INTRODUCTION TO LMI/SDP OPTIMIZATION

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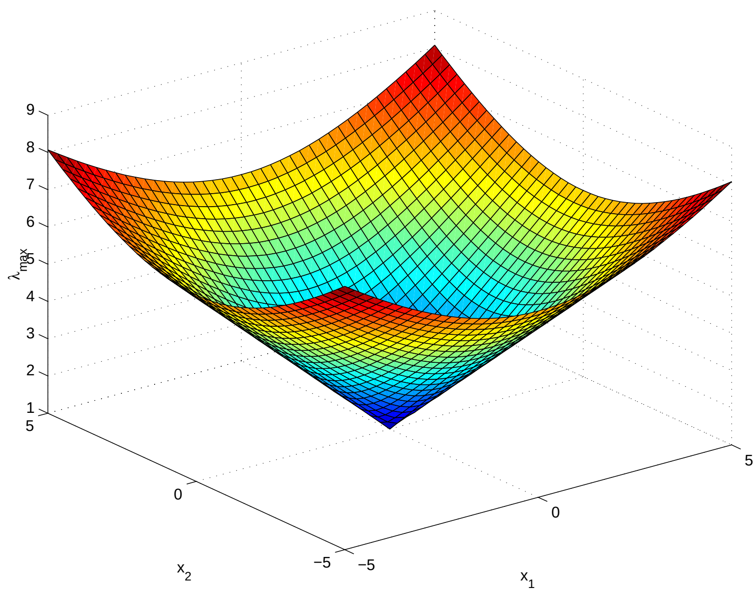

26-27 June 2017

Outline: LMI optimization

1 Introduction: What is an LMI ? What is SDP ?

Historical survey - applications - convexity - cones - polytopes

2 SDP duality

Lagrangian duality - SDP duality - KKT conditions

3 Geometry of LMI sets

Geometry - algebraic tricks

4 Solving LMIs

Interior point methods - solvers - interfaces

Lecture material

References on convex optimization:

• M.F. Anjos, J.B. Lasserre. Handbook on Semidefinite, Conic and Polynomial Optimization, Springer, 2012

• S. Boyd, L. Vandenberghe. Convex Optimization, Lecture Notes Stanford & UCLA, CA, 2002

• A. Ben-Tal, A. Nemirovskii. Lectures on Modern Convex Optimization, SIAM, 2001

• H. Wolkowicz, R. Saigal, L. Vandenberghe. Handbook of semidefinite programming, Kluwer, 2000

Modern state-space LMI methods in control:

• C. Scherer, S. Weiland. Course on LMIs in Control, Lecture Notes Delft & Eindhoven Univ Tech, NL, 2002

• S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan. Linear Matrix Inequalities in System and Control Theory, SIAM, 1994

LMI and algebraic optimization:

• J.B. Lasserre. An Introduction to Polynomial and Semi-Algebraic Optimization. Cambridge Text in Applied Mathematics, UK, 2015

• P. A. Parrilo, S. Lall. Semidefinite Programming Relaxations and Algebraic Optimization in Control, Workshop presented at the 42nd IEEE Conference on Decision and Control, Maui HI, USA, 2003

LMI OPTIMIZATION
PART 1

WHAT IS AN LMI ?
WHAT IS SDP ?

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LMI - Linear Matrix Inequality

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq \mathbf{0}$$

- $F_i \in \mathbb{S}^m$ given symmetric matrices
- $x_i \in \mathbb{R}^n$ decision variables

Fundamental property: feasible set is convex

$$\mathcal{S} = \{x \in \mathbb{R}^n \; : \; F(x) \succeq \mathbf{0}\}$$

$\mathcal{S}$ is the Spectrahedron

Nota : $\succeq 0$ ($\succ 0$) means positive semidefinite (positive definite) e.g. real nonnegative eigenvalues (strictly positive eigenvalues) and defines generalized inequalities on PSD cone

Terminology coined out by Jan Willems in 1971

$$F(P) = \begin{bmatrix} A'P + PA + Q & PB + C' \\ B'P + C & R \end{bmatrix} \succeq \mathbf{0}$$

"The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms"

Lyapunov's LMI

Historically, the first LMIs appeared around 1890 when Lyapunov showed that the autonomous system with LTI model:

$$\frac{d}{dt}x(t) = \dot{x}(t) = Ax(t)$$

is stable (all trajectories converge to zero) iff there exists a solution to the matrix inequalities

$$A'P + PA \prec \mathbf{0} \quad P = P' \succ \mathbf{0}$$

which are linear in unknown matrix P

Aleksandr Mikhailovich Lyapunov
(1857 Yaroslavl - 1918 Odessa)

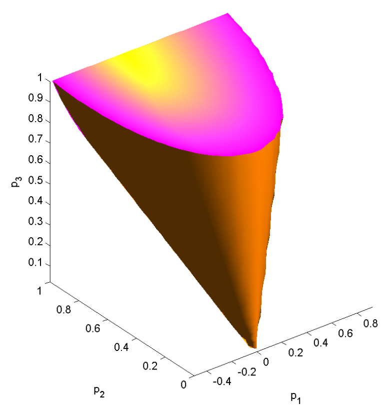
Example of Lyapunov's LMI

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$A'P + PA \prec \mathbf{0} \qquad P \succ \mathbf{0}$$

$$\begin{bmatrix} -2p_1 & 2p_1 - 3p_2 \\ 2p_1 - 3p_2 & 4p_2 - 4p_3 \end{bmatrix} \prec \mathbf{0}$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \succ \mathbf{0}$$



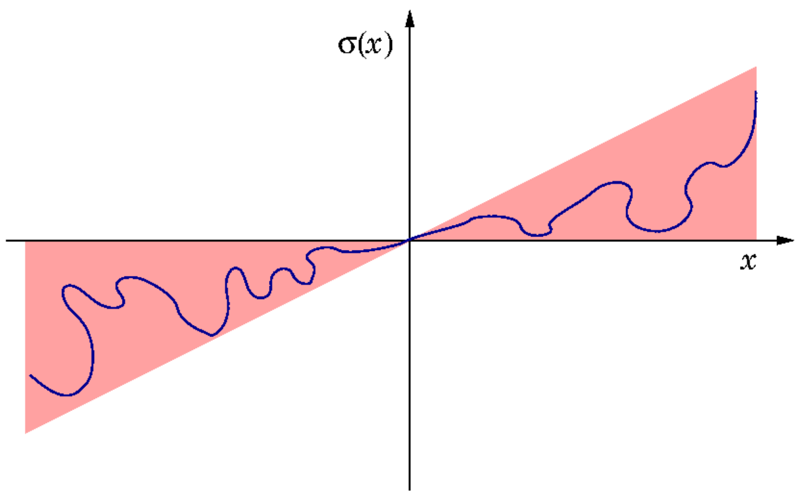
Matrices P satisfying Lyapunov LMI's

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} p_1 + \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} p_2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} p_3 \succ \mathbf{0}$$

Some history (1)

1940s - Absolute stability problem: Lu're, Postnikov et al applied Lyapunov's approach to control problems with nonlinearity in the actuator

$$\dot{x} = Ax + b\sigma(x)$$



Sector-type nonlinearity

- Stability criteria in the form of LMIs solved analytically by hand

- Reduction to Polynomial (frequency dependent) inequalities (small size)

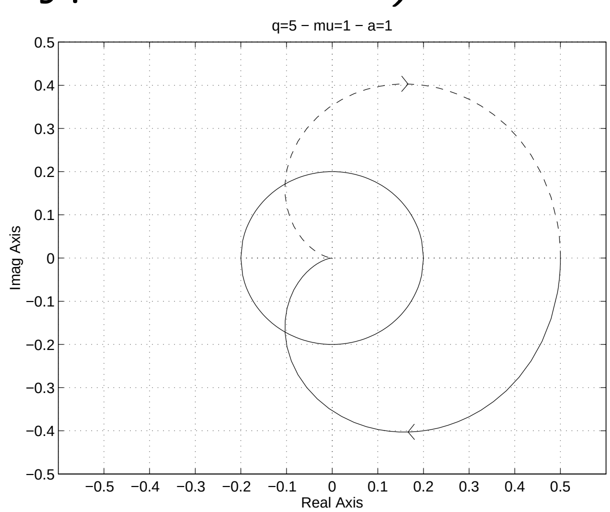
Some history (2)

1960s: Yakubovich, Popov, Kalman, Anderson et al obtained the positive real lemma

The linear system $\dot{x} = Ax + Bu, \quad y = Cx + Du$ is passive $H(s) + H(s)^* \geq 0 \ \forall \ s + s^* > 0$ iff

$$P \succ \mathbf{0} \quad \begin{bmatrix} A'P + PA & PB - C' \\ B'P - C & -D - D' \end{bmatrix} \preceq \mathbf{0}$$

- Solution via a simple graphical criterion (Popov, circle and Tsypkin criteria)



Mathieu equation: $\ddot{y} + 2\mu\dot{y} + (\mu^2 + a^2 - q\cos\omega_0 t)y = 0$
$q < 2\mu a$

Some history (3)

1971: Willems focused on solving algebraic Riccati equations (AREs)

$$A'P + PA - (PB + C')R^{-1}(B'P + C) + Q = \mathbf{0}$$

Numerical algebra

$$H = \begin{bmatrix} A - BR^{-1}C & BR^{-1}B' \\ -C'R^{-1}C & -A' + C'R^{-1}B' \end{bmatrix} \quad V = \begin{bmatrix} V_1 \\ \\ V_2 \end{bmatrix}$$

$$P_{are} = V_2 V_1^{-1}$$

By 1971, methods for solving LMIs:

- Direct for small systems
- Graphical methods
- Solving Lyapunov or Riccati equations

Some history (4)

1963: Bellman-Fan: infeasibility criteria for multiple Lyapunov inequalities (duality theory)

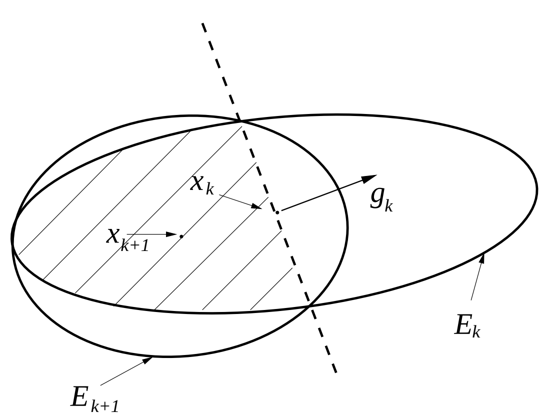
On Systems of Linear Inequalities in hermitian Matrix Variables

1975: Cullum-Donath-Wolfe: Optimality conditions, nondifferentiable criterion for multiple eigenvalues and algorithm for minimization of sum of maximum eigenvalues

The minimization of certain nondifferentiable sums of eigenvalues of symmetric matrices

1979: Khachiyan: polynomial bound on worst case iteration count for LP ellipsoid algorithm of Nemirovski and Shor

A polynomial algorithm in linear programming



Some history (5)

1981: Craven-Mond: Duality theory

Linear Programming with Matrix variables

1984: Karmarkar introduces interior-point (IP) methods for LP: improved complexity bound and efficiency

1985: Fletcher: Optimality conditions for non-differentiable optimization

Semidefinite matrix constraints in optimization

1988: Overton: Nondifferentiable optimization

On minimizing the maximum eigenvalue of a symmetric matrix

1988: Nesterov, Nemirovski, Alizadeh, Karmarkar and Thakur extend IP methods for convex programming

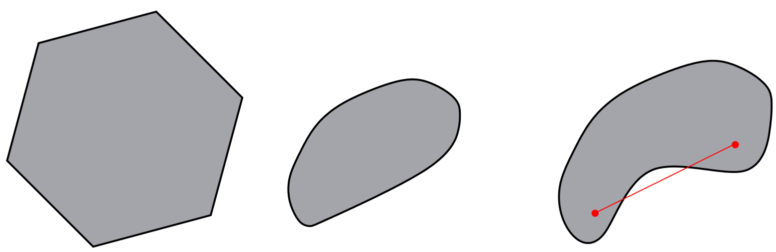
Interior-Point Polynomial Algorithms in Convex Programming

1990s: most papers on SDP are written (control theory, combinatorial optimization, approximation theory...)

Mathematical preliminaries (1)

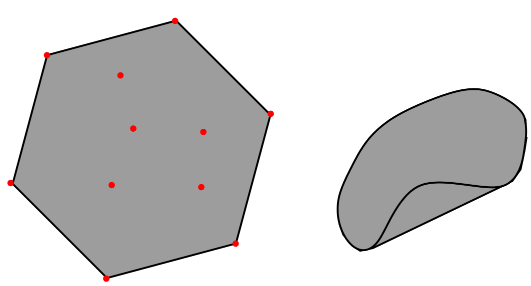
A set $\mathcal{C}$ is convex if the line segment between any two points in $\mathcal{C}$ lies in $\mathcal{C}$

$$\forall\, x_1, x_2 \in \mathcal{C} \quad \lambda x_1 + (1-\lambda) x_2 \in \mathcal{C} \quad \forall\, \lambda \quad 0 \leq \lambda \leq 1$$



The convex hull of a set $\mathcal{C}$ is the set of all convex combinations of points in $\mathcal{C}$

$$\text{co}\, \mathcal{C} = \{ \sum_i \lambda_i x_i \; : \; x_i \in \mathcal{C} \quad \lambda_i \geq 0 \quad \sum_i \lambda_i = 1 \}$$



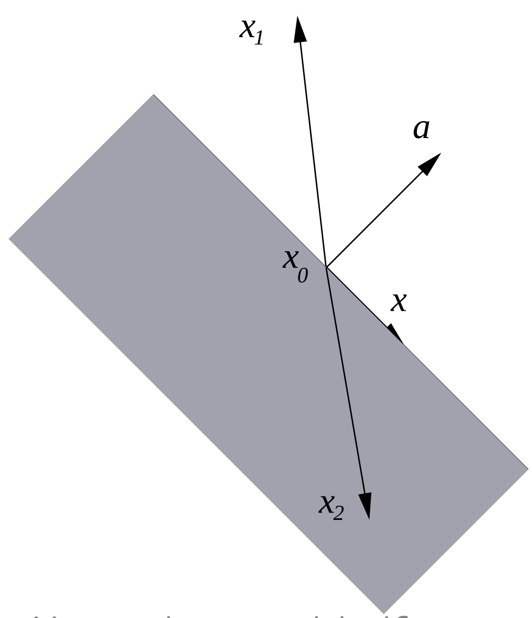
Mathematical preliminaries (2)

A hyperplane is a set of the form:

$$\mathcal{H} = \left\{ x \in \mathbb{R}^n \mid a'(x - x_0) = 0 \right\} \quad a \neq 0 \in \mathbb{R}^n$$

A hyperplane divides $\mathbb{R}^n$ into two halfspaces:

$$\mathcal{H}_- = \left\{ x \in \mathbb{R}^n \mid a'(x - x_0) \leq 0 \right\} \quad a \neq 0 \in \mathbb{R}^n$$



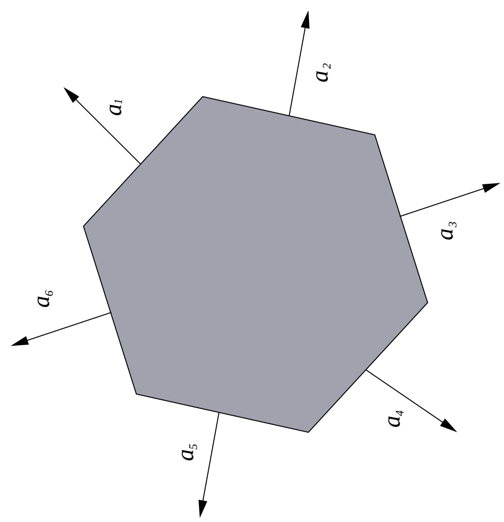
Hyperplane and halfspace
$x \in \mathcal{H}$, $x_1 \notin \mathcal{H}_-$, $x_2 \in \mathcal{H}_-$

Mathematical preliminaries (3)

A polyhedron is defined by a finite number of linear equalities and inequalities

$$\begin{aligned}\mathcal{P} &= \left\{x \in \mathbb{R}^n : \; a_j' x \leq b_j, j = 1, \cdots, m, c_i' x = d_i, i = 1, \cdots, p\right\} \\ &= \{x \in \mathbb{R}^n \; : \; Ax \preceq b, \; Cx = d\}\end{aligned}$$

A bounded polyhedron is a polytope



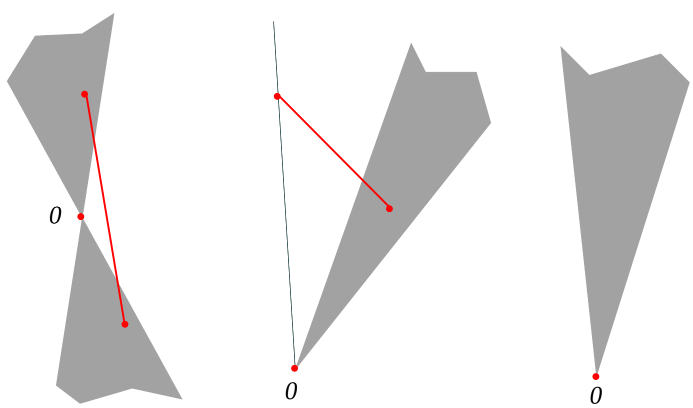
Polytope as an intersection of halfspaces

- positive orthant is a polyhedral cone
- k-dimensional simplexes in $\mathbb{R}^n$

$$\mathcal{X} = \text{co}\{v_0, \cdots, v_k\} = \left\{\sum_{i=0}^{k} \lambda_i v_i \;\; \lambda_i \geq 0 \;\; \sum_{i=0}^{k} \lambda_i = 1\right\}$$

Mathematical preliminaries (4)

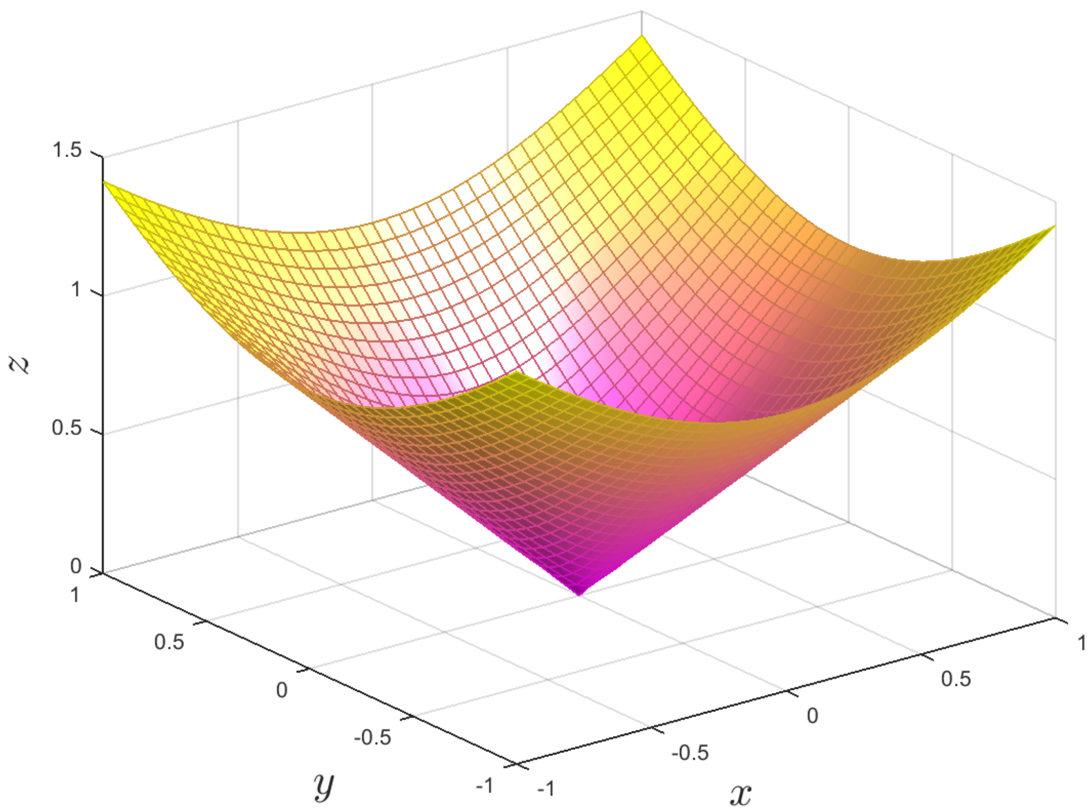
A set $\mathcal{K}$ is a cone if for every $x \in \mathcal{K}$ and $\lambda \geq 0$ we have $\lambda x \in \mathcal{K}$. A set $\mathcal{K}$ is a convex cone if it is convex and a cone



$\mathcal{K} \subseteq \mathbb{R}^n$ is called a proper cone if it is a closed solid pointed convex cone

$$a \in \mathcal{K} \quad \text{and} \quad -a \in \mathcal{K} \quad \Rightarrow \quad a = 0$$

Lorentz cone $\mathbb{L}^n$

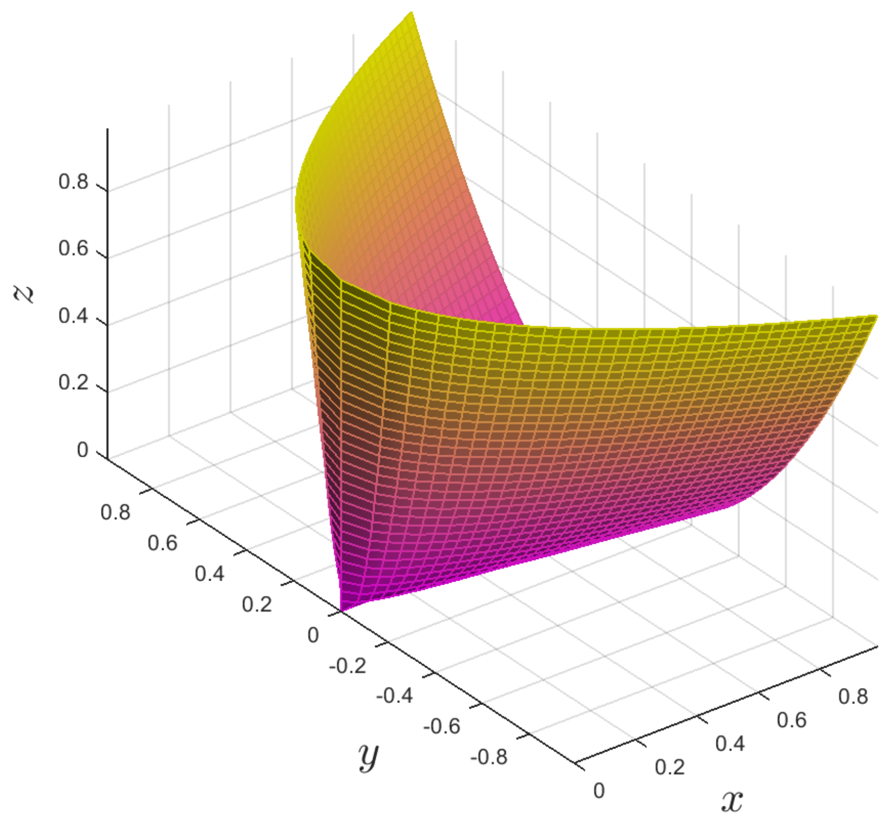


3D Lorentz cone or ice-cream cone

$$x^2 + y^2 \leq z^2 \quad z \geq 0$$

arises in quadratic programming

PSD cone $\mathbb{S}^n_+$



2D positive semidefinite cone

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0 \iff x \geq 0 \quad z \geq 0 \quad xz \geq y^2$$

arises in semidefinite programming

Mathematical preliminaries (5)

Every proper cone $\mathcal{K}$ in $\mathbb{R}^n$ induces a partial ordering $\succeq_{\mathcal{K}}$ defining generalized inequalities on $\mathbb{R}^n$

$$a \succeq_{\mathcal{K}} b \quad \Leftrightarrow \quad a - b \in \mathcal{K}$$

The positive orthant, the Lorentz cone and the PSD cone are all proper cones

- positive orthant $\mathbb{R}^n_+$: standard coordinatewise ordering (LP)

$$x \succeq_{\mathbb{R}^n_+} y \;\Leftrightarrow\; x_i \geq y_i$$

- Lorentz cone $\mathbb{L}^n$

$$x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}$$

- PSD cone $\mathbb{S}^n_+$: Löwner partial order

Mathematical preliminaries (6)

The set $\mathcal{K}^* = \{y \in \mathbb{R}^n \mid x'y \geq 0 \ \ \forall\ x \in \mathcal{K}\}$ is called the dual cone of the cone $\mathcal{K}$

- $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$

- $(\mathbb{S}^n_+)^* = \mathbb{S}^n_+$

- $\mathbb{L}^n = \left\{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\right\}$, then
$(\mathbb{L}^n)^* = \left\{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq v\right\}$ with
$\|u\|_* = \sup\{u'x \mid \|x\| \leq 1\}$

$\mathcal{K}^*$ is closed and convex, $\mathcal{K}_1 \subseteq \mathcal{K}_2 \ \Rightarrow\ \mathcal{K}_2^* \subseteq \mathcal{K}_1^*$

$\preceq_{\mathcal{K}^*}$ is a dual generalized inequality

$$x \preceq_{\mathcal{K}} y \quad \Leftrightarrow \quad \lambda' x \leq \lambda' y \ \ \forall\ \lambda \succeq_{\mathcal{K}^*} 0$$

Mathematical preliminaries (7)

$f\ :\ \mathbb{R}^n \to \mathbb{R}$ is convex if dom$f$ is a convex set and $\forall\ x,\ y \in$ dom$f$ and $0 \leq \lambda \leq 1$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

If f is differentiable: dom$f$ is a convex set and $\forall\ x,\ y \in$ dom$f$

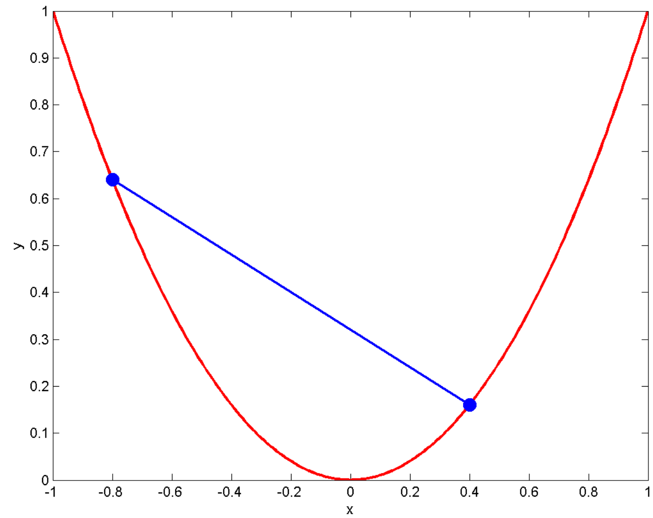
$$f(y) \geq f(x) + \nabla f(x)'(y-x)$$

If f is twice differentiable: dom$f$ is a convex set and $\forall\ x,\ y \in$ dom$f$

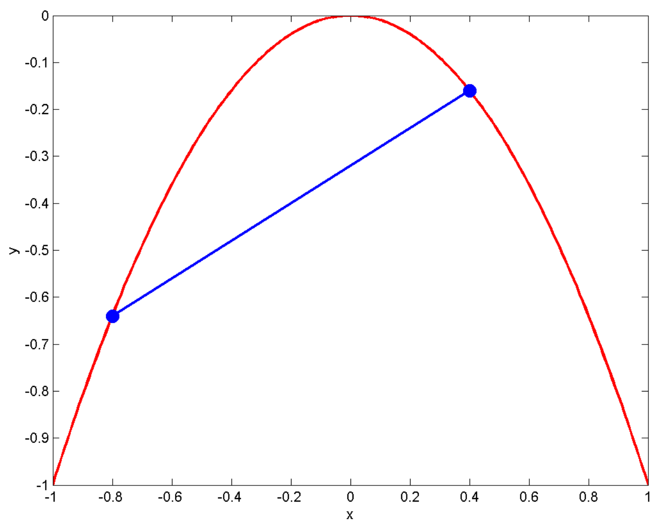
$$\nabla^2 f(x) \succeq \mathbf{0}$$

Quadratic functions:
$f(x) = (1/2)x'Px + q'x + r$ is convex if and only if $P \succeq \mathbf{0}$

Convex function $y = x^2$



Nonconvex function $y = -x^2$



Mind the sign !

LMI and SDP formalisms (1)

In mathematical programming terminology
LMI optimization = semidefinite programming (SDP)

LMI (SDP dual)

$$\begin{array}{ll} \min & c'x \\ \text{under} & F_0 + \sum_{i=1}^{n} x_i F_i \prec \mathbf{0} \end{array}$$

SDP (primal)

$$\begin{array}{ll} \min & -\mathrm{Tr}(F_0 Z) \\ \text{under} & -\mathrm{Tr}(F_i Z) = c_i \\ & Z \succeq \mathbf{0} \end{array}$$

$x \in \mathbb{R}^n$, $Z \in \mathbb{S}^m$, $F_i \in \mathbb{S}^m$, $c \in \mathbb{R}^n$, $i = 1, \cdots, n$

Nota:
In a typical control LMI

$$A'P + PA = F_0 + \sum_{i=1}^{n} x_i F_i \prec \mathbf{0}$$

individual matrix entries are decision variables

LMI and SDP formalisms (2)

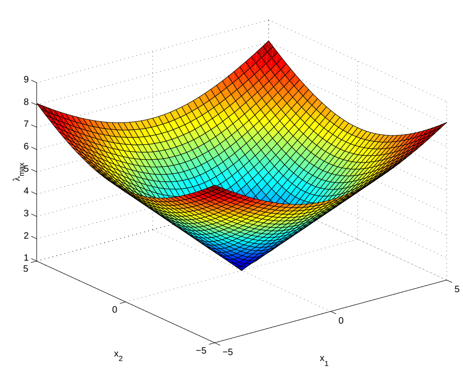
$$\exists\, x \in \mathbb{R}^n \mid \underbrace{F_0 + \sum_{i=1}^{n} x_i F_i}_{F(x)} \prec \mathbf{0} \quad \Leftrightarrow \quad \min_{x \in \mathbb{R}^n} \lambda_{max}(F(x))$$

The LMI feasibility problem is a convex and non differentiable optimization problem.

Example :

$$F(x) = \begin{bmatrix} -x_1 - 1 & -x_2 \\ -x_2 & -1 + x_1 \end{bmatrix}$$

$$\lambda_{max}(F(x)) = 1 + \sqrt{(x_1^2 + x_2^2)}$$



LMI and SDP formalisms (3)

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & \\ & b - A'x \in \mathcal{K} \end{array} \qquad \begin{array}{ll} \min & b'y \\ \text{s.t.} & \\ & Ay = c \\ & y \in \mathcal{K} \end{array}$$

Conic programming in cone $\mathcal{K}$

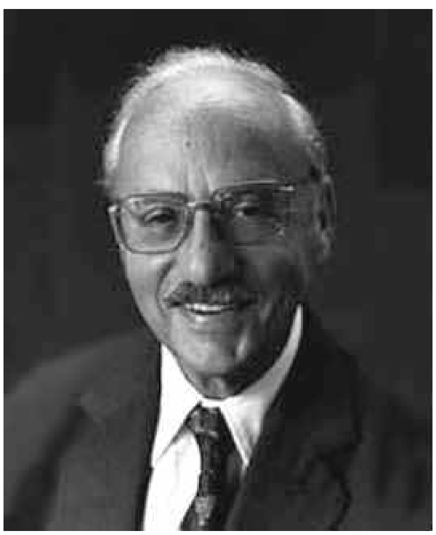
- positive orthant (LP)
- Lorentz (second-order) cone (SOCP)
- positive semidefinite cone (SDP)

Hierarchy: LP cone $\subset$ SOCP cone $\subset$ SDP cone

LMI and SDP formalisms (4)

LMI optimization = generalization of linear programming (LP) to cone of positive semidefinite matrices = semidefinite programming (SDP)

Linear programming pioneered by
- Dantzig and its simplex algorithm (1947, ranked in the top 10 algorithms by SIAM Review in 2000)
- Kantorovich (co-winner of the 1975 Nobel prize in economics)



George Dantzig
(1914 Portland, Oregon)

Leonid V Kantorovich
(1921 St Petersburg - 1986)

Unfortunately, SDP has not reached maturity of LP or SOCP so far..

Applications of SDP

• control systems
• robust optimization
• signal processing
• sparse Principal Component Analysis
• structural design (trusses)
• geometry (ellipsoids)
• Euclidean distance matrices (sensor network localization, molecular conformation)
• graph theory and combinatorics (MAXCUT, Shannon capacity)
• facility layout problem (single-row facility layout problem, VLSI floorplanning)

and many others...

See Helmberg's page on SDP
`www-user.tu-chemnitz.de/~helmberg/semidef.html`

Robust optimization (1)

In many real-life applications of optimization problems, exact values of input data (constraints) are seldom known

- Uncertainty about the future
- Approximations of complexity by uncertainty
- Errors in the data
- variables may be implemented with errors

$$\begin{array}{ll} \text{min} & f_0(x,u) \\ \text{under} & f_i(x,u) \leq 0 \quad i = 1, \cdots, m \end{array}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables and $u \in \mathbb{R}^p$ is the parameters vector.

- Stochastic programming
- Sensitivity analysis
- Interval arithmetic
- Worst-case analysis

$$\begin{array}{lll} \min\limits_{x} & \sup\limits_{u \in \mathcal{U}} & f_0(x,u) \\ \text{under} & & \sup\limits_{u \in \mathcal{U}} f_i(x,u) \leq 0 \quad i = 1, \cdots, m \end{array}$$

Robust optimization (2)

Case study by Ben Tal and Nemirovski:
[Math. Programm. 2000]
90 LP problems from NETLIB + uncertainty
quite small (just 0.1%) perturbations of "obviously uncertain" data coefficients can make the "nominal" optimal solution x^* heavily infeasible
Remedy: robust optimization, with robustly feasible solutions guaranteed to remain feasible at the expense of possible conservatism
Robust conic problem: [Ben Tal Nemirovski 96]

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & c'x \\ \text{s.t.} & Ax - b \in \mathcal{K}, \quad \forall \; (A, b) \in \mathcal{U} \end{array}$$

This last problem, the so-called robust counterpart is still convex, but depending on the structure of $\mathcal{U}$, can be much harder that original conic problem

Robust optimization (3)

Uncertainty	Problem	Optimization Problem
polytopic	LP	LP
ellipsoid		SOCP
LMI		SDP
polytopic	SOCP	SOCP
ellipsoid		SDP
LMI		NP-hard

Examples of applications:

Robust LP: Robust portfolio design in finance [Lobo 98], discrete-time optimal control [Boyd 97], robust synthesis of antennae arrays [Lebret 94], FIR filter design [Wu 96]

Robust SOCP: robust least-squares in identification [El Ghaoui 97], robust synthesis of antennae arrays and FIR filter synthesis

Robust optimization (4)
Robust LP as a SOCP

Robust counterpart of robust LP

$$\min_{x \in \mathbb{R}^n} \quad c'x$$

s.t.

$$a_i'x \leq b_i, \quad i = 1, \cdots m,$$
$$\forall \ a_i \in \mathcal{E}_i$$
$$\mathcal{E}_i = \{\overline{a}_i + P_i u \mid ||u||_2 \leq 1 \text{ and } P_i \succeq 0\}$$

Note that

$$\max_{a_i \in \mathcal{E}_i} \ a_i'x = \overline{a}_i'x + ||P_i x||_2 \leq b_i$$

SOCP formulation

$$\min_{x \in \mathbb{R}^n} \quad c'x$$

s.t.

$$\overline{a}_i'x + ||P_i x|| \leq b_i, \quad i = 1, \cdots m,$$

Robust optimization (5)
Example of Robust LP

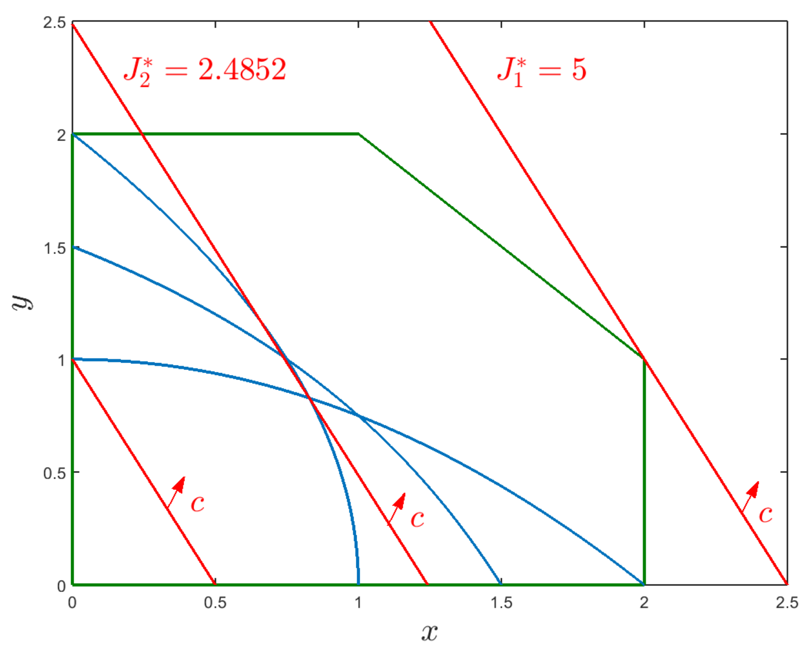
$$J_1^* = \max_{x,y} \quad 2x + y$$
$$\text{s.t.} \quad x \geq 0, \; y \geq 0$$
$$x \leq 2$$
$$y \leq 2$$
$$x + y \leq 3$$

$$J_2^* = \max_{x,y} \quad 2x + y$$
$$\text{s.t.} \quad x \geq 0, \; y \geq 0$$
$$\sqrt{x^2 + y^2} \leq 2 - x$$
$$\sqrt{x^2 + y^2} \leq 2 - y$$
$$\sqrt{x^2 + y^2} \leq 3 - x - y$$

$$(x^*, y^*) = (2, 1) \qquad J_1^* = 5$$

$$(x^*, y^*) = (0.8284, 0.8284) \qquad J_2^* = 2.4852$$

$$\mathcal{E}_1 = \mathcal{E}_2 = \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T + \mathbf{1}_2 u \mid ||u||_2 \leq 1 \right\}$$
$$\mathcal{E}_3 = \left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T + \mathbf{1}_2 u \mid ||u||_2 \leq 1 \right\}$$



Combinatorial optimization (1)

Combinatorics: Graph theory, polyhedral combinatorics, combinatorial optimization, enumerative combinatorics...

Definition: Optimization problems in which the solution space is discrete (finite collection of objects) or a decision-making problem in which each decision has a finite (possibly many) number of feasibilities

Depending upon the formalism

- 0-1 Linear Programming problems: 0-1 Knapsack problem,...
- Propositional logic: Maximum satisfiability problems...
- Constraints satisfaction problems: Airline crew assignment, maximum weighted stable set problem...
- Graph problems: Max-Cut, Shannon or Lovasz capacity of a graph, bandwidth problems, equipartition problems...

Combinatorial optimization (2)

SDP relaxation of QP in binary variables

$$(BQP) \quad \max_{x \in \{-1,1\}} \quad x'Qx$$

Noticing that $x'Qx = \text{trace}(Qxx')$
we get the equivalent form

$$(BQP) \quad \max_{X} \quad \text{trace}(QX)$$
$$s.t. \quad \begin{array}{l} \text{diag}(X_{ii}) = e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}' \\ X \succeq 0 \\ \text{rank}(X) = 1 \end{array}$$

Dropping the non convex rank constraint leads to the SDP relaxation:

$$(SDP) \quad \max_{X} \quad \text{trace}(QX)$$
$$s.t. \quad \begin{array}{l} \text{diag}(X_{ii}) = e = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}' \\ X \succeq 0 \end{array}$$

Interpretation: lift from $\mathbb{R}^n$ to $\mathbb{S}^n$

Combinatorial optimization (3)

Example

$$(BQP) \quad \min_{x \in \{-1,1\}} \quad x'Qx = x_1x_2 - 2x_1x_3 + 3x_2x_3$$

with $Q = \begin{bmatrix} 0 & 0.5 & -1 \\ 0.5 & 0 & 1.5 \\ -1 & 1.5 & 0 \end{bmatrix}$

SDP relaxation

$$(SDP) \quad \min_{X} \quad \text{trace}(QX) = X_1 - 2X_2 + 3X_3$$

$$\text{s.t.} \quad X = \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & 1 & X_3 \\ X_2 & X_3 & 1 \end{bmatrix} \succeq 0$$

$$X^* = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{rank}(X^*) = 1$$

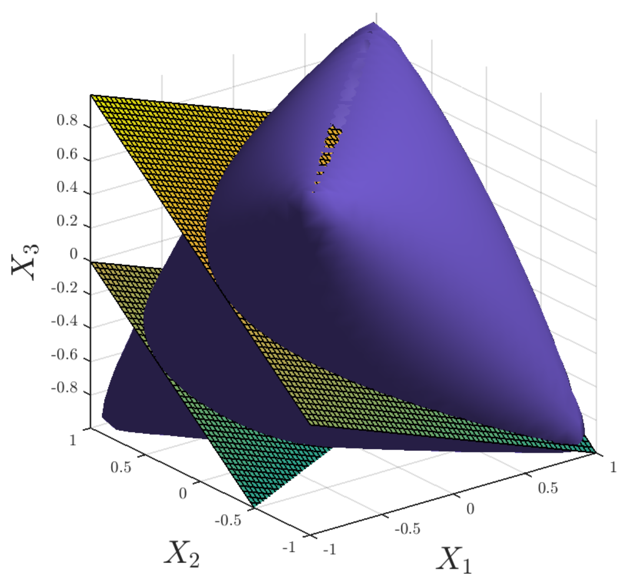
From $X^* = x^* x^{*\prime}$, we recover the optimal solution of (BQP)

$$x^* = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}'$$

Example (continued)

Visualization of the feasible set of (SDP) in (X_1, X_2, X_3) space :

$$X = \begin{bmatrix} 1 & X_1 & X_2 \\ X_1 & 1 & X_3 \\ X_2 & X_3 & 1 \end{bmatrix} \succeq 0$$



Optimal vertex is $\begin{bmatrix} -1 & 1 & -1 \end{bmatrix}$

LMI OPTIMIZATION
PART 2

Lagrangian and SDP duality

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Duality

- Versatile notion
- Theoritical results and numerical methods
- Certificates of infeasibility

Lagrangian duality has many applications and interpretations (price or tax, game, geometry...)

Applications of SDP duality:
• numerical solvers design
• problems reduction
• new theoretical insights into control problems

In the sequel we will recall some basic facts about Lagrangian duality and SDP duality

Lagrangian duality

Let the primal problem

$$p^\star = \min_{x \in \mathbb{R}^n} \; f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad i = 1, \cdots, m$$
$$h_i(x) = 0 \quad i = 1, \cdots, p$$

Define Lagrangian $L(.,.,.)\ \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

where λ, μ are Lagrange multipliers vectors or dual variables

Let the Lagrange dual function

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \mu)$$

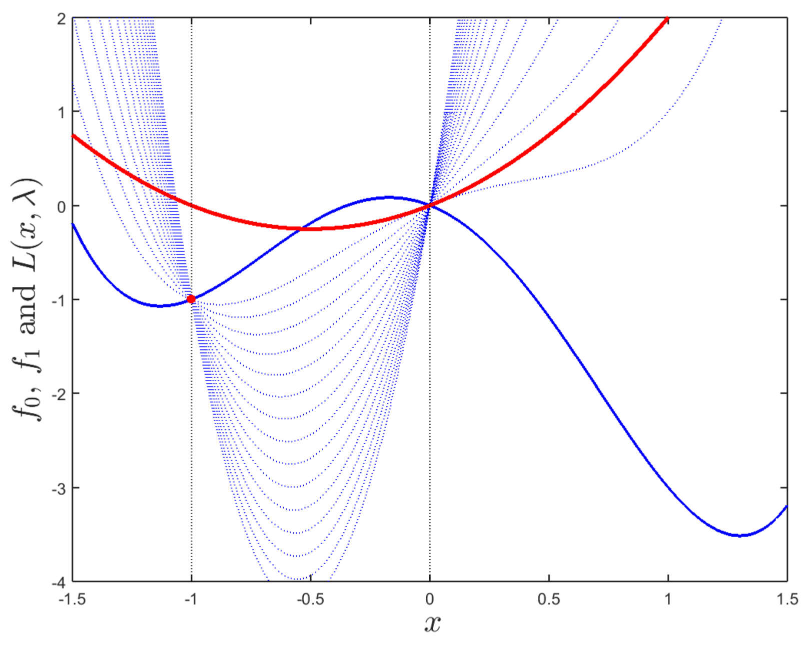
- g is always concave
- $g(\lambda, \mu) = -\infty$ if there is no finite infimum

Lagrangian duality (2)

A pair (λ, μ) s.t. $\lambda \succeq \mathbf{0}$ and $g(\lambda, \mu) > -\infty$ is dual feasible

For any primal feasible x and dual feasible pair (λ, μ)

$$g(\lambda, \mu) \leq p^* \leq f_0(x)$$



$$\min_x \quad x^4 - 3x^2 - x$$
$$\text{under} \quad x(x+1) \leq 0$$

Lagrangian duality (3)

Lagrange dual problem

$$d^\star = \max_{\lambda,\mu} \; g(\lambda,\mu) \quad \text{s.t.} \quad \lambda \succeq \mathbf{0}$$

The Lagrange dual problem is a convex optimization problem

Primal	Dual
$\inf_{x\in\mathbb{R}^n} \sup_{\lambda,\mu} L(x,\lambda,\mu)$ s.t. $\lambda \succeq \mathbf{0}$	$\sup_{\lambda,\mu} \inf_{x\in\mathbb{R}^n} L(x,\lambda,\mu)$ s.t. $\lambda \succeq \mathbf{0}$

A Lagrangian relaxation consists in solving the dual problem instead of the primal problem

Weak and strong duality

Weak duality (max-min inequality):

$$p^\star \geq d^\star$$

because

$$g(\lambda, \mu) \leq f_0(x) + \sum_{i=1}^{m} \lambda_i \underbrace{f_i(x)}_{\leq 0} + \sum_{i=1}^{p} \mu_i \underbrace{h_i(x)}_{=0} \leq f_0(x)$$

for any primal feasible x and dual feasible λ, μ

The difference $p^\star - d^\star \geq 0$ is called duality gap

Strong duality (saddle-point property):

$$p^\star = d^\star$$

Sometimes, constraint qualifications ensure that strong duality holds
Example: Slater's condition = strictly feasible convex primal problem

$$f_i(x) < 0,\ i = 1, \cdots, m \quad h_i(x) = 0,\ i = 1, \cdots, p$$

Geometric interpretation of duality (1)

Consider the primal optimization problem

$$p^\star = \min_{x \in \mathbb{R}} \; f_0(x) \quad \text{s.t.} \quad f_1(x) \leq 0$$

with Lagrangian and dual function

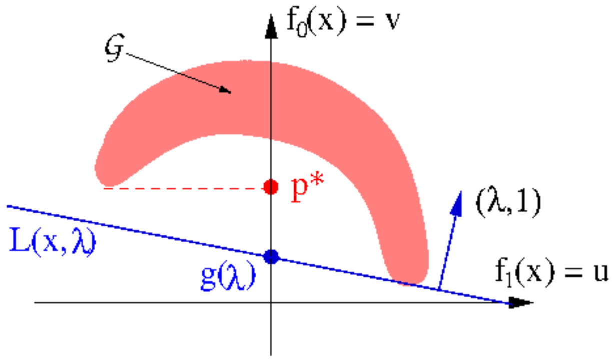
$$L(x, \lambda) = f_0(x) + \lambda f_1(x) \quad g(\lambda) = \inf_x L(x, \lambda)$$

The dual problem is given by:

$$d^\star = \max_{\lambda} \; g(\lambda) \quad \text{s.t.} \quad \lambda \succeq \mathbf{0}$$

Geometric interpretation of duality (2)

Set of values $\mathcal{G} = (f_1(x),\ f_0(x)),\ \forall\ x \in \mathcal{D}$



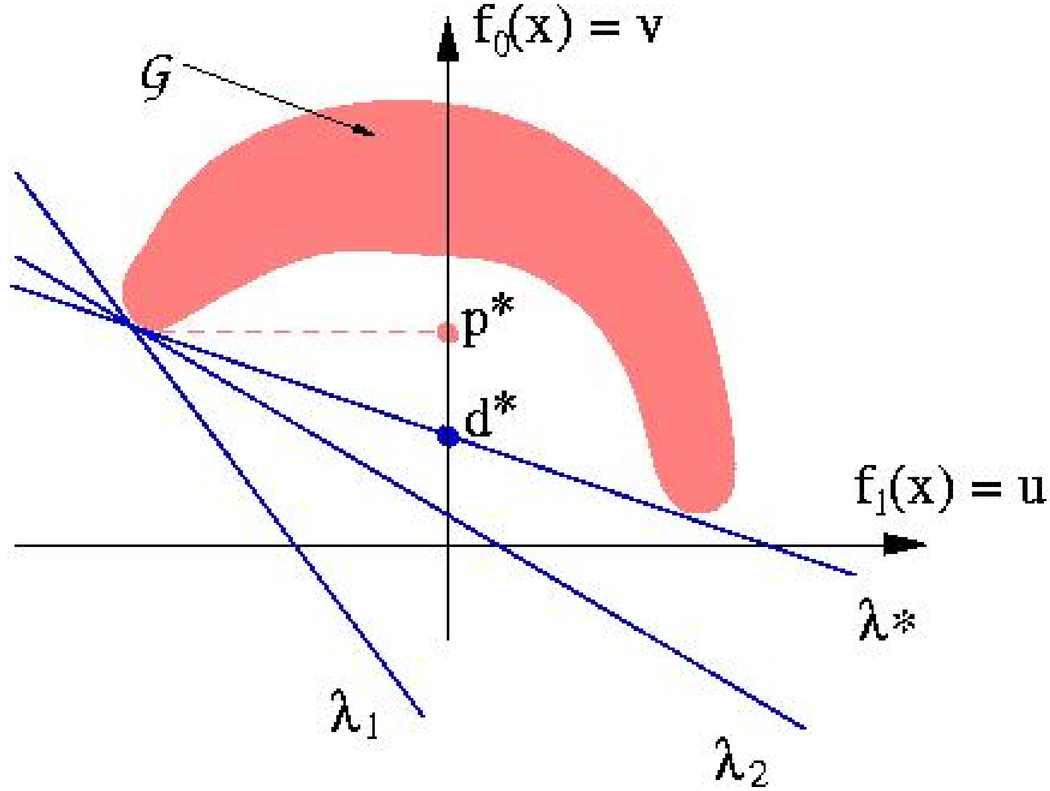
$$L(x,\lambda) = f_0(x) + \lambda f_1(x) = \left[\begin{array}{cc} \lambda & 1 \end{array}\right] \left[\begin{array}{c} f_1(x) \\ f_0(x) \end{array}\right]$$

$$g(\lambda) = \inf_{x\in\mathcal{D}} L(\lambda, x) = \inf_{x\in\mathcal{D}} \left\{ \left[\begin{array}{cc} \lambda & 1 \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right]\ (u,v) \in \mathcal{G} \right\}$$

Supporting hyperplane with slope $-\lambda$

$$\left[\begin{array}{cc} \lambda & 1 \end{array}\right] \left[\begin{array}{c} u \\ v \end{array}\right] \geq g(\lambda)\ (u,v) \in \mathcal{G}$$

Geometric interpretation of duality (3)



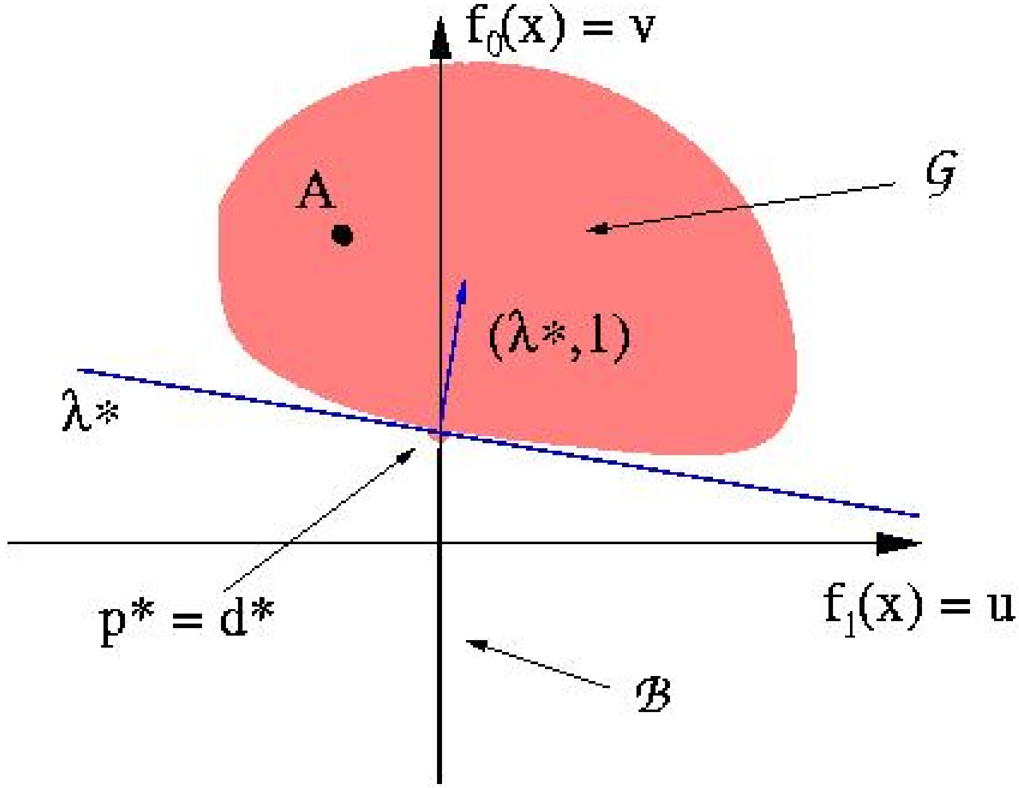
Three supporting hyperplanes, including the optimum $\lambda^\star$ yielding $d^\star < p^\star$
No strong duality here

$$p^* - d^* > 0$$

Duality gap $\neq 0$

Geometric interpretation of duality (4)

$$\mathcal{B} = \{(0, s) \in \mathbb{R} \times \mathbb{R} \ : \ s < p^*\}$$



- Separating hyperplane theorem for $\mathcal{G}$ and $\mathcal{B}$
- The separating hyperplane is a supporting hyperplane to $\mathcal{G}$ in $(0, p^*)$
- Slater's condition ensures the hyperplane is non vertical

Optimality conditions

Suppose that strong duality holds, let $x^\star$ be primal optimal and $(\lambda^\star, \mu^\star)$ be dual optimal,

$$\begin{aligned} f_0(x^\star) &= g(\lambda^\star, \mu^\star) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^\star f_i(x) + \sum_{i=1}^p \mu_i^\star h_i(x) \right) \\ &\leq f_0(x^\star) + \sum_{i=1}^m \lambda_i^\star f_i(x^\star) + \sum_{i=1}^p \mu_i^\star h_i(x^\star) \\ &\leq f_0(x^\star) \end{aligned}$$

$$\lambda_i^\star f_i(x^\star) = 0 \quad i = 1, \cdots, m$$

This is complementary slackness condition

$$\lambda_i^\star > 0 \;\Rightarrow\; f_i(x^\star) = 0 \quad \text{or} \quad f_i(x^\star) < 0 \;\Rightarrow\; \lambda_i^\star = 0$$

In words, the ith optimal Lagrange multiplier is zero unless the ith constraint is active at the optimum

KKT optimality conditions

f_i, h_i are differentiable and strong duality holds

$$
\begin{gathered}
h_i(x^\star) = 0,\ i = 1, \cdots, p, \quad \text{(primal feasible)} \\
f_i(x^\star) \leq 0,\ i = 1, \cdots, m, \ \text{(primal feasible)} \\
\lambda_i^\star \succeq \mathbf{0},\ i = 1, \cdots, m, \ \text{(dual feasible)} \\
\lambda_i^\star f_i(x^\star) = 0,\ i = 1, \cdots, m, \ \text{(complementary)} \\
\nabla f_0(x^\star) + \sum_{i=1}^{p} \lambda_i^\star \nabla f_i(x^\star) + \sum_{i=1}^{p} \mu_i^\star \nabla h_i(x^\star) = 0
\end{gathered}
$$

Necessary Karush-Kuhn-Tucker conditions satisfied by any primal and dual optimal pair $x^\star$ and $(\lambda^\star, \mu^\star)$

For convex problems, KKT conditions are also sufficient

Feasibility of inequalities (1)

$$\exists \; x \in \mathbb{R}^n \;\; : \;\; \begin{cases} f_i(x) \leq 0 & i = 1, \cdots, m \\ h_i(x) = 0 & i = 1, \cdots, p \end{cases}$$

Dual function: $g(.,.) \; : \; \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

The dual feasibility problem is

$$\exists \; (\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p \;\; : \;\; \begin{cases} g(\lambda, \mu) > 0 \\ \lambda \succeq \mathbf{0} \end{cases}$$

Theorem of weak alternatives

At most, one of the two (primal and dual) is feasible

If the dual problem is feasible then the primal problem is infeasible

Feasibility of inequalities (2)

Proof of the theorem of alternatives

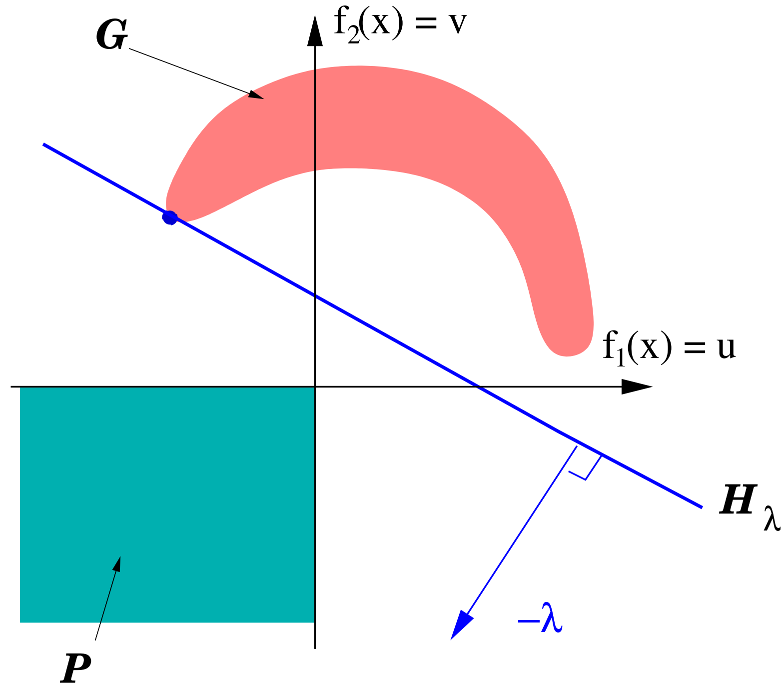
Suppose $\overline{x} \in \mathcal{D}$ is a feasible point for the primal problem

$$\begin{aligned} g(\lambda, \mu) &= \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x) \\ &\leq \sum_{i=1}^{m} \lambda_i \underbrace{f_i(\overline{x})}_{\leq 0} + \sum_{i=1}^{p} \mu_i \underbrace{h_i(\overline{x})}_{=0} \\ &\quad \forall\ (\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p \end{aligned}$$

and so $g(\lambda, \mu) \leq 0$ for all $\lambda \succeq 0$
If f_i are convex functions, h_i are affine functions and some type of constraint qualification holds:
Theorem of strong alternatives
Exactly one of the two alternative holds

A dual feasible pair (λ, μ) gives a certificate (proof) of infeasibility of the primal

Feasibility of inequalities (3)
Geometric interpretation



$$P = \left\{ (u,v) \in \mathbb{R}^2 \;:\; \begin{bmatrix} u \\ v \end{bmatrix} \preceq 0 \right\}$$

$$H_\lambda = \left\{ (u,v) \in \mathbb{R}^2 \;:\; \lambda' \begin{bmatrix} u \\ v \end{bmatrix} = g(\lambda) \right\}$$

If $g(\lambda) > 0$ and $\lambda \succeq 0$ then H_λ is a separating hyperplane for P from

$$G = \left\{ \begin{bmatrix} f_1(x) & f_2(x) \end{bmatrix} \;:\; x \in \mathbb{R}^n \right\}$$

Conic duality (1)

Let the primal:

$$p^\star = \min_{x\in\mathbb{R}^n} \; f_0(x) \quad \text{s.t.} \quad f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots m$$

Lagrange dual function: $g(.) \; : \; \mathbb{R}^m \to \mathbb{R}$

$$g(\lambda) = \inf_{x\in\mathcal{D}} f_0(x) + \sum_{i=1}^{m} \lambda_i' f_i(x)$$

Lagrange dual problem:

$$d^\star = \max_{\lambda\in\mathbb{R}^m} \; g(\lambda) \quad \text{s.t.} \quad \lambda_i \succeq_{\mathcal{K}_i^*} \mathbf{0}, \quad i = 1, \cdots, m$$

Conic duality (2)

- Weak duality
- Strong duality:

- if primal is s.f. with finite $p^\star$ then $d^\star$ is reached by dual
- if dual is s.f. with finite $d^\star$ then $p^\star$ is reached by primal
- if primal and dual are s.f. then $p^\star = d^\star$

- Complementary slackness:

$$\lambda_i^{\star\prime} f_i(x^\star) = 0$$

$$\lambda_i^\star \succ_{\mathcal{K}_i^\star} \mathbf{0} \Rightarrow f_i(x^\star) = 0$$

$$f_i(x^\star) \prec_{\mathcal{K}_i} \mathbf{0} \Rightarrow \lambda_i^\star = \mathbf{0}$$

- KKT conditions:

$$f_i(x^\star) \preceq_{\mathcal{K}_i} \mathbf{0}$$

$$\lambda_i^\star \succeq_{\mathcal{K}_i^\star} \mathbf{0}$$

$$\nabla f_0(x^\star) + \sum_{i=1}^{m} \nabla f_i(x^\star)' \lambda_i^\star = \mathbf{0}$$

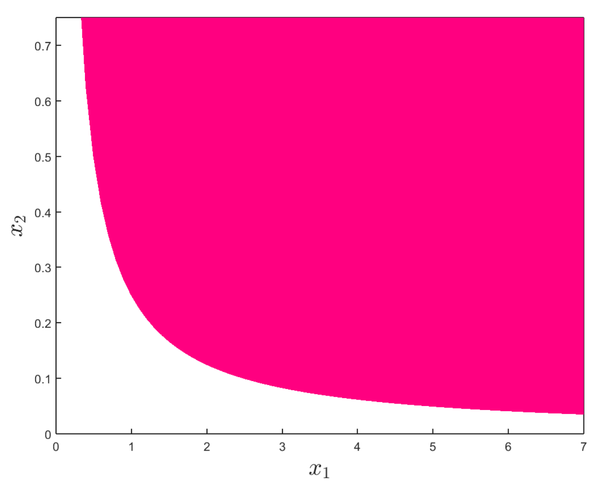
Example of conic duality

Consider the primal conic program

$$\begin{array}{ll} \min & x_1 \\ \text{s.t.} & \begin{bmatrix} x_1 - x_2 \\ 1 \\ x_1 + x_2 \end{bmatrix} \succeq_{\mathbb{L}^3} \mathbf{0} \Leftrightarrow \begin{array}{l} x_1 + x_2 > 0 \\ 4x_1x_2 \geq 1 \end{array} \end{array}$$

with dual

$$\begin{array}{ll} \max & -\lambda_2 \\ \text{s.t.} & \begin{cases} \lambda_1 + \lambda_3 = 1 \\ -\lambda_1 + \lambda_3 = 0 \\ \lambda \in \mathbb{L}^3 \end{cases} \Leftrightarrow \begin{array}{l} \lambda_1 = \lambda_3 = 1/2 \\ 1/2 \geq \sqrt{1/4 + \lambda_2{}^2} \end{array} \end{array}$$



The primal is strictly feasible and bounded below with $p^\star = 0$ which is not reached since dual problem is infeasible $d^\star = -\infty$

SDP duality (1)

Primal SDP:

$$p^\star = \min_{x\in\mathbb{R}^n} \; c'x \quad \text{s.t.} \quad F_0 + \sum_{i=1}^n x_i F_i \preceq \mathbf{0}$$

Lagrange dual function:

$$\begin{aligned} g(Z) &= \inf_{x\in\mathcal{D}} \left(c'x + \operatorname{tr} ZF(x) \right) \\ &= \begin{cases} \operatorname{tr} F_0 Z & \text{if } \operatorname{tr} F_i Z + c_i = 0 \quad i = 1, \cdots, n \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Dual SDP:

$$\begin{aligned} d^\star = \max_{Z\in\mathbb{S}^m} \quad & \operatorname{tr} F_0 Z \\ \text{s.t.} \quad & \operatorname{tr} F_i Z + c_i = 0 \quad i = 1, \cdots, n \\ & Z \succeq \mathbf{0} \end{aligned}$$

Complementary slackness:

$$\operatorname{tr} F(x^\star) Z^\star = 0 \iff F(x^\star) Z^\star = Z^\star F(x^\star) = \mathbf{0}$$

SDP duality (2)
KKT optimality conditions

$$
\begin{array}{rcll}
F_0 + \sum_{i=1}^{n} x_i F_i + Y & = & \mathbf{0} & Y \succeq \mathbf{0} \\
\forall\ i \ \text{trace}\, F_i Z + c_i & = & 0 & Z \succeq \mathbf{0} \\
Z^\star F(x^\star) = -Z^\star Y^\star & = & \mathbf{0} &
\end{array}
$$

Nota:

Since $Y^\star \succeq \mathbf{0}$ and $Z^* \succeq \mathbf{0}$ then

$$\text{trace}\, F(x^\star) Z^\star = \mathbf{0} \Longleftrightarrow F(x^\star) Z^\star = Z^\star F(x^\star) = \mathbf{0}$$

Theorem:

Under the assumption of strict feasibility for the primal and the dual, the above conditions form a system of necessary and sufficient optimality conditions for the primal and the dual

Example of SDP duality gap

Consider the primal semidefinite program

$$
\begin{array}{ll}
\min & x_1 \\
\text{s.t.} & \begin{bmatrix} 0 & x_1 & 0 \\ x_1 & -x_2 & 0 \\ 0 & 0 & -1 - x_1 \end{bmatrix} \preceq \mathbf{0}
\end{array}
$$

with dual

$$
\begin{array}{ll}
\max & -z_6 \\
\text{s.t.} & \begin{bmatrix} z_1 & (1 - z_6)/2 & z_4 \\ (1 - z_6)/2 & 0 & z_5 \\ z_4 & z_5 & z_6 \end{bmatrix} \succeq 0
\end{array}
$$

In the primal $x_1 = 0$ (x_1 appears in a row with zero diagonal entry) so the primal optimum is $x_1^\star = 0$

Similarly, in the dual necessarily $(1 - z_6)/2 = 0$ so the dual optimum is $z_6^\star = 1$

There is a nonzero duality gap here $(p^\star = 0) > (d^\star = -1)$

Conic theorem of alternatives

$$f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \qquad \mathcal{K}_i \subseteq \mathbb{R}^{k_i}$$

Lagrange dual function

$$g(\lambda) = \inf_{x \in \mathcal{D}} \sum_{i=1}^{m} \lambda_i' f_i(x) \quad \lambda_i \in \mathbb{R}^{k_i}$$

Weak alternatives:

$$1 - \ f_i(x) \preceq_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots, m$$

$$2 - \ \lambda_i \succeq_{\mathcal{K}_i^\star} \mathbf{0} \qquad g(\lambda) > 0$$

Strong alternatives:

f_i $\mathcal{K}_i$-convex and $\exists\ x \in \text{relint}\mathcal{D}$

$$1 - \ f_i(x) \prec_{\mathcal{K}_i} \mathbf{0} \quad i = 1, \cdots, m$$

$$2 - \ \lambda_i \succeq_{\mathcal{K}_i^\star} \mathbf{0} \qquad g(\lambda) \geq 0$$

Theorem of alternatives for LMIs

For the LMI feasible set

$$F(x) = F_0 + \sum_i x_i F_i \prec \mathbf{0}$$

Exactly one statement is true
1- $\exists\ x$ s.t. $F(x) \prec 0$
2- $\exists\ \mathbf{0} \neq Z \succeq 0$ s.t.
trace $F_0 Z \geq \mathbf{0}$ and trace $F_i Z = \mathbf{0}$ for $i = 1, \cdots, n$

Useful for giving certificate of infeasibility of LMIs

Rich literature on theorems of alternatives for generalized inequalities, e.g. nonpolyhedral convex cones

Elegant proofs of standard results (Lyapunov, ARE) in linear systems control

S-procedure (1)

S-procedure: also frequently useful in robust and nonlinear control, also an outcome of the theorem of alternatives

1- if $x'A_1x \geq 0, \cdots, x'A_mx \geq 0$
then $x'A_0x \geq 0 \ \forall \ x \in \mathbb{R}^n$

2- $\exists \ \tau_j \geq 0$ s.t. $x'A_0x - \sum_{j=1}^{m} \tau_j x'A_jx \geq 0$

The S-procedure consists in replacing 1 by 2

The converse also holds (no duality gap)
- when $m = 1$ for real quadratic forms and $\exists \ x \mid x'A_1x > 0$ (from the theorem of alternatives)
- when $m = 2$ for complex quadratic forms

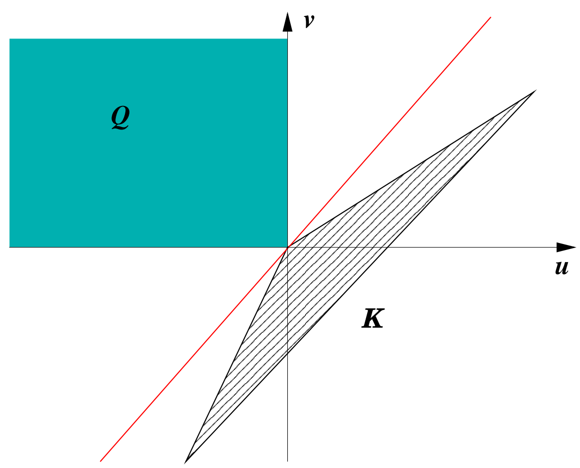
S-procedure (2)
Sketch of the proof for $m = 1$

Dines theorem:

For $(A_0, A_1) \in \mathbb{S}_n$ then

$$\mathcal{K} = \left\{ (u, v) = (x' A_0 x, x' A_1 x) \; : \; x \in \mathbb{R}^n \right\}$$

is a closed convex cone of $\mathbb{R}^2$



Suppose if $v = x' A_1 x \geq 0$ then $u = x' A_0 x \geq 0$
Defining $\mathcal{Q} = \{v \geq 0, \; u < 0\}$ then $\mathcal{K} \cap \mathcal{Q} = \emptyset$

Separating Hyperplane Theorem:

$\tau_1 u - \tau_2 v < 0$	$(u, v) \in \mathcal{Q}$	$\tau_2 \geq 0$	$\tau_1 > 0$
$\forall \; (u, v) \in \mathcal{K}$	$\exists \; \tau = \tau_2 / \tau_1 \geq 0$	$u - \tau v \geq 0$	

S-procedure (3)
Counter-example $m = 3$ and $n = 2$

Let the quadratic forms

$$f_1(x,y) = -x^2 + 2y^2 \quad f_2(x,y) = 2x^2 - y^2$$

$$f_0(x,y) = xy$$

then

$$\begin{aligned} Q &= \{(x,y) \mid f_1(x,y) \geq 0 \text{ and } f_2(x,y) \geq 0\} \\ &= \left\{(x,y) \mid 1/\sqrt{2} \leq \left|\frac{x}{y}\right| \leq \sqrt{2}\right\} \end{aligned}$$

and

$$(x,y) = (1,1) \mid f_1(x,y) > 0 \text{ and } f_2(x,y) > 0$$

$$f_0(x,y) \geq 0 \quad \forall \ (x,y) \in \ Q$$

But $\nexists$ $(\tau_1, \tau_2) \succeq 0$ s.t.

$$xy - \tau_1(-x^2 + 2y^2) - \tau_2(2x^2 - y^2) \geq 0$$

Finsler's (Debreu) lemma (1)

The following statements are equivalent

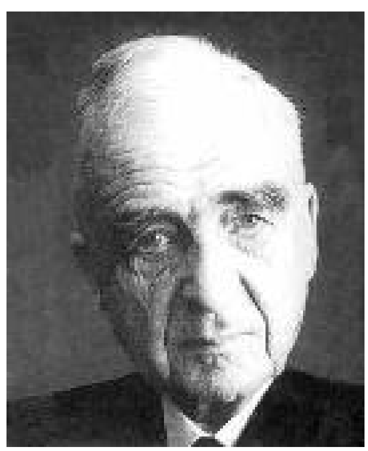
1 − $x'A_0x > 0 \ \forall \ x \neq 0 \in \mathbb{R}^n, \ n \geq 3,$ s.t. $x'A_1x = 0$

2 − $A_0 + \tau A_1 \succ 0$ for some $\tau \in \mathbb{R}$

Theorem of alternatives

1 − $\exists \ \tau \in \mathbb{R} \mid \tau A_1 + A_0 \succ \mathbf{0}$

2 − $\exists \ Z \in \mathbb{S}^n_+ \ : \ \mathrm{tr}(ZA_1) = 0$ and $\mathrm{tr}(A_0 Z) \leq 0$



Paul Finsler
(1894 Heilbronn - 1970 Zurich)

Finsler's (Debreu) lemma (2)
Counter-examples

Counter-example 1:

$$f_0(x) = x_1^2 - 2x_2^2 - x_3^2 \quad f_1(x) = x_1 - x_2$$

$$f_0(x) \leq 0 \text{ if } f_1(x) = 0$$

But, no τ exists s.t. $f_0(x) + \tau f_1(x) \leq 0$

$$x' \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \tau \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x \leq 0$$

Pick out $x = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}'$ and $x = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}'$

Counter-example 2:

$$f_0(x) = 2x_1x_2 \quad f_1(x) = x_1^2 - x_2^2$$

$f_0(x) > 0$ for $x \mid f_1(x) = 0$ but no $\tau \in \mathbb{R}$ exists s.t.

$$f_0(x) + \tau f_1(x) = x' \begin{bmatrix} \tau & 1 \\ 1 & -\tau \end{bmatrix} x > 0$$

Elimination lemma

The following statements are equivalent

$$1- \quad H^{\perp}AH^{\perp *} \succ 0 \quad \text{or} \quad HH^{*} \succ 0$$

$$2- \quad \exists\ X \mid A + XH + H^{\star}X^{\star} \succ 0$$

Theorem of alternatives

$1- \ \exists\ X \in \mathbb{C}^{m\times n} \mid HX + (XH)^{*} + A \succ \mathbf{0}$

$2- \ \exists\ Z \in \mathbb{S}^{n}_{+} \ :\ ZH = 0$ and $\mathrm{tr}(AZ) \leq 0$

Nota: For $H \in \mathbb{C}^{n\times m}$ with rank r, $H^{\perp} \in \mathbb{C}^{(n-r)\times n}$ s.t.

$$H^{\perp}H = 0 \quad H^{\perp}H^{\perp *} \succ 0$$

LMI OPTIMIZATION
PART 3

GEOMETRY OF LMI SETS

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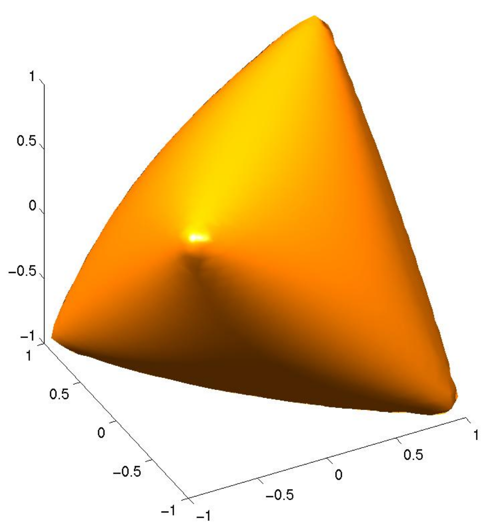


26-27 June 2017

Geometry of LMI sets

Given $F_i \in \mathbb{S}^m$ we want to characterize the shape in $\mathbb{R}^n$ of the LMI set

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succeq 0\}$$

Matrix $F(x)$ is PSD iff its principal minors $f_i(x)$ are nonnegative

Principal minors are multivariate polynomials of indeterminates x_i

So the LMI set can be described as

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ f_i(x) \geq 0, \ i = 1, \ldots, n\}$$

which is a semialgebraic set

Moreover, it is a convex set

Example of 2D LMI feasible set

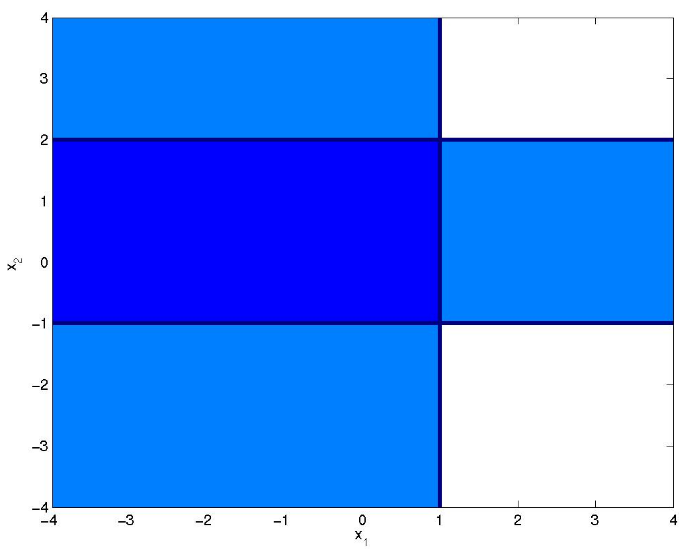
$$F(x) = \begin{bmatrix} 1-x_1 & x_1+x_2 & x_1 \\ x_1+x_2 & 2-x_2 & 0 \\ x_1 & 0 & 1+x_2 \end{bmatrix} \succeq 0$$

Feasible iff all principal minors nonnegative

System of polynomial inequalities $f_i(x) \geq 0$

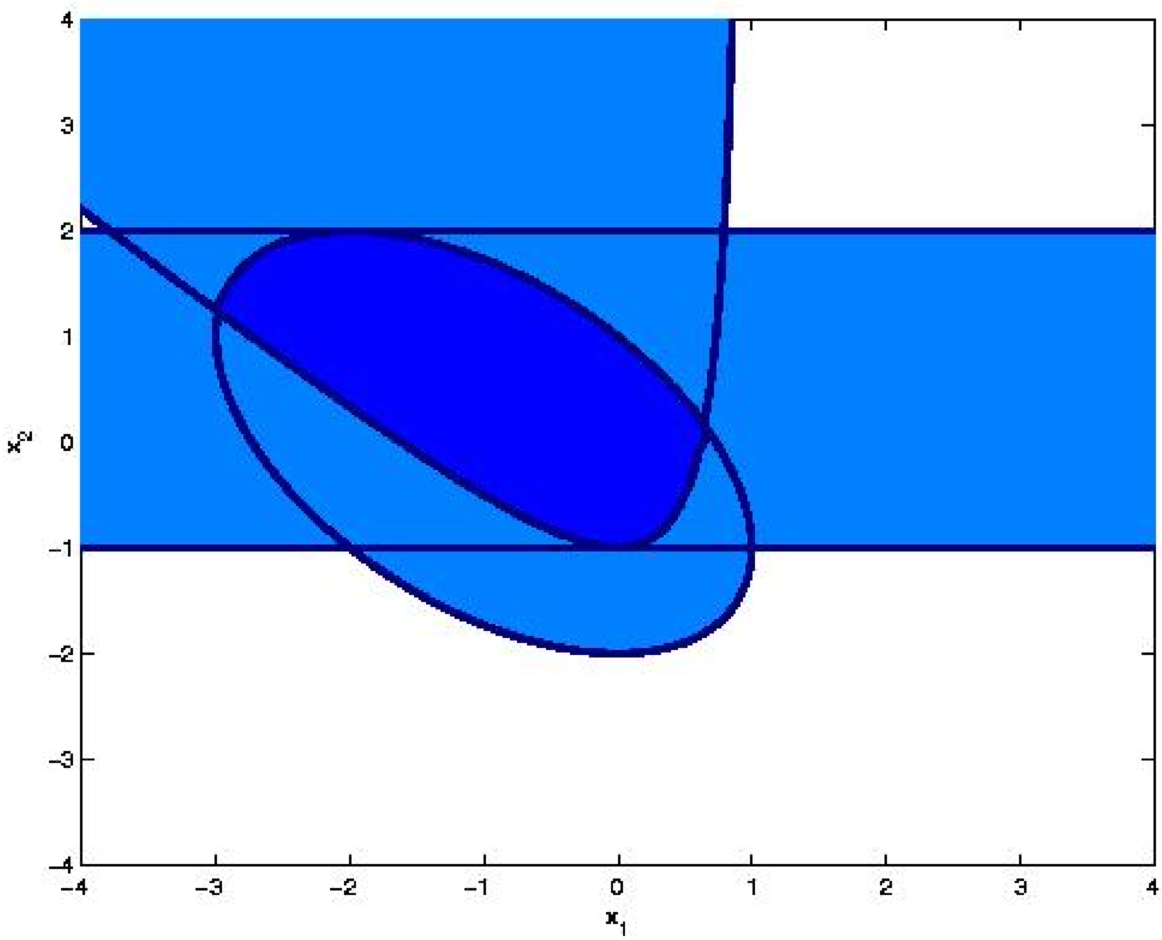
1st order minors

$f_1(x) = 1 - x_1 \geq 0$
$f_2(x) = 2 - x_2 \geq 0$
$f_3(x) = 1 + x_2 \geq 0$



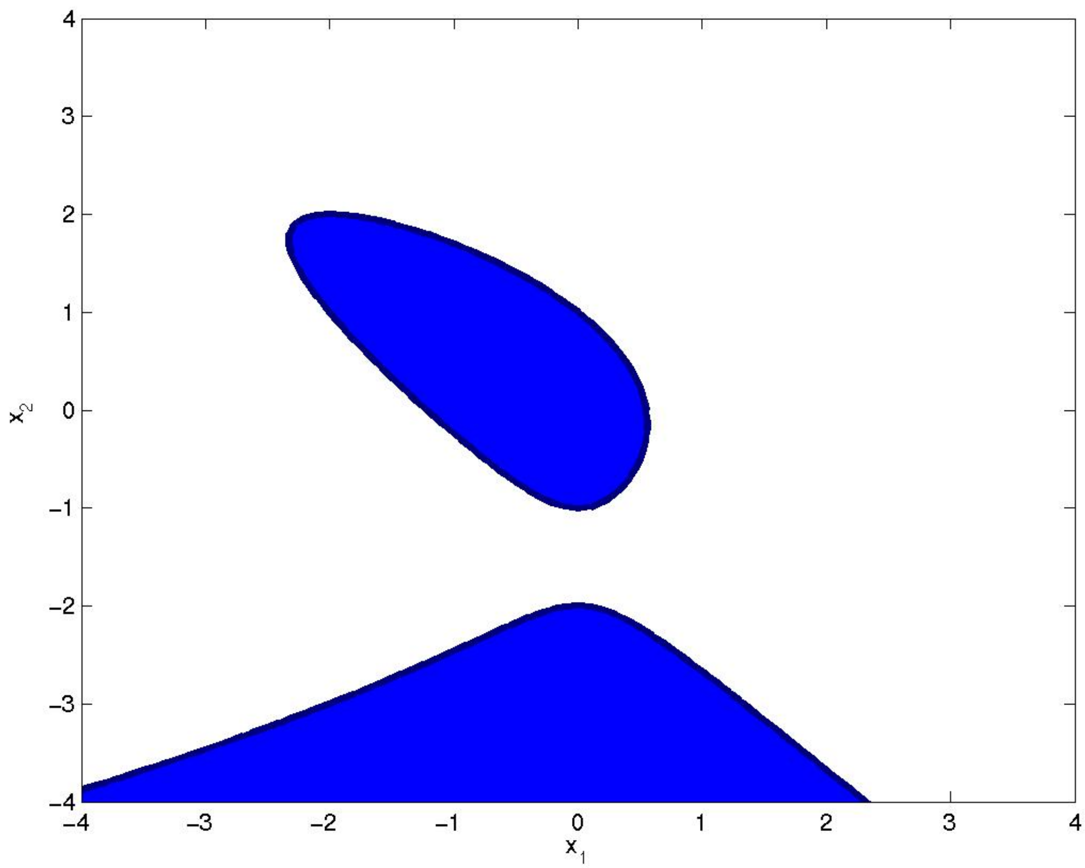
2nd order minors

$$f_4(x) = (1 - x_1)(2 - x_2) - (x_1 + x_2)^2 \geq 0$$
$$f_5(x) = (1 - x_1)(1 + x_2) - x_1^2 \geq 0$$
$$f_6(x) = (2 - x_2)(1 + x_2) \geq 0$$

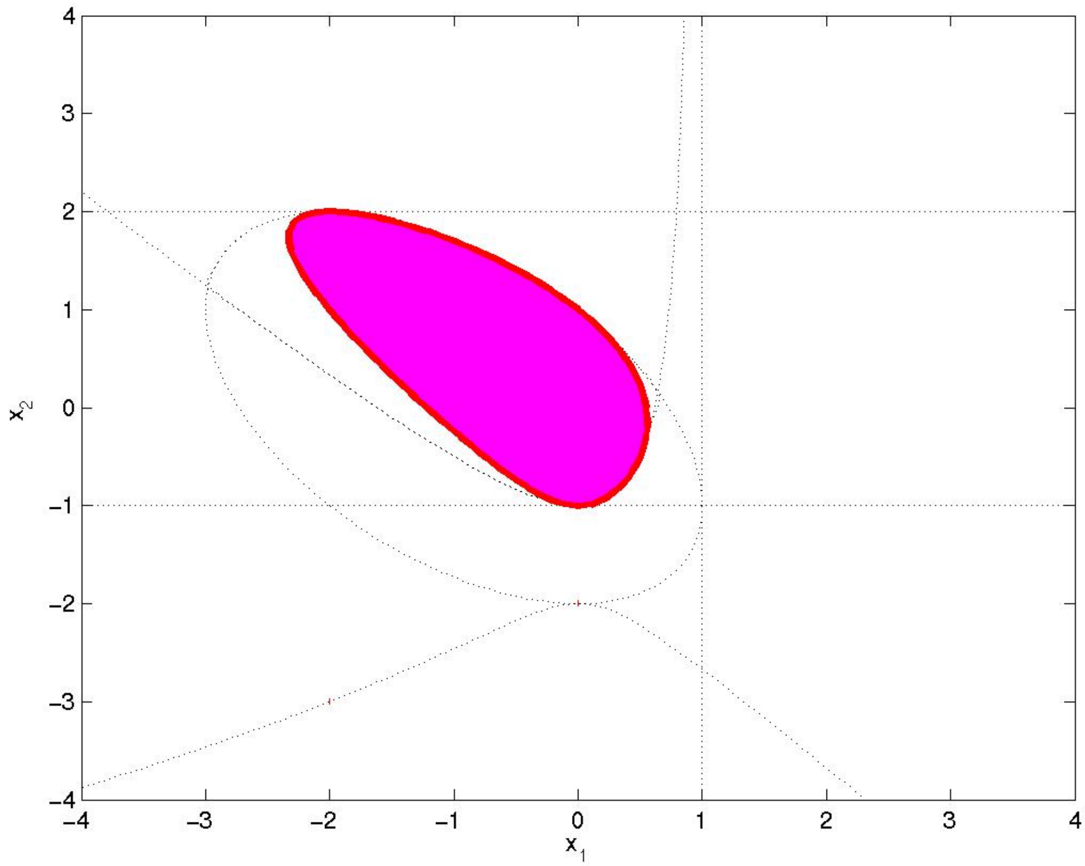


3rd order minor

$$
\begin{aligned}
f_7(x) \;=\; & (1 + x_2)((1 - x_1)(2 - x_2) - (x_1 + x_2)^2) \\
& -x_1^2(2 - x_2) \geq 0
\end{aligned}
$$



LMI feasible set = intersection of semialgebraic sets $f_i(x) \geq 0$ for $i = 1, \ldots, 7$

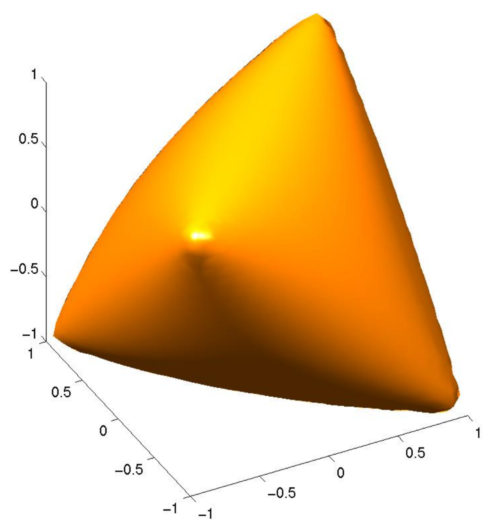


Example of 3D LMI feasible set

LMI set

$$\mathcal{S} = \{x \in \mathbb{R}^3 \ : \ \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{bmatrix} \succeq 0\}$$

arising in SDP relaxation of MAXCUT



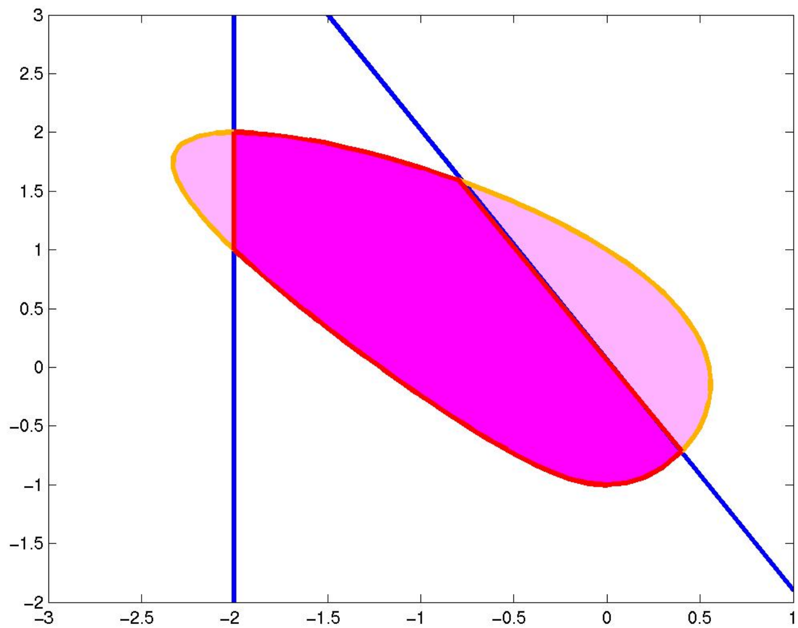
Semialgebraic set

$$\mathcal{S} = \{x \in \mathbb{R}^3 \ : \ \begin{array}{l} 1 + 2x_1x_2x_3 - (x_1^2 + x_2^2 + x_3^2) \geq 0, \\ x_1^2 \leq 1, \ x_2^2 \leq 1, \ x_3^2 \leq 1\} \end{array}$$

Intersection of LMI sets

Intersection of LMI feasible sets

$$F(x) \succeq 0 \quad x_1 \geq -2 \quad 2x_1 + x_2 \leq 0$$



is also an LMI

$$\begin{bmatrix} F(x) & 0 & 0 \\ 0 & x_1 + 2 & 0 \\ 0 & 0 & -2x_1 - x_2 \end{bmatrix} \succeq 0$$

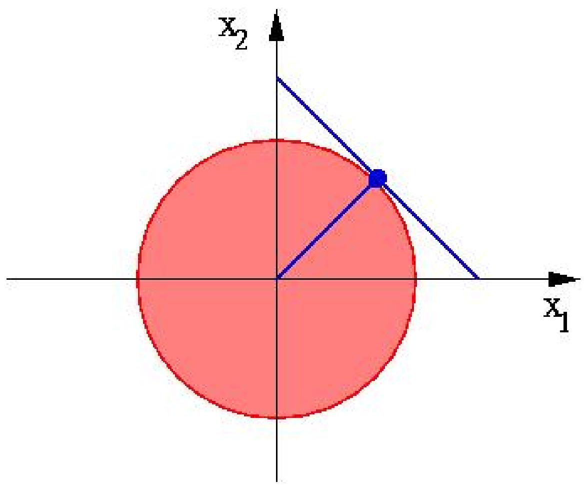
Reformulations

Linear LMI constraint = projection in subspace

Using explicit subspace basis, more efficient formulations (less decision variables) can be obtained

Example: original problem

$$\begin{array}{ll} \max & 2x_1 + 2x_2 \\ \text{s.t.} & \begin{bmatrix} 1+x_1 & x_2 \\ x_2 & 1-x_1 \end{bmatrix} \succeq 0 \end{array}$$



with dual

$$\begin{array}{ll} \min & \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z \\ \text{s.t.} & \text{trace} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} Z = 2 \\ & \text{trace} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} Z = 2 \\ & Z \succeq 0 \end{array}$$

Reformulations (2)

Denoting

$$Z = \begin{bmatrix} z_{11} & z_{21} \\ z_{21} & z_{22} \end{bmatrix}$$

the linear trace constraints on Z can be written

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Particular solution and explicit null-space basis

$$\begin{bmatrix} z_{11} \\ z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bar{z}$$

so we obtain the equivalent dual problem
with less variables

$$\begin{array}{ll} \min & 2\bar{z} \\ \text{s.t.} & \begin{bmatrix} \bar{z}-1 & -1 \\ -1 & \bar{z}+1 \end{bmatrix} \succeq 0 \end{array}$$

and primal

$$\begin{array}{ll} \max & \text{trace} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{X} \\ \text{s.t.} & \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bar{X} = 2 \\ & \bar{X} \succeq 0 \end{array}$$

Nonlinear matrix ineqalities
Schur complement

We can use the Schur complement to convert a non-linear matrix inequality into an LMI

$$A(x) - B(x)C^{-1}(x)B'(x) \succeq 0$$
$$C(x) \succ 0$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} A(x) & B(x) \\ B(x) & C(x) \end{bmatrix} \succeq 0$$
$$C(x) \succ 0$$

Issai Schur
(1875 Mogilyov - 1941 Tel Aviv)

COURSE ON LMI OPTIMIZATION
PART 5

SOLVING LMIs

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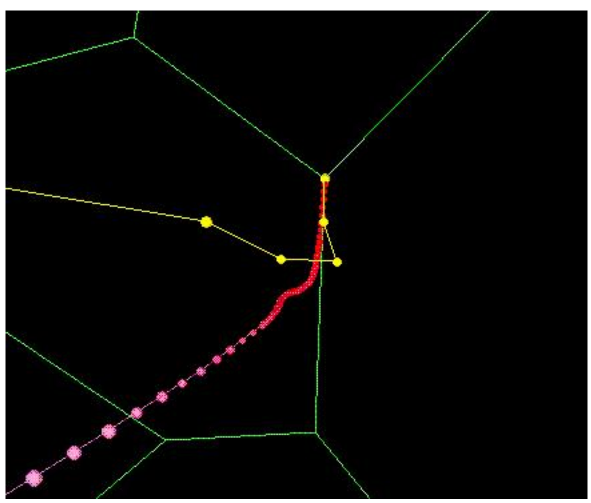


26-27 June 2017

History

Convex programming

- Logarithmic barrier function [K. Frisch 1955)]
- Method of centers ([P. Huard 1967]

Interior-point (IP) methods for LP

- Ellipsoid algorithm [Khachiyan 1979]
polynomial bound on worst-case iteration count
- IP methods for LP [Karmarkar 1984]
improved complexity bound and efficiency - About 50% of commercial LP solvers

IP methods for SDP

- Self-concordant barrier functions [Nesterov, Nemirovski 1988], [Alizadeh 1991]
- IP methods for general convex programs (SDP and LMI)
Academic and commercial solvers (MATLAB)

Interior point methods (1)

For the optimization problem

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & f_0(x) \\ \text{s.t.} & f_i(x) \geq 0 \; i = 1, \cdots, m \end{array}$$

where the $f_i(x)$ are twice continuously differentiable convex functions

Sequential minimization techniques: Reduction of the initial problem into a sequence of unconstraint optimization problems
[Fiacco - Mc Cormick 68]

$$\min_{x \in \mathbb{R}^n} f_0(x) + \mu\phi(x)$$

where $\mu > 0$ is a parameter sequentially decreased to 0 and the term $\phi(x)$ is a barrier function
Barrier functions go to infinity as the boundary of the feasible set is approached

Interior point methods (2)
Descent methods

To solve an unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

we produce a minimizing sequence

$$x_{k+1} = x_k + t_k \Delta x_k$$

where $\Delta x_k \in \mathbb{R}^n$ is the step or search direction and $t_k \geq 0$ is the step size or step length

A descent method consists in finding a sequence $\{x_k\}$ such that

$$f(x^\star) \leq \cdots f(x_{k+1}) < f(x_k)$$

where $x^\star$ is the optimum

General descent method

0. $k = 0$; given starting point x_k
1. determine descent direction Δx_k
2. line search: choose step size $t_k > 0$
3. update: $k = k + 1$; $x_k = x_{k-1} + t_{k-1} \Delta x_{k-1}$
4. go to step 1 until a stopping criterion is satisfied

Interior point methods (3)
Newton's method

A particular choice of search direction is the Newton step

$$\Delta x = -\nabla^2 f(x)^{-1} \nabla f(x)$$

where
- $\nabla f(x)$ is the gradient
- $\nabla^2 f(x)$ is the Hessian

This step $y = \Delta x$ minimizes the second-order Taylor approximation

$$\widehat{f}(x+y) = f(x) + \nabla f(x)' y + y' \nabla^2 f(x) y/2$$

and it is the steepest descent direction for the quadratic norm defined by the Hessian

Quadratic convergence near the optimum

Interior point methods (4)
Conic optimization

For the conic optimization problem

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & f_0(x) \\ \text{s.t.} & f_i(x) \preceq_{\mathcal{K}} 0 \; i = 1, \cdots, m \end{array}$$

suitable barrier functions are called self-concordant

Smooth convex 3-differentiable functions f with second derivative Lipschitz continuous w.r. to the local metric induced by the Hessian

$$|f'''(x)| \leq 2f''(x)^{3/2}$$

- goes to infinity as the boundary of the cone is approached
- can be efficiently minimized by Newton's method
- Each convex cone $\mathcal{K}$ possesses a self-concordant barrier
- Such barriers are only computable for some special cones

Barrier function for LP (1)

For LP and positive orthant $\mathbb{R}^n_+$, the logarithmic barrier function

$$\phi(y) = -\sum_{i=1}^{n} \log(y_i) = \log \prod_{i=1}^{n} y_i^{-1}$$

is convex in the interior $y \succ 0$ of the feasible set and is instrumental to design IP algorithms

$$\begin{array}{ll} \max\limits_{\mu \in \mathbb{R}^p} & b'y \\ \text{s.t.} & c_i - a_i y \succeq \mathbf{0}, \; i = 1, \cdots, m, \; (y \in \mathcal{P}) \end{array}$$

$$\phi(y) = -\log \prod_{i=1}^{m} (c_i - a_i y) = -\sum_{i=1}^{m} \log(c_i - a_i y)$$

The optimum

$$y_c = \arg \left[\min_y \phi(y) \right]$$

is called the analytic center of the polytope

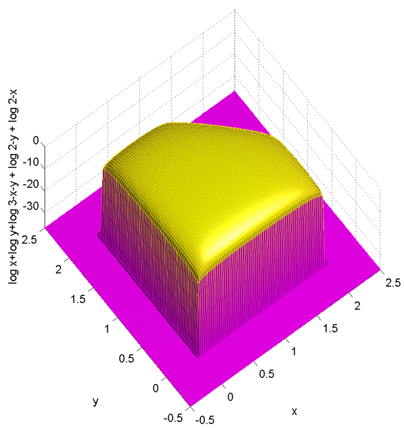
Barrier function for LP (2)
Example

$$J_1^* = \max_{x,y} \quad 2x + y$$

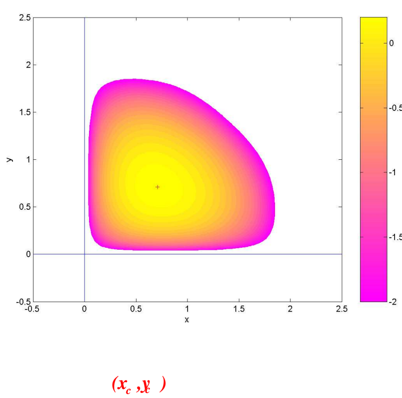
$$\text{s.t.} \qquad x \geq 0 \qquad y \geq 0 \qquad x \leq 2$$

$$y \leq 2 \qquad x + y \leq 3$$

$$\phi(x,y) = -\log(xy) - \log(2-x) - \log(2-y) - \log(3-x-y)$$



$$(x_c, y_c) = \left(\frac{6-\sqrt{6}}{5}, \frac{6-\sqrt{6}}{5}\right)$$



Barrier function for an LMI (1)

Given an LMI constraint $F(x) \succeq 0$

Self-concordant barriers are smooth convex 3-differentiable functions $\phi : \mathbb{S}^n_+ \to \mathbb{R}$ s.t. for $\overline{\phi}(\alpha) = \phi(X + \alpha H)$ for $X \succ 0$ and $H \in \mathbb{S}^n$

$$|\overline{\phi}'''(0)| \leq 2\overline{\phi}''(0)^{3/2}$$

Logarithmic barrier function

$$\phi(x) = -\log\det F(x) = \log\det F(x)^{-1}$$

This function is analytic, convex and self-concordant on $\{x : F(x) \succ 0\}$

The optimum

$$x_c = \arg\left[\min_x \phi(x)\right]$$

is called the analytic center of the LMI

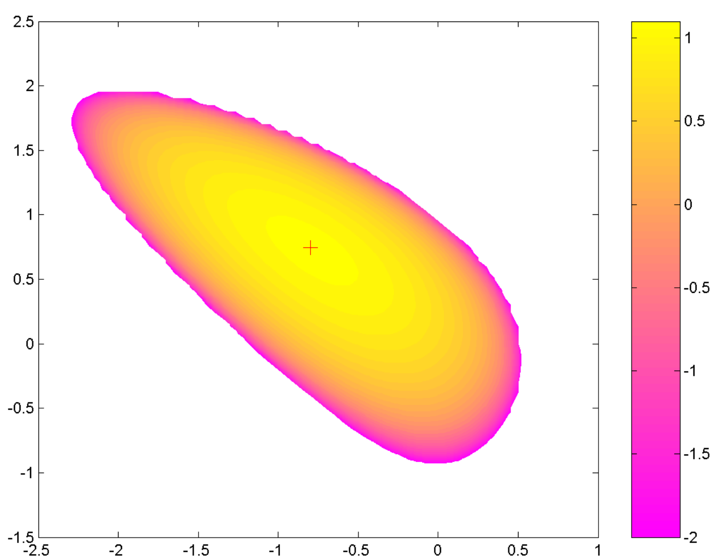
Barrier function for an LMI (2)
Example (1)

$$F(x_1,x_2) = \begin{bmatrix} 1-x_1 & x_1+x_2 & x_1 \\ x_1+x_2 & 2-x_2 & 0 \\ x_1 & 0 & 1+x_2 \end{bmatrix} \succeq 0$$

Computation of analytic center:

$$\nabla_{x_1} \log\det F(x) = 2 + 3x_2 + 6x_1 + x_2^2 = 0$$

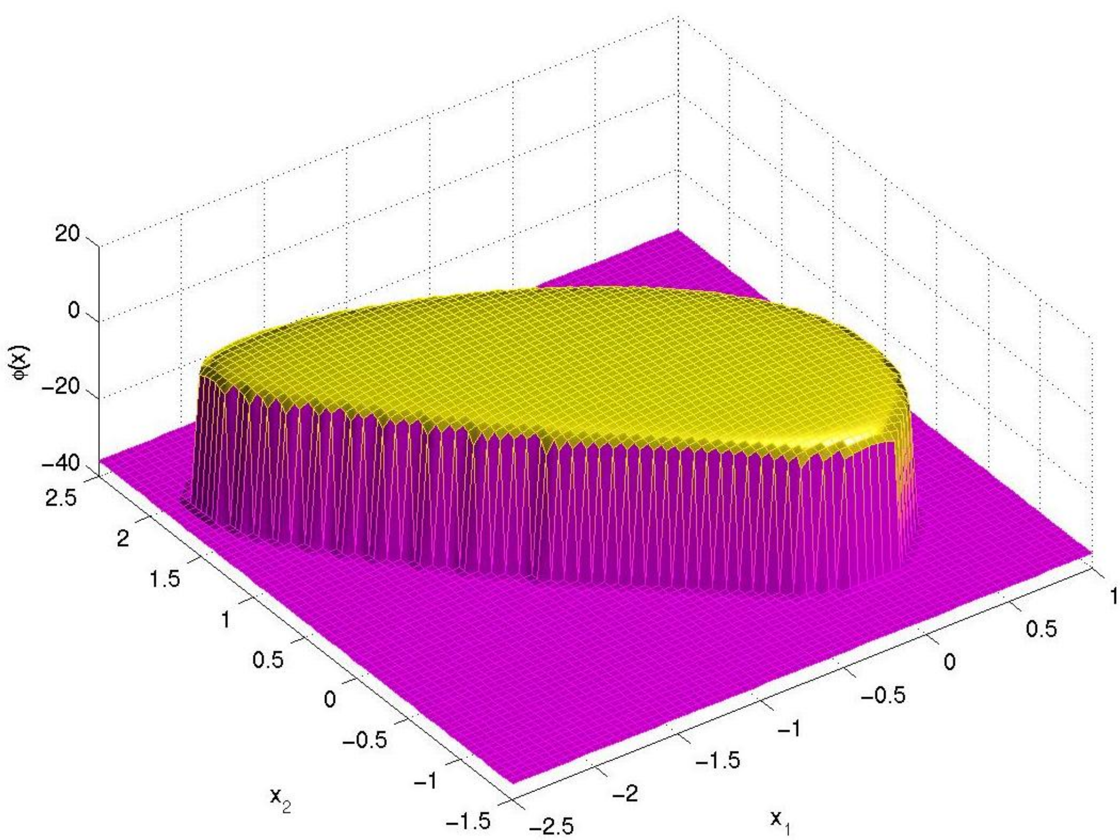
$$\nabla_{x_2} \log\det F(x) = 1 - 3x_1 - 4x_2 - 3x_2^2 - 2x_1x_2 = 0$$



$$x_{1c} = -0.7989 \qquad x_{2c} = 0.7458$$

Barrier function for an LMI (3)
Example (2)

The barrier function $\phi(x)$ is flat in the interior of the feasible set and sharply increases toward the boundary



IP methods for SDP (1)

Primal / dual SDP

$$\min_{Z} \quad -\text{trace}(F_0 Z)$$
$$\text{s.t.} \quad -\text{trace}(F_i Z) = c_i$$
$$Z \succeq 0$$

$$\min_{x,\, Y} \quad c'x$$
$$\text{s.t.} \quad Y + F_0 + \sum_{i=1}^{m} x_i F_i = \mathbf{0}$$
$$Y \succeq 0$$

Remember KKT optimality conditions

$$F_0 + \sum_{i=1}^{m} x_i F_i + Y \quad = \quad \mathbf{0} \quad Y \succeq \mathbf{0}$$

$$\forall\ i\ \text{trace}\, F_i Z + c_i \quad = \quad \mathbf{0} \quad Z \succeq \mathbf{0}$$

$$Z^\star F(x^\star) = -Z^\star Y^\star \quad = \quad \mathbf{0}$$

IP methods for SDP (2)
The central path

Perturbed KKT optimality conditions = Centrality conditions

$$
\begin{aligned}
F_0 + \sum_{i=1}^{m} x_i F_i + Y &= \mathbf{0} \quad & Y \succeq \mathbf{0} \\
\forall\ i\ \text{trace}\, F_i Z + c_i &= \mathbf{0} \quad & Z \succeq \mathbf{0} \\
ZY &= \mu \mathbf{1}
\end{aligned}
$$

where $\mu > 0$ is the centering parameter or barrier parameter

For any $\mu > 0$, centrality conditions have a unique solution $Z(\mu), x(\mu), Y(\mu)$ which can be seen as the parametric representation of an analytic curve: The central path

The central path exists if the primal and dual are strictly feasible and converges to the analytic center when $\mu \to 0$

IP methods for SDP (3)
Primal methods

$$\begin{array}{ll} \min_{Z} & -\text{trace}(F_0 Z) - \mu \log\det Z \\ \text{s.t.} & \text{trace}(F_i Z) = -c_i \end{array}$$

where parameter μ is sequentially decreased to zero

Follow the primal central path approximately:
Primal path-following methods

The function $f_p^{\mu}(Z)$

$$f_p^{\mu}(Z) = -\frac{1}{\mu}\text{trace}(F_0 Z) - \log\det\ Z$$

is the primal barrier function and the primal central path corresponds to the minimizers $Z(\mu)$ of $f_p^{\mu}(Z)$

- The projected Newton direction ΔZ
- Updating of the centering parameter μ

IP methods for SDP (4)
Dual methods (1)

$$\begin{aligned} \min_{x,Y} \quad & c'x - \mu \log\det Y \\ \text{s.t.} \quad & Y + F_0 + \sum_{i=1}^{m} x_i F_i = \mathbf{0} \end{aligned}$$

where parameter μ is sequentially decreased to zero

The function $f_d^\mu(x, Y)$

$$f_d^\mu(x, Y) = \frac{1}{\mu} c'x - \log\det Y$$

is the dual barrier function and the dual central path corresponds to the minimizers $(x(\mu), Y(\mu))$ of $f_d^\mu(x, Y)$

$Y_k \succeq 0$ ensured via Newton process:

- Large decreases of μ require damped Newton steps
- Small updates allow full (deep) Newton steps

Dual methods (2)
Newton step for LMI

The centering problem is

$$\min \phi(x) = \frac{1}{\mu} c' x - \log\det(-F(x))$$

and at each iteration Newton step Δx satisfies the linear system of equations (LSE)

$$H \Delta x = -g$$

where gradient g and Hessian H are given by

$$\begin{aligned} H_{ij} &= \text{trace}\, F(x)^{-1} F_i F(x)^{-1} F_j \\ g_i &= c_i/\mu - \text{trace}\, F(x)^{-1} F_i \end{aligned}$$

LSE typically solved via Cholesky factorization or QR decomposition (near the optimum)

Nota: Expressions for derivatives of $\phi(x) = -\log\det F(x)$
Gradient:

$$\begin{aligned} (\nabla \phi(x))_i &= -\text{trace}\, F(x)^{-1} F_i \\ &= -\text{trace}\, F(x)^{-1/2} F_i F(x)^{-1/2} \end{aligned}$$

Hessian:

$$\begin{aligned} (\nabla^2 \phi(x))_{ij} &= \text{trace}\, F(x)^{-1} F_i F(x)^{-1} F_j \\ &= \mu\text{trace}\left(F(x)^{-1/2} F_i F(x)^{-1/2}\right)\left(F(x)^{-1/2} F_j F(x)^{-1/2} \right. \end{aligned}$$

IP methods for SDP (4)
Primal-dual methods (1)

$$\begin{array}{ll} \min\limits_{x,Y,Z} & \text{trace}\, YZ - \mu \log\det YZ \\ \text{s.t.} & -\text{trace}\, F_i Z = c_i \\ & Y + F_0 + \sum_{i=1}^{m} x_i F_i = \mathbf{0} \end{array}$$

Minimizers $(x(\mu), Y(\mu), Z(\mu))$ satisfy optimality conditions

$$\begin{aligned} \text{trace}\, F_i Z &= -c_i \\ \sum_{i=1}^{m} x_i F_i + Y &= -F_0 \\ YZ &= \mu I \\ Y, Z &\succeq \mathbf{0} \end{aligned}$$

The duality gap:

$$-\text{trace}(F_0 Z) - c'x = \text{trace}(YZ) \geq 0$$

is minimized along the central path

IP methods for SDP (5)
Primal-dual methods (2)

For primal-dual IP methods, primal and dual directions ΔZ, Δx and ΔY must satisfy non-linear and over determined system of conditions

$$\begin{aligned}
\text{trace}(F_i \Delta Z) &= 0 \\
\sum_{i=1}^{m} \Delta x_i F_i + \Delta Y &= 0 \\
(Z + \Delta Z)(Y + \Delta Y) &= \mu I \\
Z + \Delta Z &\succeq \mathbf{0} \\
\Delta Z &= \Delta Z' \\
Y + \Delta Y &\succeq \mathbf{0}
\end{aligned}$$

These centrality conditions are solved *approximately* for a given $\mu > 0$, after which μ is reduced and the process is repeated

Key point is in linearizing and symmetrizing the latter equation

IP methods for SDP (6)
Primal-dual methods (3)

The non linear equation in the centrality conditions is replaced by

$$H_P(\Delta Z Y + Z \Delta Y) = \mu \mathbf{1} - H_P(ZY)$$

where H_P is the linear transformation

$$H_P(M) = \frac{1}{2}\left[PMP^{-1} + P^{-1'}M'P'\right]$$

for any matrix M and the scaling matrix P gives the symmetrization strategy.

Following the choice of P, long list of primal-dual search directions, (AHO, HRVW, KSH, M, NT...), the most known of which is Nesterov-Todd's

Algorithms differ in how the symmetrized equations are solved and how μ is updated (long step methods, dynamic updates of for predictor-corrector methods)

IP methods in general

Generally for LP, QP or SDP primal-dual methods outperform primal or dual methods
General characteristics of IP methods:

- Efficiency: About 5 to 50 iterations, almost independent of input data (problem), each iteration is a least-squares problem (well established linear algebra)
- Theory: Worst-case analysis of IP methods yields polynomial computational time
- Structure: Tailored SDP solvers can exploit problem structure

For more information see the Linear, Cone and SDP section at

`www.optimization-online.org`

and the Optimization and Control section at

`fr.arXiv.org/archive/math`

SDP solvers

Primal-dual algorithms:

- SeDuMi (J. Sturm, I. Polik)
- SDPT3 (K.C. Toh, R. Tütüncü, M. Todd)
- CSDP (B. Borchers)
- SDPA (M. Kojima and *al.*)
- SMCP (E. Andersen and L. Vandenberghe)
- MOSEK (E. Andersen)

Bundle methods:

- ConicBundle (C. Helmberg)

Dual-scaling potential reduction algorithms:

- DSDP (S. Benson, Y. Ye)

Barrier method and augmented Lagrangian:

- PENSDP (M. Kočvara, M. Stingl)
- SDPLR (S. Burer, R. Monteiro)

Matrices as variables

Generally, in control problems we do not encounter the LMI in canonical or semidefinite form but rather with matrix variables

Lyapunov's inequality

$$A'P + PA < 0 \quad P = P' > 0$$

can be written in canonical form

$$F(x) = F_0 + \sum_{i=1}^{m} F_i x_i < 0$$

with the notations

$$F_0 = 0 \quad F_i = A'B_i + B_i A$$

where B_i, $i = 1, \ldots, n(n+1)/2$ are matrix bases for symmetric matrices of size n

Most software packages for solving LMIs however work with canonical or semidefinite forms, so that a (sometimes time-consuming) pre-processing step is required

LMI solvers

Available under the Matlab environment

Projective method: project iterate on ellipsoid within PSD cone = least squares problem
• LMI Control Toolbox (P. Gahinet, A. Nemirovski)
exploits structure with rank-one linear algebra
warm-start + generalized eigenvalues
originally developed for INRIA's Scilab

LMI parser to SDP solvers
• YALMIP (Y. Löfberg)

See Helmberg's page on SDP
`www-user.tu-chemnitz.de/~helmberg/semidef.html`
and Mittelmann's page on optimization software with benchmarks
`plato.la.asu.edu/guide.html`