Delta t_{j,l}) \geqslant K_{i,j}, \end{cases} \tag{16}$$

$\circ$

3. Problem resolution in the Switching Systems Framework

3.1 Transformation of the Optimal Control Problem

The analysis of the geostationary orbital perturbations done in the existing literature (see e.g. Campan and Bousse,[5] Sidi[11] or Soop[12]) allow to determine some rule of thumb for geostationary SK strategies. The North-South out-of-plane effect of the Sun and the Moon attraction are the most predominant forces that have to be corrected each half orbit, once in the South direction and half an orbit later in the North direction. According to this rule of thumb, the control scheme that should be applied is such that the two North thrusters are firing simultaneously, as well as the two South thrusters of the considered propulsion system. Nevertheless, this resulting control sequence does not fulfil the operational constraint (a).

In the reference Gazzino et al.,[8] a two-step decomposition method have been used in order to solve the station keeping problem while satisfying the operational constraints (a)-(d). However this decomposition method suffers from the drawback of necessitating a fine tuning of a lot of parameters. The aim of the proposed contribution is to add a third step that can help to simplify this parameter tuning step.

From the method of Gazzino et al.,[8] the control sequence that has been created is a combination of coast arcs, of firing arcs for the North-East thruster, of firing arcs for the South-East thruster, of firing arcs of the South-East thruster and of firing arcs for the South-West thruster, that is to say, to combination of the five subsystems:

$$\dot{x}(t) = \begin{cases} A(t)x(t) + D(t), \\ A(t)x(t) + D(t) + B(t)\Gamma F_{max}\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, \\ A(t)x(t) + D(t) + B(t)\Gamma F_{max}\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T, \\ A(t)x(t) + D(t) + B(t)\Gamma F_{max}\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T, \\ A(t)x(t) + D(t) + B(t)\Gamma F_{max}\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T. \end{cases} \tag{17}$$

By denoting U_i a $\mathbb{R}^4$ vector with a 1 at the ith position, and by extension U_0 the $\mathbb{R}^4$ zero vector, all the dynamics functions:

$$f_i(x(t)) = A(t)x(t) + D(t) + B(t)\Gamma F_{max}U_i,\ i \in \{0, \ldots, 4\}, \tag{18}$$

can represent a subsystem of the overall system. Hence, the time horizon optimisation $[0, T]$ can be split into K intervals T_k for which the control vector V_k is one of the five admissible control vectors U_i, $i \in \{1, \ldots, 4\}$. With these considerations, all the dynamics functions f_k can represent a subsystem of the overall system. Hence, the commutation from V_k to V_{k+1} between the intervals T_k and T_{k+1} at commutation time t_k can be viewed as a commutation between two of the five subsystems. In a switched system framework, the system dynamics can be written as:

$$\text{for } t \in T_k, \ \exists i_k \in \{0, \ldots, 4\}, \ \dot{x}(t) = f_{i_k}(x(t)), \tag{19}$$

where the following equality holds : $V_k = U_{i_k}$.

In accordance with the operational constraints (c)-(d) two thrusting arcs must be separated by a coasting arc. Thus the control sequence has the following structure:

$$U_0, \ V_{j_1}, \ U_0, \ V_{j_2}, \ U_0, \ldots, \ U_0, \ V_{j_P}, \ U_0 \tag{20}$$

where $V_{j_k} = U_j$ with $j \in \{1, \ldots, 4\}$ and P is the optimal number of thrusts. There are therefore $2P$ switches and $2P + 1$ intervals on which the control is constant. It is possible to write $T = t_{2P+1}$ as if the final time were the last commutation time. In order to satisfy the operational constraints (c) and (d), the length of the firing arcs must respect the constraints:

- $t_{2k+1} - t_{2k} \geqslant T_s$ if the coasting arc lies between two firing arcs of the same thruster,

- $t_{2k+1} - t_{2k} \geqslant T_d$ if the coasting arc lies between two firing arcs of two different thrusters.

The operational constraint (b) is then written as:

$$t_{2k} - t_{2k-1} \geqslant T_l. \tag{21}$$

With this reformulation using the five subsystems of the overall system, the rectangular signal of the thrust can be parametrised by its start and end times as:

$$F_i(t) = F_{max} \sum_{j=1}^{P_i} \text{RectangleFunction}(t, t_{\text{on}}, t_{\text{off}}). \tag{22}$$

The performance index is thus rewritten as:

$$J = F_{max} \sum_{i=1}^{P} \|\Gamma_i\|_1 \ (t_{2i} - t_{2i-1}), \tag{23}$$

since odd intervals are coast arcs and the even intervals are firing arcs.

The Optimal Control Problem 1 is thus rewritten in the following form:

Problem 2 Find the optimal switching sequence $\{t_k\}$, $k \in \{1, \ldots, 2P\}$, with P fixed, solution of the minimisation problem:

$$\min_{\{t_k\}} J = F_{max} \sum_{i=1}^{P} (t_{2i} - t_{2i-1}),$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = f_{i_k}(x)(t), \\ \|[0 \ 1 \ 0]C(t)x(t)\| \leqslant \delta, \\ \|[0 \ 0 \ 1]C(t)x(t)\| \leqslant \delta, \\ x(0) = 0, \\ t_{2k} - t_{2k-1} \geqslant T_l, \\ t_{2k+1} - t_{2k} \geqslant T_g, \end{cases} \tag{24}$$

with $T_g = T_s$ for the firing arcs of the same thruster and $T_g = T_d$ for the firing arcs of two different thrusters. ∘

The Problem 2 is an OCP for which the decision variables are the commutation times. It has been partly solved in Gazzino et al.[8] by a numerical two-step decomposition approach but with no guarantee of optimality of the firing times thus obtained. The first step of this method consists in removing the operational constraints and in solving the simplified SK OCP with a direct method initialised by an indirect method. The trajectory obtained after this first step is SK-feasible but not operationally feasible. In order to recover rectangle functions from the continuous control profile, a threshold parameter ς has to be chosen carefully. The second step aims at creating an operationally feasible trajectory from the one computed at step 1. To this end, two equivalence schemes have been created, the first one being called the Consumption Based Equivalence (CBE) and the second one called the Effect Based Equivalence (EBE). The SK-feasible and operationally feasible trajectory resulting from step 2 is very sensitive to the value of the threshold parameters ς. Therefore, our objective is to improve this preliminary solution by using the method of Xu and Antsaklis[13] based on the switched systems theory.

3.2 Optimisation of the Commutation Times with the Switched Systems Theory

The reference Xu and Antsaklis[13] presents a technique to optimise the commutation times between the different subsystems of a system that is used in order to optimise the switching times between the coasting and the firing arcs. This method resorts a time change of coordinates and a parametrisation of the commutation times. However it is mandatory to assume that the order of the sequences of the active subsystems is a feasible one, obtained by application of the result of Gazzino et al.[8] Seeking this on and off times is equivalent to find an optimal ~~subsystems~~ commutation sequence.

A time change of coordinates is used to parametrise the switching times:

$$t = t_k + \Delta_k(\tau - k) \ \text{ if } t \in [t_k, t_{k+1}], \tag{25}$$

with $\Delta_k = t_{k+1} - t_k$. The system dynamics is rewritten with the new time variable:

$$\frac{\partial x(\tau)}{\partial \tau} = \begin{cases} \Delta_{2k-2}[A(t(\tau))x(\tau) + D(t(\tau))] & \text{if } \tau \in [2k-2, 2k-1], \\ \Delta_{2k-1}[A(t(\tau))x(\tau) + D(t(\tau)) + B(t(\tau))\Gamma F_{max}U_k] & \text{if } \tau \in [2k-1, 2k], \\ \ \vdots \\ \Delta_{2P}[A(t(\tau))x(\tau) + D(t(\tau))] & \text{if } \tau \in [2P, 2P+1], \end{cases} \tag{26}$$

for $k = 1, \ldots, P$. The state vector can be now considered as a function of the new time variable τ and of the switching times t_k: $x = x(\tau, \{t_k\})$.

In order to handle state constraints, in the reference Gazzino et al.,[7] where a fuel-optimal rendezvous problem has been solved with the Xu and Antsaklis[13] technique, a target term has been added to the objective function. However, as the station keeping constraint are expressed as trajectory state constraints, ~~this constraint is~~ handled in the objective function by adding a penalisation term:

$$\psi(t,x(t)) = \frac{1}{2} \left[C_2(t)x(t) - \delta \right]^2 \left[\text{sign}(C_2(t)x(t) - \delta) + 1 \right] + \frac{1}{2} \left[C_2(t)x(t) + \delta \right]^2 \left[\text{sign}(-C_2(t)x(t) - \delta) + 1 \right]$$
$$+ \frac{1}{2} \left[C_3(t)x(t) - \delta \right]^2 \left[\text{sign}(C_3(t)x(t) - \delta) + 1 \right] + \frac{1}{2} \left[C_3(t)x(t) + \delta \right]^2 \left[\text{sign}(-C_3(t)x(t) - \delta) + 1 \right], \tag{27}$$

where $C_2(t) = [0\ 1\ 0]C(t)$ and $C_3(t) = [0\ 0\ 1]C(t)$. Applying the temporal change of coordinates, the new objective function taking the station keeping window penalisation term into account reads:

$$J(x, \{t_k\}) = F_{max} \sum_{i=1}^{P}(t_{2i} - t_{2i-1}) + \mu \sum_{k=1}^{2P+1}(t_k - t_{k-1}) \int_{k-1}^{k} \psi(t(\tau), x)d\tau, \tag{28}$$

with μ a parameter that has to be chosen. The station OCP reads thus in the switching systems framework:

Problem 3 Find the optimal, switching sequence $\{t_k\}$, $k \in \{1, \ldots, 2P\}$, with P fixed, solution of the minimisation problem:

$$\min_{x, \{t_k\}} J(x, \{t_k\}) = F_{max} \sum_{i=1}^{P}(t_{2i} - t_{2i-1}) + \mu \sum_{k=1}^{2P+1}(t_k - t_{k-1}) \int_{k-1}^{k} \psi(t(\tau), x)d\tau,$$
$$\text{such that}$$
$$\begin{cases} \dot{x}(\tau) = \begin{cases} \Delta_{2k-2}[A(t(\tau))x(\tau) + D(t(\tau))] & \text{if } \tau \in [2k-2, 2k-1], \\ \Delta_{2k-1}[A(t(\tau))x(\tau) + D(t(\tau)) + B(t(\tau))\Gamma F_{max}U_k] & \text{if } \tau \in [2k-1, 2k] \end{cases} \\ x(0) = 0, \\ t_{2k} - t_{2k-1} \geqslant T_l, \\ t_{2k+1} - t_{2k} \geqslant T_g, \end{cases} \tag{29}$$

with $T_g = T_s$ for the firing arcs of the same thruster and $T_g = T_d$ for the firing arcs of two different thrusters. ∘

As the switching times determine the structure of the control profile, the state trajectory is recovered by propagation of the system dynamics from the initial condition.

Problem 3 can be solved with a descent algorithm. To this end, it is necessary to compute the derivative of the performance index in order to obtain the descent direction. The computation of derivative requires to distinguish the

odd and the even commutation times:

$$
\begin{aligned}
\frac{dJ}{dt_{2l-1}} = &-F_{max} + \mu \int_{2l-2}^{2l-1} \psi(t(\tau), x(\tau) d\tau \\
&+ \mu(t_{2l-1} - t_{2l-2}) \int_{2l-2}^{2l-1} \left[(\tau - 2l + 2)\frac{\partial \psi}{\partial t}(t(\tau), x(\tau)) + \frac{\partial \psi}{\partial x}^t (t(\tau), x(\tau))\frac{\partial x}{\partial t_{2l-1}}(\tau) \right] d\tau \\
&- \mu \int_{2l-1}^{2l} \psi(t(\tau), x(\tau) d\tau \\
&+ \mu(t_{2l} - t_{2l-1}) \int_{2l-1}^{2l} \left[(-\tau + 2l + 2)\frac{\partial \psi}{\partial t}(t(\tau), x(\tau)) + \frac{\partial \psi}{\partial x}^t (t(\tau), x(\tau))\frac{\partial x}{\partial t_{2l-1}}(\tau) \right] d\tau,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\frac{dJ}{dt_{2l}} = &\, F_{max} - \mu \int_{2l}^{2l+1} \psi(t(\tau), x(\tau) d\tau \\
&+ \mu(t_{2l+1} - t_{2l}) \int_{2l}^{2l+1} \left[(-\tau + 2l + 1)\frac{\partial \psi}{\partial t}(t(\tau), x(\tau)) + \frac{\partial \psi}{\partial x}^t (t(\tau), x(\tau))\frac{\partial x}{\partial t_{2l-1}}(\tau) \right] d\tau \\
&+ \mu \int_{2l-1}^{2l} \psi(t(\tau), x(\tau) d\tau \\
&+ \mu(t_{2l} - t_{2l-1}) \int_{2l-1}^{2l} \left[(\tau - 2l + 1)\frac{\partial \psi}{\partial t}(t(\tau), x(\tau)) + \frac{\partial \psi}{\partial x}^t (t(\tau), x(\tau))\frac{\partial x}{\partial t_{2l-1}}(\tau) \right] d\tau.
\end{aligned}
\tag{31}
$$

The derivatives of the state vector with respect to the commutation times ~~verifies~~ the following relation:

$$
\frac{d}{d\tau}\left(\frac{\partial x(\tau)}{\partial t_k} \right) = \frac{\partial}{\partial t_k}\left(\frac{dx(\tau)}{d\tau} \right).
\tag{32}
$$

It is thus possible to derive a dynamic equation for $\frac{\partial x(\tau)}{\partial t_k}$ by differentiation of the dynamics equation of the state vector $x(\tau)$ with respect to the commutation time t_k. Hence the function $\tau \mapsto \frac{\partial x(\tau)}{\partial t_k}$ is obtained by integrating the following dynamics equations:

$$
\frac{d}{d\tau}\left(\frac{\partial x(\tau)}{\partial t_{2l-1}} \right) =
\begin{cases}
(t_{2k-1} - t_{2k-2})A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l-1}} & \text{if } \tau \in [2k-2, 2k-1], \\
& k \neq l \\[2mm]
(t_{2k} - t_{2k-1})A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l-1}} & \text{if } \tau \in [2k-1, 2k], \\
& k \neq l \\[2mm]
A(t(\tau))x(\tau) + D(t(\tau)) & \\
\quad + (t_{2l-1} - t_{2l-2})\left[(\tau - 2l + 2)\left(\dfrac{\partial A(t)}{\partial t}x(\tau) + \dfrac{\partial D(t)}{\partial t} \right) \right. & \text{if } \tau \in [2l-2, 2l-1], \\
\quad \left. + A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l-1}} \right] & \\[2mm]
- [A(t(\tau))x(\tau) + D(t(\tau)) + B(t(\tau))U_l] & \\
\quad + (t_{2l} - t_{2l-1})\left[(-\tau + 2l + 2)\left(\dfrac{\partial A(t)}{\partial t}x(\tau) + \dfrac{\partial D(t)}{\partial t} + \dfrac{\partial B(t)}{\partial t}U_l \right) \right. & \text{if } \tau \in [2l-1, 2l], \\
\quad \left. + A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l-1}} \right] &
\end{cases}
\tag{33}
$$

and

$$\frac{d}{d\tau}\left(\frac{\partial x(\tau)}{\partial t_{2l}}\right) = \begin{cases} (t_{2k-1} - t_{2k-2})A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l}} & \text{if } \tau \in [2k-2, 2k-1], \\ & k \neq l \\[2ex] (t_{2k} - t_{2k-1})A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l}} & \text{if } \tau \in [2k-1, 2k], \\ & k \neq l \\[2ex] -[A(t(\tau))x(\tau) + D(t(\tau))] & \\ \quad + (t_{2l+1} - t_{2l})\left[(-\tau + 2l + 1)\left(\dfrac{\partial A(t)}{\partial t}x(\tau) + \dfrac{\partial D(t)}{\partial t}\right)\right. & \text{if } \tau \in [2l, 2l+1], \\ \quad \left. +A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l}}\right] & \\[2ex] A(t(\tau))x(\tau) + D(t(\tau)) + B(t(\tau))U_l & \\ \quad + (t_{2l} - t_{2l-1})\left[(\tau - 2l + 1)\left(\dfrac{\partial A(t)}{\partial t}x(\tau) + \dfrac{\partial D(t)}{\partial t} + \dfrac{\partial B(t)}{\partial t}U_l\right)\right. & \text{if } \tau \in [2l-1, 2l]. \\ \quad \left. +A(t(\tau))\dfrac{\partial x(\tau)}{\partial t_{2l}}\right] & \end{cases}$$
(34)

The derivative of the penalisation function with respect to the state vector is:

$$\frac{\partial \psi}{\partial x}(t, x(\tau)) = C_2(t)\left[C_2(t)x(t) - \delta\right]^2\left[\text{sign}(C_2(t)x(t) - \delta) + 1\right] + C_2(t)\left[C_2(t)x(t) + \delta\right]^2\left[\text{sign}(-C_2(t)x(t) - \delta) + 1\right]$$
$$+ C_3(t)\left[C_3(t)x(t) - \delta\right]^2\left[\text{sign}(C_3(t)x(t) - \delta) + 1\right] + C_3(t)\left[C_3(t)x(t) + \delta\right]^2\left[\text{sign}(-C_3(t)x(t) - \delta) + 1\right],$$
(35)

and its derivative with respect to time is:

$$\frac{\partial \psi}{\partial x}(t, x(\tau)) = \frac{dC_2(t)}{dt}x\left[C_2(t)x(t) - \delta\right]^2\left[\text{sign}(C_2(t)x(t) - \delta) + 1\right] + \frac{dC_2(t)}{dt}x\left[C_2(t)x(t) + \delta\right]^2\left[\text{sign}(-C_2(t)x(t) - \delta) + 1\right]$$
$$+ \frac{dC_3(t)}{dt}x\left[C_3(t)x(t) - \delta\right]^2\left[\text{sign}(C_3(t)x(t) - \delta) + 1\right] + \frac{dC_3(t)}{dt}x\left[C_3(t)x(t) + \delta\right]^2\left[\text{sign}(-C_3(t)x(t) - \delta) + 1\right],$$
(36)

The SK problem can therefore be solved according to the following algorithm:

Algorithm 1 Optimise the commutation times $\{t_k\}$

Require: $x(0)$ the initial position
Ensure: the trajectory is operationally and SK-feasible

    {Step 1: simple SK problem}
1: Remove the operational constraints (a)-(d) from Problem 1
2: $u_i(t)_{i=1,\dots,4, t\in[0,T]} \leftarrow$ the optimal control for the SK-feasible trajectory computed by a indirect method initialised by a direct method (see Gazzino et al.[8])
3: $\varsigma \leftarrow$ threshold parameter to transform the continuous control profile in a rectangle profile
4: $\{t_{i,k}, \Delta t_{i,k}\} \leftarrow$ the middle times $t_{i,k}$ and the half durations $\Delta t_{i,k}$ of the thrusts,

    {Step 2: equivalence schemes}
5: $\{t_{i,k}^\star, \Delta t_{i,k}^\star\} \leftarrow$ the middle times and half durations computed by the CBE or the EBE.

    {Step 3: commutation times optimisation with an interior point algorithm}
6: $\{t_k\} \leftarrow$ the commutation times computed from $\{t_{i,k}^\star, \Delta t_{i,k}^\star\}$
7: $\{t_k^{(0)}\} \leftarrow \{t_k\}$
8: $\{t_k^\star\} \leftarrow$ the solution of Problem 3 with an interior point algorithm

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4. Numerical Results

The proposed commutation times optimisation technique is applied for the station keeping of a 4850 kg satellite equipped with the electric propulsion system presented in Section 2.2. Two cases are studied. For the first one,

the threshold parameter ς could be well adjusted so that the trajectory obtained after the step 2 is SK-feasible and operationally feasible. The proposed technique optimises the commutation times in order to decrease the consumption of the Sk strategy.

For the second case, the parameter ς was harder to determine, resulting in a trajectory operationally feasible but not SK-feasible. The proposed technique manages to optimise the commutation times so that the ensuing trajectory is SK and operationally feasible.

4.1 Commutation Time Optimisation for a SK-Feasible Trajectory

The two-step decomposition technique of Gazzino et al.[8] is used first in order to obtain a SK and operationally feasible trajectory. Solving the problem without the operational constraints (step 1) shows that when they are removed, the optimal control profile is such that the North thrusters and the South thrusters have simultaneous thrusts respectively, which is also the optimal configuration according to the station keeping literature. Therefore, the step 2 is mandatory. It was possible to choose ς = 0.8 for the CBE and ς = 0.9 for the EBE. The μ coefficient of Equation (28) is chosen to be 1.10^7.

Figures 1 and 2 depict the control profiles obtained by the proposed method starting respectively from the CBE scheme and from the EBE scheme. Figures 3 and 4 exhibit the trajectories resulting from these control profiles in the latitude-longitude plane. The two equivalence schemes lead to different consumption values (see Table 1), and the proposed optimisation of the switching times leads to a substantial fuel saving.
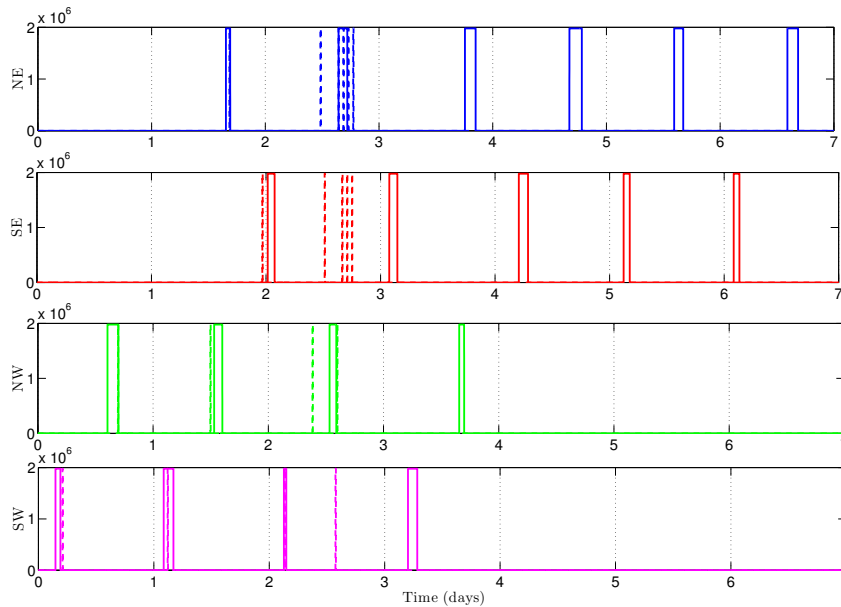

Figure 1: Control profiles:
–: solution of the consumption based equivalence with threshold 0.8,
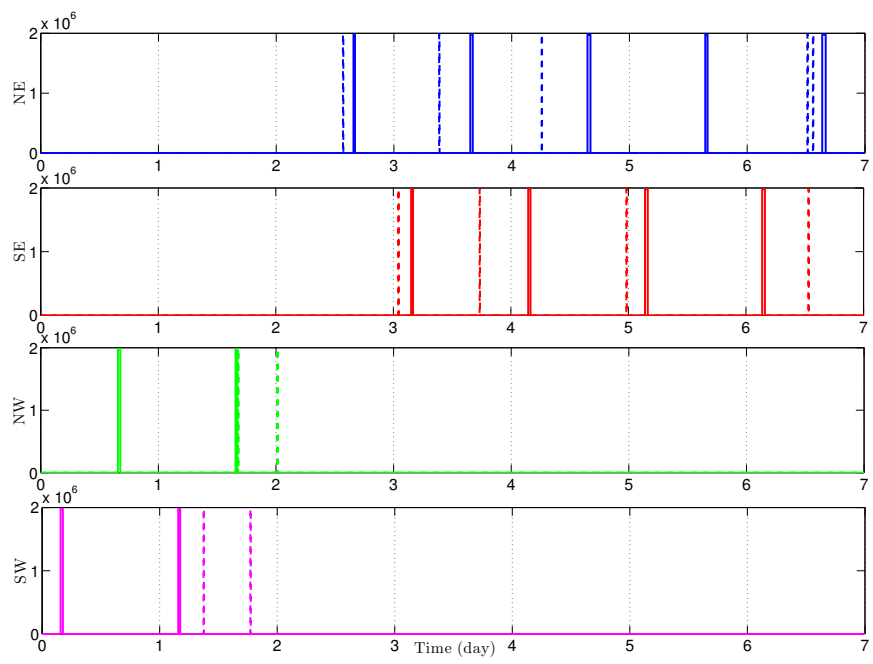- -: solution of the switched systems optimisation.

Figure 2: Control profiles:
−: solution of the consumption based equivalence with threshold 0.8,
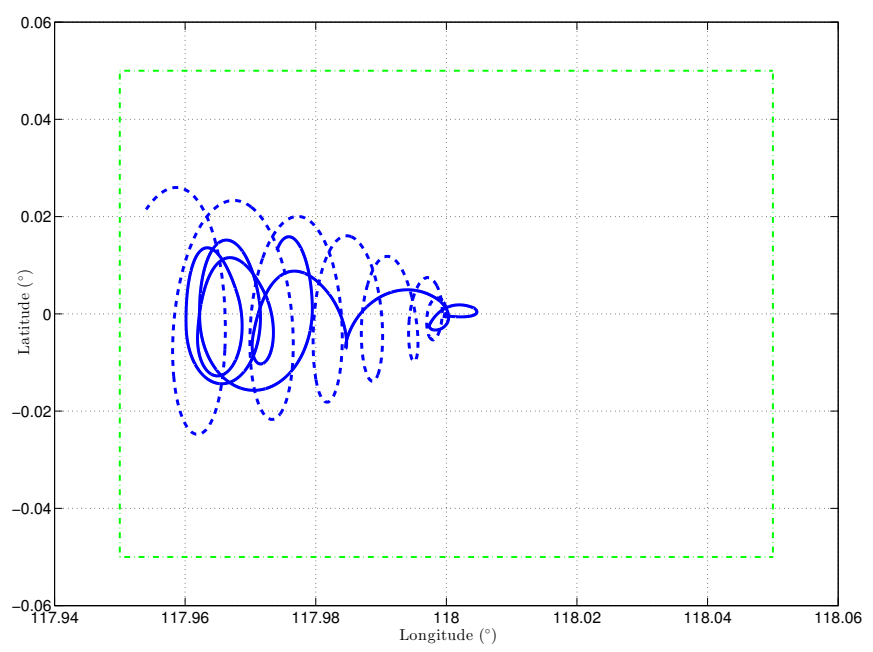- -: solution of the switched systems optimisation.

Figure 3: Latitude vs. longitude:
−: solution of the consumption based equivalence with threshold 0.8,
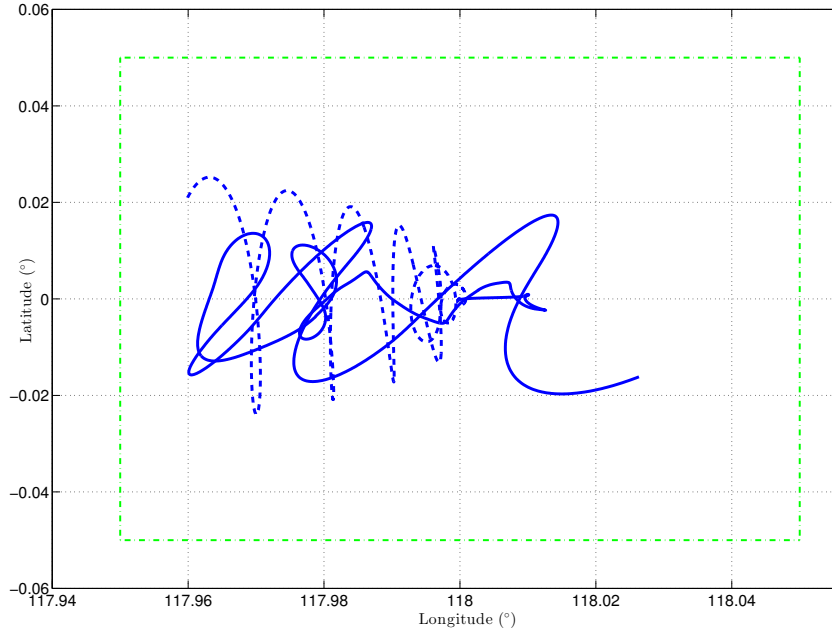- -: solution of the switched systems optimisation.

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Figure 4: Latitude vs. longitude:
–: solution of the effect based equivalence with threshold 0.9,
- -: solution of the switched systems optimisation.

Equivalence scheme	CBE	EBE
Two-step solution	8.88 m/s	1.72 m/s
Threshold value	0.8	0.9
Three-step solution	0.45 m/s	0.31 m/s

Table 1: Consumptions for the two-step and the three-step solutions.

4.2 Commutation Time Optimisation for a non SK-Feasible Trajectory

The CBE and EBE schemes enforce the operational constraints but do not tackle the SK constraints. Therefore the threshold parameter ς has to be precisely adjusted so that a SK-feasible trajectory can be computed at the end of step 2. Nevertheless, it may happen that the threshold parameter ς is so hard to determine that the computed middle times and thrust half durations lead to a non SK-feasible trajectory. The proposed technique based on the switched systems manages to optimise the commutation time so that the trajectory at the end of step 3 is both operationally feasible and SK-feasible.

Figure 6 depict the state trajectory at the end of step 2 with $\varsigma = 0.9$. In this case, the SK constraints are not respected. Despite this, the step 3 enforces the SK constraints while making sure that the operational constraints are still respected. Figure 5 displays the corresponding control profiles. Table 2 shows the decrease in consumption realised by step 3.
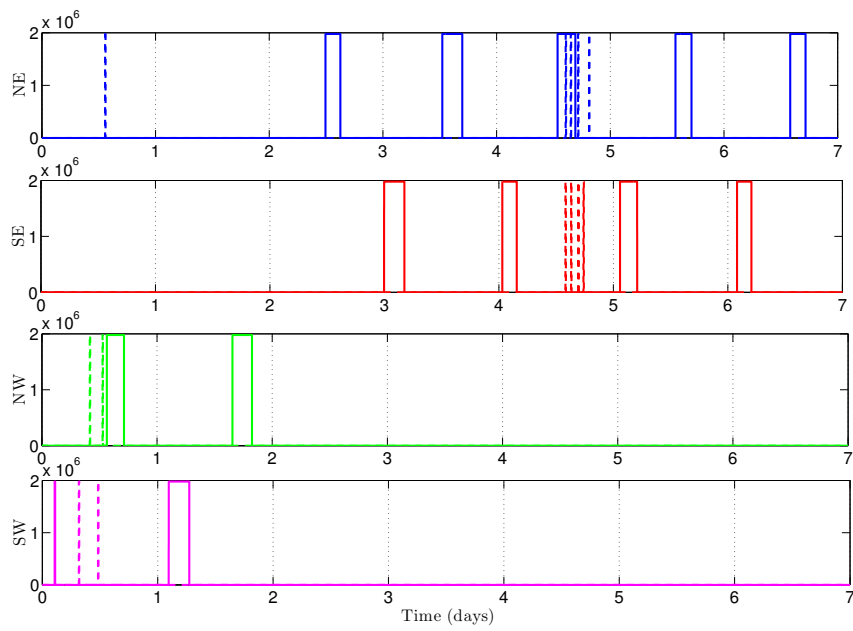
Figure 5: Control profiles:
−: solution of the consumption based equivalence with threshold 0.9,
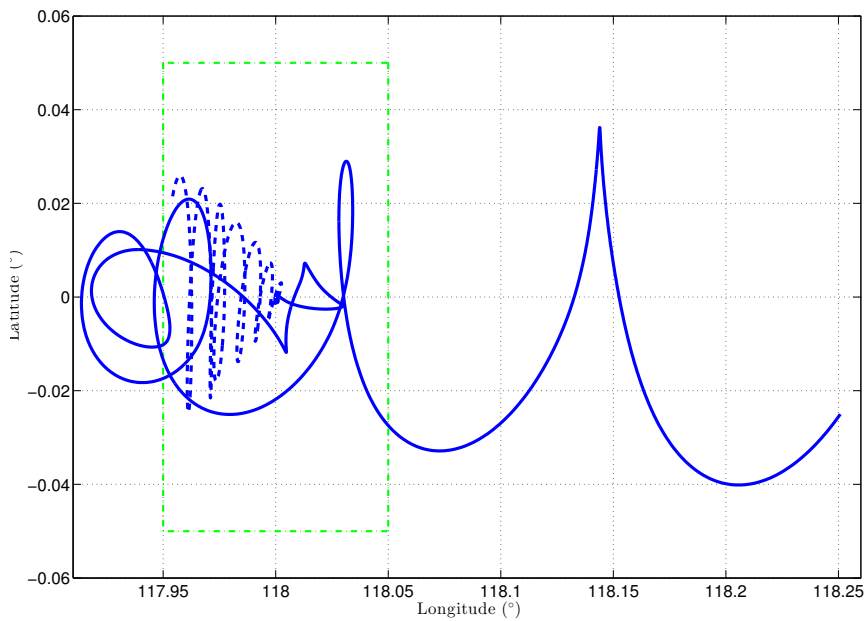- -: solution of the switched systems optimisation.

Figure 6: Latitude vs. longitude:
−: solution of the consumption based equivalence with threshold 0.9,
- -: solution of the switched systems optimisation.

Equivalence scheme	CBE
Two-step solution	12.37 m/s
Threshold value	0.9
Three-step solution	0.31 m/s

Table 2: Consumptions for the two-step and the three-step solutions in the case were the SK constraints are not respected at the end of step 2.

5. Conclusion

In this paper, an optimisation of the commutation times originating in the theory of the switched systems has been used in order to optimise the on-off firing sequence for the minimum-fuel station keeping of a satellite equipped with electric thrusters. In fact, the system can be naturally decomposed into several subsystems, one per thruster whose control is on and one corresponding to a coasting arc. As the proposed method requires to know beforehand a feasible firing sequence, a two-step decomposition technique has been used as an initial guess for the optimal sequence of firing arcs.. The optimisation of the switching times allow to overcome the drawbacks of the previous steps.

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