Optimal Control for Minimum-Fuel Geostationary Station Keeping of Satellites Equipped with Electric Propulsion

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Abstract: In this paper, a control scheme is elaborated to perform the station keeping of a geostationary satellite equipped with electric propulsion. The use of electric thrusters imposes to take into account some additional non linear constraints that make the overall station keeping optimal control problem difficult to solve directly. That is why we choose here to decompose the station keeping problem in two control problems. The first one consists in solving a classical optimal control problem with an indirect method initialized by a direct method without using the thrusters operational constraints. The second problem deals with the thrusters operating constraints, that are taken into account by two different ways. Simulation results validate the effect of the optimal control thrusts obtained with these methods.

Keywords: electric propulsion, GEO satellite station keeping, optimal control problem, Pontryagin Maximum Principle, indirect methods, direct collocation methods

1. INTRODUCTION

Due to orbital disturbing forces, any satellite in Geostationary Earth Orbit (GEO) drifts outside its station keeping (SK) window (a rectangular box of a given geographical longitude and latitude range). Performing an accurate SK strategy is therefore necessary to compensate for the induced environmental secular and periodic disturbances and GEO satellite are equipped with electric and/or chemical thrusters.

Chemical propulsion systems have been and are widely used. For these propulsion systems with high thrust capabilities, SK control laws are usually designed assuming an impulsive idealisation of the thrust, as described for example in Campan et al. (1995), Soop (1994) or Sidi (1997). The idea of using electric propulsion for station keeping dates back to the sixties (see, e.g. Boucher (1963), Molitor (1964), Barrett (1967) and Hunziker (1970)) and some theoretical developments have been presented in the eighties byAnzel (1988), Eckstein (1980), Eckstein and Hechler (1980), Eckstein et al. (1981). Nowadays the electric propulsion is a viable alternative to the chemical one, in particular in the case of SK of GEO satellite (Peukert and Wollenhaupt (2014)), to take into account thrust operations constraints: large on board power needs, mission requirements restricting the duration of use of the electric power system, impossibility to perform SK maneuvers at eclipse epochs. Moreover, its bigger specific impulsion and the consequent savings in fuel consumption, leading to a reduction of the satellite mass and enabling increased payload capacity and/or improving satellite longevity. However it produces very low thrust. As a matter of fact, it is well-known that if SK maneuvers for GEO satellites are small, the satellite lifetime requires a large fuel quantity to perform the long time orbital maneuvers by continuous thruster firings.

Considering these technological and operational features, optimal control strategies for electric SK, taking various constraints into account (minimum elapsed time between two consecutive firing, on-off profile of the thrusters, thrust allocation) have to be carefully designed. The problem of station keeping is in general expressed as an optimal control problem even if the above mentioned constraints, inherent to the use of electric thrusters, prevent us to solve it with classical methods.

Several types of methods may be used to solve optimal SK control problem. When the thrust is considered as impulsive or in case simple models are used to describe the disturbing forces, analytical models provide sufficient control laws, as in Pessina et al. (2015), Romero et al. (2007) or Sukhanov and Prado (2012). Otherwise, it is necessary to resort to numerical methods, such as direct collocation based methods as described in Elnagar et al.

(1995), Hargraves and Paris (1987), Hull (1997), Betts (1998), Betts and Huffman (1998) and Betts (2000). For this family of approaches, the state and the control variables are discretized to produce a non linear programming problem and get an optimal control open loop. In order to counteract uncertainties affecting the control law, the use of Model Predictive Control algorithms is proposed in Weiss and Di Cairano (2015) and Weiss et al. (2015). To deal with on-off models of thrusts, the references Vazquez et al. (2011), Vazquez et al. (2013), Vazquez et al. (2014) and Vazquez et al. (2015) use the Pulse Width Modulation technique to generate rectangular profiles from a continuous one. Losa et al. (2005) has formulated a method based on differential inclusion, and a first avenue for the use of decomposition methods to solve the problem is given in Losa et al. (2006).

In this paper, the idea is to decompose the overall SK optimal control problem into two sub problems for which appropriate solution methods are designed. In a first step, an indirect method based on the application of Prontryagin Maximum Principle (PMP)with mixed control-state constraints is applied to solve a simplified optimal SK control problem, i.e. without considering some hard constraints on the control law (thrust constraints such as latency between two bursts of the same thruster and no simultaneous thrusting for instance). The solution of the Two Point Boundary Value Problem (TPBVP) derived from the optimal necessary conditions is initialized by an approximate solution given by a collocation based direct method. In a second step, a numerical approach is used to enforce all the thrust constraints left apart at the first step. A realistic numerical example illustrates the efficiency of the proposed approach.

2. PROBLEM STATEMENT

2.1 Minimum-Fuel Station Keeping Problem

Let consider a satellite equipped with 4 electric thrusters mounted on the anti-nadir face. The position of the satellite on its orbit is described with the equinoctial orbital elements as defined in Battin (1999):

$$\boldsymbol{x}_{eoe} = [a \ e_x \ e_y \ i_x \ i_y \ \ell_{M\Theta}]^t \in \mathbb{R}^6, \qquad (1)$$

where a is the semi-major axis, (e_x, e_y) the eccentricity vector components, (i_x, i_y) the inclination vector components, $\ell_{M\Theta} = \omega + \Omega + M - \Theta$ is the mean longitude where Ω is the right ascension of the ascending node, ω is the perigee's argument, M is the mean anomaly and $\Theta(t)$ is the right ascension of the Greenwich meridian. The dynamics of the satellite may be represented by the following non linear state-space model:

$$\begin{cases} \dfrac{d\boldsymbol{x}_{eoe}}{dt} = \boldsymbol{f_L}(\boldsymbol{x}_{eoe}, t) + f_G(\boldsymbol{x}_{eoe}, t)\boldsymbol{u}, \\ \boldsymbol{y}_{geo} = \boldsymbol{g}(\boldsymbol{x}_{eoe}, t), \end{cases} \qquad (2)$$

where $\boldsymbol{f_L} \in \mathbb{R}^6$ is the Lagrange contribution part of the external force model described by the CNES ORANGE

model (cf. Campan and Brousse (1994)) and $\boldsymbol{f_G} \in \mathbb{R}^{6 \times 3}$ is the Gauss contribution part. $\boldsymbol{y}_{geo} = [r \ \phi \ \lambda]^t \in \mathbb{R}^3$ is the geographical position of the satellite expressed in term of radius, latitude and longitude.

In order to deal with the station keeping problem, the relative state of the satellite with respect to the station keeping state is rather used. The station keeping state is defined as follows:

$$\boldsymbol{x}_{sk} = [a_{sk} \ 0 \ 0 \ 0 \ 0 \ \ell_{M\Theta_{sk}}]^t, \qquad (3)$$

where a_{sk} is the synchronous semi-major axis and $\ell_{M\Theta_{sk}}$ is the station mean longitude. The associated station keeping geographical position is $\boldsymbol{y}_{sk} = [r_{sk} \ 0 \ \lambda_{sk}]^t$.
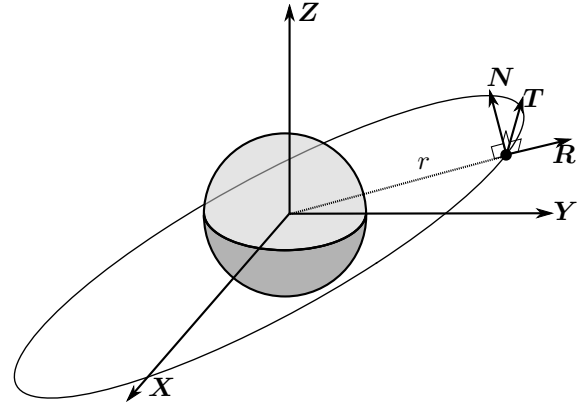

Fig. 1. Local Orbital Frame RTN.

The relative dynamics equations are developed by linearization of equation (2) about the station keeping point (3). By denoting $\boldsymbol{x} = \boldsymbol{x}_{eoe} - \boldsymbol{x}_{sk}$ and $\boldsymbol{y} = \boldsymbol{y}_{geo} - \boldsymbol{y}_{sk}$, the relative state and output model for the SK problem reads:

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = A(t)\boldsymbol{x} + \boldsymbol{D}(t) + B(t)\boldsymbol{u} \\ \boldsymbol{y} = C(t)\boldsymbol{x} \end{cases}, \qquad (4)$$

where matrices $A \in \mathbb{R}^{6 \times 6}$, $B \in \mathbb{R}^{6 \times 3}$, $C \in \mathbb{R}^{3 \times 6}$ and $\boldsymbol{D} \in \mathbb{R}^6$ are obtained from the linearization of functions $\boldsymbol{f_L}$, $\boldsymbol{u} \mapsto f_G \boldsymbol{u}$ and $\boldsymbol{g}$.

$\boldsymbol{u} = [u_R \ u_T \ u_N]^t \in \mathbb{R}^3$ is the control vector expressed in the local orbital RTN frame (also written RSW) defined in Vallado (1997) by (see Figure 1):

- $\boldsymbol{N}$ is the unit vector along the kinetic momentum;
- $\boldsymbol{R}$ is the unit vector along the direction Earth's center - satellite;
- $\boldsymbol{T}$ makes orthogonal direct basis.

Remembering that 4 thrusters are available to realize the control, the transcription of the station keeping problem expressed in terms of the control vector $\boldsymbol{u}$ would require solving of an allocation problem in a second stage to find a right combination of thrusts. An alternative, used in this paper, consists in directly considering the 4 thrusts provided by the 4 engines in the satellite dynamic. The control $\boldsymbol{u}(t)$ is a linear combination of the 4 thrusts such that $\boldsymbol{u} = \Gamma \boldsymbol{F}$, where $\Gamma = [\boldsymbol{\Gamma}_1 \mid \boldsymbol{\Gamma}_2 \mid \boldsymbol{\Gamma}_3 \mid \boldsymbol{\Gamma}_4] \in \mathbb{R}^{3 \times 4}$ and $\boldsymbol{F} = [F_1 \ F_2 \ F_3 \ F_4]^t \in [0, F_{max}]^4$. The thrust direction matrices $\boldsymbol{\Gamma}_j \in \mathbb{R}^3$ are defined such that:

$$\boldsymbol{\Gamma_j} = \frac{1}{m} \begin{bmatrix} -\sin\theta_j \cos\alpha_j \\ -\sin\theta_j \sin\alpha_j \\ -\cos\theta_j \end{bmatrix} \tag{5}$$

Two other transformations are proposed. Firstly, for the sake of simplicity, we will normalize the thrust vector: $\boldsymbol{F} = F_{max} \, \tilde{\boldsymbol{F}}$ with $\tilde{\boldsymbol{F}} \in [0,1]^4$. Secondly, as the thrusts are on-off, the thrust profile is modeled as a rectangular signal that is parametrized by the date $t_{i,j}$ corresponding to the middle instant of the thrust and by its half width duration denoted $\Delta t_{i,j}$ as shown on Figure 2:

$$F_i(t) = F_{max} \sum_{j=1}^{P_i} \text{RectangleFunction}(t, t_{i,j}, \Delta t_{i,j}). \tag{6}$$

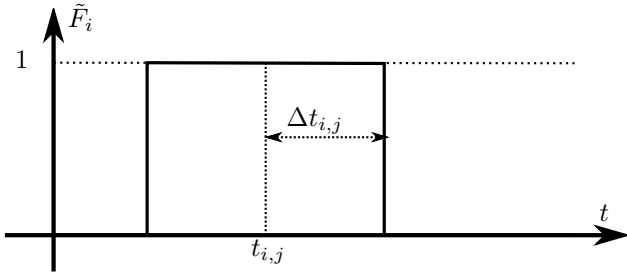where P_i is the number of thrusts of thruster i.

Fig. 2. Parametrization of the jth thrust with middle time $t_{i,j}$ and half-width duration $\Delta t_{i,j}$ for thruster i.

The main goal of the station keeping problem is to maintain the longitude and the latitude of the satellite in a box defined by its size δ by acting on the orbital parameter via the 4 thrusters. The associated optimal control problem is in general defined over a fixed horizon for the computation of optimal open loop control laws. In this context, optimality means that a minimum fuel-solution is looked for to extend the operational life time of the satellite. The minimum-fuel station keeping problem may therefore be defined as the following fixed-time constrained optimal control problem :

Problem 1. Find:

$$\min_{F_i} J \tag{7}$$

s.t.

$$\dot{\boldsymbol{x}}(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{D}(t) + \bar{B}(t)\,\tilde{\boldsymbol{F}}(t), \tag{8a}$$

$$\boldsymbol{G}(\boldsymbol{y}(t), t) \leqslant 0, \tag{8b}$$

$$\boldsymbol{H}(\tilde{\boldsymbol{F}}(t)) \leqslant 0, \tag{8c}$$

where:

$$J = \int_0^T \sum_{\text{thruster } i=1}^4 \sum_{j=1}^{P_i} \left(|u_{R_{ij}}(t)| + |u_{T_{ij}}(t)| + |u_{N_{ij}}(t)| \right) dt \tag{9}$$

$$= F_{max} \int_0^T \sum_{\text{thruster } i=1}^4 ||\boldsymbol{\Gamma}_i||_1 \sum_{j=1}^{P_i} |\tilde{F}_{i,j}(t)| dt \tag{10}$$

$$= 2F_{max} \sum_{\text{thruster } i=1}^4 ||\boldsymbol{\Gamma}_i||_1 \sum_{j=1}^{P_i} \Delta t_{i,j}, \tag{11}$$

with $\bar{B} = F_{max} B\Gamma$, equation (8b) is the constraint on the geographical position and equation (8c) is the constraint

on the control variables. T is the station keeping horizon.

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Note that the optimal solution of this problem is not affected by the constant parameters such as F_{max} and $||\boldsymbol{\Gamma}_i||_1$ that will only scale the value of the optimum. They will therefore be removed from the formulation of the problem for matter of simplicity.

The other constraints that have to be taken into account are presented in the next section.

The constraints on the output variables of Problem 1 are imposed so that the satellite stay in the station keeping geographical box. This box is defined in the plane (ϕ, λ) of width $2\delta \times 2\delta$ centered on the station keeping geographical position $\boldsymbol{y}_{sk}$. These constraints are written as $|y_2(t)| \leqslant \delta$ and $|y_3(t)| \leqslant \delta$. Using the definition of the output vector $\boldsymbol{y}(t) = C(t)\boldsymbol{x}(t)$, these constraints on the output variables are transformed into constraints on the state variables:

$$|[0\ 1\ 0]C(t)\boldsymbol{x}(t)| \leqslant \delta \text{ and } |[0\ 0\ 1]C(t)\boldsymbol{x}(t)| \,\forall t \in [0, T] \tag{12}$$

Boundary conditions have to be set. The initial position is chosen to be at the center of the station keeping box, what is written $\boldsymbol{x}(0) = 0$. The station keeping has to be performed on the time interval $[0, T]$. In order to use the same control law on the intervals $[kT, (k+1)T]$, $k \in \mathbb{N}^*$, it is convenient to add the terminal condition : $\boldsymbol{x}(T) = 0$.

2.2 Operational Constraints on Actuation

Beside the station keeping geographical constraints and the usual bounds on the maximum thrust, some additional technological operational constraints on the actuation described in Losa (2014) have to be taken into account:

(i) thrusters cannot have simultaneous thrusts;
(ii) a thrust must last at least $T_l : 2\Delta t_{i,j} \geqslant T_l$;
(iii) two successive thrusts of a given thruster must be separated of an interval of latency equal to T_s;
(iv) two thrusts of two different thrusters must be separated by an interval of latency equal to T_d.

In order to give a tractable mathematical expression for constraints (iii) and (iv), let us define for the thruster i the ordered sequence of firing times $(t_{i,k})_{k=1\ldots P_i}$ in increasing order. The constraint for the time latency between the thrust k of thruster i and the thrust l of thruster j is thus mathematically expressed as:

$$|t_{i,k} - t_{j,l}| - (\Delta t_{i,k} + \Delta t_{j,l}) \geqslant K_{i,j}, \tag{13}$$

for $k = 1 \ldots P_i$ and $l = 1 \ldots P_j$, where $K_{i,j} = T_s$ if $i = j$ (constraint (iii)) and $K_{i,j} = T_d$ otherwise (constraint (iv)).

The operational constraint (13) raises some difficult mathematical issues. Firstly, due to the parametrization of the rectangular functions by $t_{i,j}$ and $\Delta t_{i,j}$, the firing times and durations are both optimization variables and intrinsic variables of the problem to solve. Secondly, this constraint is non convex and logical, and is therefore difficult to be tackled within an optimal control problem.

In addition, some other convenient constraints are enforced in order to make sure that the thrusts will not begin before 0 and end after T:

$$t_{i,j} - \Delta t_{i,j} \geqslant 0 \ \text{ and } \ t_{i,j} + \Delta t_{i,j} \leqslant T. \quad (14)$$

2.3 Decomposition of the Overall Problem

Considering all the operational constraints described above, the minimum-fuel SK problem to solve may be summarized as the following optimal control problem:

Problem 2. Find the sequence of dates $\{t_{i,j}\}$ and durations $\{\Delta t_{i,j}\}$, for $i = 1 \ldots 4$, $j = 1 \ldots P_i$ (with P_i fixed) solutions of the minimization problem:

$$\min_{t_{i,j}, \Delta t_{i,j}} J = \sum_{i=1}^{4} \sum_{j=1}^{P_i} \Delta t_{i,j}, \quad (15)$$

with the constraints:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{D}(t) + \bar{B}(t)\,\tilde{\boldsymbol{F}}(t), \\ \boldsymbol{x}(0) = 0, \ \boldsymbol{x}(T) = 0, \\ |[0\ 1\ 0]C(t)\boldsymbol{x}(t)| \leqslant \delta, \ |[0\ 0\ 1]C(t)\boldsymbol{x}(t)| \leqslant \delta, \\ \Delta t_{i,j} \geqslant T_l, \ t_{i,j} - \Delta t_{i,j} \geqslant 0, \ t_{i,j} + \Delta t_{i,j} \leqslant T, \\ |t_{i,k} - t_{j,l}| - (\Delta t_{i,k} + \Delta t_{j,l}) \geqslant K_{i,j}, \end{cases} \quad (16)$$

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It is possible to solve the whole problem with direct methods such as collocation methods as described in Elnagar et al. (1995), Hargraves and Paris (1987), Hull (1997), Betts (1998), Betts and Huffman (1998) and Betts (2000). However, to obtain a solution with this method, the number of bursts per thruster has to be known beforehand. To solved this problem in its whole generality, it is necessary to solve a non linear mixed integer optimization problem with respect to the thrusts number for each thruster. Moreover, the solution is very sensitive to the number of collocation points and the initial guess of the optimisation variables.

An alternative strategy for solving the station keeping problem is thus developed in the sequel in order to overcome the drawbacks of using only a direct. From a mathematical point of view, the problem is difficult to solve. First, we have constraints on state vector and second, the constraint (13) is difficult to take into account as explained above. Instead of solving directly and at once the Problem 2, the resolution is thus split into two different steps:

(A) The thrusters constraints (13) and (14) are removed so that only an optimal control problem (OCP) with state and control constraints remains. This particular OCP is tackled via a hybrid approach relying on an indirect method initialised by a direct method solution dedicated to the search of adjoint variables as described in Bonnard et al. (2005), Grimm and Markl (1997), Bulirsch et al. (1993) and von Stryk and Bulirsch (1992). The Pontryagin Maximum Principle (PMP) is applied to derive necessary conditions of optimality and the associated Two Points Boundary Value Problem (TPBVP). The TPBVP is then solved

via a collocation method for which the results of the direct collocation step serves as first guess.

(B) As the result of the first step produces a control law that does not necessarily respect the thruster operational constraints, a second part is needed in order to obtain modified results of the first part fitting the thrusters constraints.

To sum up, instead of solving the overall problem at once, this problem is split into two smaller and simpler problems that are more easily solved in general as in Losa et al. (2006). Note that a different method is used here. Section 3 will be devoted to the first part described above: solution of the simplified OCP via the hybrid method. Then in Section 4, the thrusters constraints are considered and a complete realizable solution is obtained via the resolution of a second minimization problem.

3. SOLUTION OF THE SIMPLIFIED OCP (STEP (A))

Removing the operational actuation constraints (13) and (14) allows to redefine the optimal control problem. In particular, the thrust functions $\tilde{F}_i$ are not a priori modeled as rectangular functions parametrized by $t_{i,j}$ and $\Delta t_{i,j}$ and the simplified OCP to be solved reads as :

Problem 3. The simplified OCP to be solved is finding:

$$\min_{\tilde{\boldsymbol{F}}(t) \in [0;1]^4} J = \int_0^T \sum_{i=1}^{4} \tilde{F}_i(t) dt, \quad (17)$$

with the constraints:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{D}(t) + \bar{B}(t)\,\tilde{\boldsymbol{F}}(t), \\ |[0\ 1\ 0]C(t)\boldsymbol{x}(t)| \leqslant \delta, \ |[0\ 0\ 1]C(t)\boldsymbol{x}(t)| \leqslant \delta, \\ \boldsymbol{x}(0) = 0, \boldsymbol{x}(T) = 0, \end{cases} \quad (18)$$

$\circ$

Problem 3 is a minimum-fuel linear OCP with constraints both on the state and the control vectors and defined on a fixed horizon. To solve this problem, a direct collocation method finds a first approximation of the optimal solution. This approximated solution is then used to initialize the solution of the TPBV problem obtained by the application of Pontryagin Maximum Principle (PMP) and the derivation of the necessary optimality conditions. As described in the reference Bonnard et al. (2005), using an indirect method to solve an OCP leads to a very precise solution but that is very sensitive to the initial conditions. A good guess for this initial condition is therefore needed. It is provided by the direct collocation method.

3.1 Finding an Initial Guess via a Direct Method

Suboptimal solution of the problem 3 is found by applying a classical collocation method. The time interval $[0, T]$ is divided into N equidistant points:

$$0 = \tau_1 < \tau_2 < \cdots < \tau_k < \cdots < \tau_{N-1} < \tau_N = T, \quad (19)$$

with $\tau_{k+1} - \tau_k = \Delta\tau$ so that a grid of N points defines the discretization of Problem 3, with $\boldsymbol{x}_k = \boldsymbol{x}(\tau_k)$, $\tilde{\boldsymbol{F}}_k = \tilde{\boldsymbol{F}}(\tau_k)$ and $\boldsymbol{y}_k = \boldsymbol{y}(\tau_k)$. the collocation method aims at finding

the value of the control at the points τ_k of the grid. The continuous control is then obtained by interpolation.

The collocation problem is mathematically formulated as follows:

Problem 4. The collocation problem to be solved to initialize the optimal control problem 3 is finding:

$$\min_{\tilde{F}_{k,i}, x_k} \ J = \sum_{i=1}^{4} \sum_{k=1}^{N} \frac{\tilde{F}_{k,i} + \tilde{F}_{k+1,i}}{2} \Delta\tau, \qquad (20)$$

subject to the constraints:

$$\begin{cases} \boldsymbol{R}_k(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \tilde{\boldsymbol{F}}_k, \tilde{\boldsymbol{F}}_{k+1}) = 0, \\ |[0\ 1\ 0]C(\tau_k)\boldsymbol{x}_k| \leqslant \delta, \ |[0\ 0\ 1]C(\tau_k)\boldsymbol{x}_k| \leqslant \delta \end{cases} \qquad (21)$$

where the defects $\boldsymbol{R}_k$ at each point τ_k are defined by Hull (1997):

$$\begin{aligned} \boldsymbol{R}_k(\boldsymbol{x}_k, &\boldsymbol{x}_{k+1}, \tilde{\boldsymbol{F}}_k, \tilde{\boldsymbol{F}}_{k+1}) = \\ &\left[Id + \frac{\Delta\tau}{6}\left(-2A_m + \frac{\Delta\tau}{2}A_m A_{k+1} - A_{k+1} \right) \right] \boldsymbol{x}_{k+1} \\ &- \left[Id + \frac{\Delta\tau}{6}\left(2A_m + \frac{\Delta\tau}{2}A_m A_k + A_k \right) \right] \boldsymbol{x}_k \\ &+ \frac{\Delta\tau}{6}\left[-\bar{B}_{k+1} + \frac{\Delta\tau}{2}A_m \bar{B}_{k+1} - 2\bar{B}_m \right] \tilde{\boldsymbol{F}}_{k+1} \\ &- \frac{\Delta\tau}{6}\left[\bar{B}_k + \frac{\Delta\tau}{2}A_m \bar{B}_k + 2\bar{B}_m \right] \tilde{\boldsymbol{F}}_k \\ &+ \frac{\Delta\tau}{6}\left[-4\boldsymbol{D}_m + \left(\frac{\Delta\tau}{2}A_m + Id \right)(\boldsymbol{D}_{k+1} - \boldsymbol{D}_k) \right], \end{aligned} \qquad (22)$$

where Id is the identity matrix, $M_k = M(\tau_k)$, $M_{k+1} = M(\tau_{k+1})$, $M_m = M\left(\frac{\tau_k + \tau_{k+1}}{2} \right)$ for $M \in \{A, \bar{B}, D\}$. ∘

The defects are obtained by discretizing the differential constraint of Problem 3 using the 4th order Simpson algorithm described in Hull (1997). Note 20 trap 17.

The trajectory $\{\boldsymbol{x}_k, k = 1 \dots N\}$ verifying the geographical constraints and the set of the Lagrange multipliers of the defect constraints $\{\boldsymbol{\lambda}_k, k = 1 \dots N\}$ are thus obtained. The reference Bonnard et al. (2005) shows that these Lagrange parameters are an approximation of the adjoint vector for the OCP problem 3. Therefore, $(\boldsymbol{x}_1, \boldsymbol{\lambda}_1)$ can be used as an initial condition for the TPBV problem obtained from the PMP and from the derivation of the necessary optimality conditions.

3.2 Solving the Simplified OCP via an Indirect Method

The idea is now to tackle the usual issues of the direct methods (sub optimality rough precision and sensitivity of the solution to the initial guess) by combining it with an indirect approach based on the PMP. To handle the state geographical constraints, the technique is based on a penalty function as described in Naidu (2002). The following penalty function is added to the cost function of Problem 3 (see equation (18)) and is chosen such that:

$$\boldsymbol{g}(\boldsymbol{x}(t)) = \sum_{i=1}^{4} \frac{1}{2}\left[g_i(\boldsymbol{x}(t)) \right]^2 \ \left(\operatorname{sign}(g_i(\boldsymbol{x}(t))) + 1 \right). \qquad (23)$$

The Hamiltonian of the system becomes:

$$\begin{aligned} \mathcal{H}(\boldsymbol{x}(t), \tilde{\boldsymbol{F}}(t), \boldsymbol{\lambda}(t)) &= \sum_{i=1}^{4} \tilde{F}_i(t) \\ &+ \boldsymbol{\lambda}(t)^t \left[A(t)\,\boldsymbol{x}(t) + \boldsymbol{D}(t) + \bar{B}(t)\,\tilde{\boldsymbol{F}}(t) \right] \\ &+ \mu\,\boldsymbol{g}(\boldsymbol{x}(t)), \quad (24) \end{aligned}$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^6$ is the adjoint vector, and μ is a constant parameter that can be chosen as large as needed.

Applying the PMP described in Bonnard et al. (2005) and Naidu (2002), the minimization condition:

$$\mathcal{H}\left(\boldsymbol{x}(t), \tilde{\boldsymbol{F}}(t), \boldsymbol{\lambda}(t) \right) = \min_{\boldsymbol{u} \in [0,1]^4} \mathcal{H}\left(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t) \right) \qquad (25)$$

leads to the following switching conditions:

$$\tilde{F}(t) = \frac{1}{2}\left[1 - \operatorname{sign}\left(1 + \lambda^t(t)\bar{B}(t) \right) \right]. \qquad (26)$$

It is assumed that singular arcs are not possible optimal solution for this problem. Another way to take into account the state constraint in the derivation of the optimality conditions with the PMP is presented in Hartl et al. (1995).

By applying the first order necessary conditions:

$$\dot{\boldsymbol{x}}(t) = \frac{\partial\mathcal{H}}{\partial\boldsymbol{\lambda}} \text{ and } \dot{\boldsymbol{\lambda}}(t) = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{x}}, \qquad (27)$$

and the transversality conditions as boundary conditions, a two points boundary value problem is obtained and formulated as Problem 5.

Problem 5. The TPBV problem to be solved is defined as:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A(t)\boldsymbol{x}(t) + \boldsymbol{D}(t) \\ &+ \bar{B}(t)\frac{1}{2}\left[1 - \operatorname{sign}\left(1 + \boldsymbol{\lambda}^t(t)\bar{B}(t) \right) \right], \\ \boldsymbol{x}(0) &= 0, \quad \boldsymbol{x}(T) = 0 \\ \dot{\boldsymbol{\lambda}}(t) &= -A(t)^t \boldsymbol{\lambda}(t) - \mu\frac{d\boldsymbol{g}}{d\boldsymbol{x}}(\boldsymbol{x}(t)), \\ \boldsymbol{\lambda}(0) &\text{ and } \boldsymbol{\lambda}(T) \text{ free.} \end{aligned} \qquad (28)$$

∘

To solve this problem, the initial guess used is the one obtained as a solution of Problem 4.

The initial conditions for the state and adjoint vectors being well chosen thanks to the solution of Problem 4, Problem 5 is easily solvable. However, this solution does not satisfy in general the operational actuation constraints described in section 2.2, the next section proposes an additional complementary step for which a direct method is applied on an auxiliary problem enforcing the actuation constraints on a new and equivalent control law, while preserving the structure and overall effect of the thrust.

4. ENFORCING THE OPERATIONAL THRUSTERS CONSTRAINTS (STEP (B))

The solution of the TPBV problem 3 consists in a series of rectangle signals naturally exhibited as a result from application of the PMP. However, this control law does not respect the actuation constraints left apart in the first part. For instance, simultaneous activation of two different

thrusters may occur. The aim of this second part is to find an equivalent control law satisfying the required non linear operational constraints. To do so, it is first necessary to clearly define the meaning of equivalent control laws in the sequel. Two different notions of equivalent control laws will be used hereafter.

The first notion of equivalence between two control laws relies on the fuel consumption argument: the goal is to compute a raw control profile that has the same fuel consumption as the profile obtained by solving the TPBV problem. Let $\tilde{F}_{BVP}$ be the control obtained by solving Problem 5. Finding a consumption based equivalent control for the satellite is then equivalent to solve Problem 6 defined as follows:

Problem 6. Find:

$$\min_{t_{i,j}, \Delta t_{i,j}} \sum_{i=1}^{4} \left(\|\tilde{F}_{\text{BVP},i}(t)\|_1 - \sum_{j=1}^{P_i} \Delta t_{i,j} \right), \qquad (29)$$

subject to the constraints:

$$\begin{cases} \Delta t_{i,j} \geqslant T_l, \\ |t_{i,k} - t_{j,l}| - (\Delta t_{i,k} + \Delta t_{j,l}) \geqslant K_{i,j}, \\ t_{i,j} - \Delta t_{i,j} \geqslant 0, \quad t_{i,j} + \Delta t_{i,j} \leqslant T. \end{cases} \qquad (30)$$

∘

This problem is a non linear optimization problem where $\|\tilde{F}_{\text{BVP},i}(t)\|_1$ is the L^1 norm of the ith component of the the solution of Problem 5.

The second way to obtain an equivalent control respecting the actuation constraints is to define an "effect-based" equivalent control. If the system differential equation is given by:

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{D}(t) + \bar{B}(t)\tilde{\boldsymbol{F}}(t), \qquad (31)$$

then the state vector at time t is given by:

$$\boldsymbol{x}(t) = \Phi(t,t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau) \left[\boldsymbol{D}(\tau) + B(\tau)\Gamma\boldsymbol{u}(\tau) \right] d\tau, \qquad (32)$$

where $\Phi(t,t_0)$ is the transition matrix at time t from time t_0, implicitly defined by the homogeneous differential equation $\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$, $\Phi(t_0,t_0) = Id$. To get a control profile that has the same effect at time T as the solution of Problem 5 and respecting the actuation constraints, Problem 7 has to be solved and is defined as follows:

Problem 7.

$$\min_{t_{i,j}, \Delta t_{i,j}} \left| \sum_{i=1}^{4} \left(\int_0^T \left[\Phi(T,\tau)\bar{B}(\tau)\tilde{\boldsymbol{F}}_{\text{BVP}}(\tau) \right]_i d\tau \right. \right.$$
$$\left. \left. - \sum_{j=1}^{P_i} \Delta t_{i,j} \int_{t_{i,j}-\Delta t_{i,j}}^{t_{i,j}+\Delta t_{i,j}} [\Phi(T,\tau)\bar{B}(\tau)]_i d\tau \right) \right|, \qquad (33)$$

such that the constraints (30) are satisfied, ∘

where $[\boldsymbol{h}]_i$ stands for the ith component of any vector $\boldsymbol{h}$. This problem is a non linear optimization problem. Both Problem 6 and 7 can be solved by classical non linear optimization solvers.

5. NUMERICAL RESULTS

In this section, simulation results obtained with the proposed methodology are presented. Let consider a satellite of mass 4850 kg equipped with 4 electric thrusters oriented in the directions North-East, North-West, South-East and South-West. This satellite has to be controlled in order to remain close to its geostationary position at a fixed longitude $\bar{\lambda}$. The decomposition presented in Sections 3 and 4 is now illustrated numerically. Parameter μ has to be chosen to compute numerical results. To analyze the results of the first part, we use $\mu = 1.10^4$. For the numerical analysis of the second part, in order to make the satellite stay in the station keeping window, two different values of μ have to be chosen for the two different equivalent schemes: for the consumption-based one, $\mu = 1.10^4$ is still acceptable, whereas for the effect-based one $\mu = 1.10^{12}$ is chosen.

On Figure 3, the geographical parameters obtained by solving Problem 4, and the geographical parameters obtained by solving the TPBV problem 5 are drawn. It is recalled that the solution of Problem 4 is used as initial guess for solving Problem 5. The commutation function associated to the solution of Problem 5 is shown in Figure 4. Figure 5 shows the control profile obtained after solving Problem 5. As explained above, the thrusters constraints are not satisfied, as thrusters NE and SE are simultaneously active, as well as thrusters NW and SW. This issue requires to perform the second step to make the control profile realizable with respect to the thrusters operational constraints.
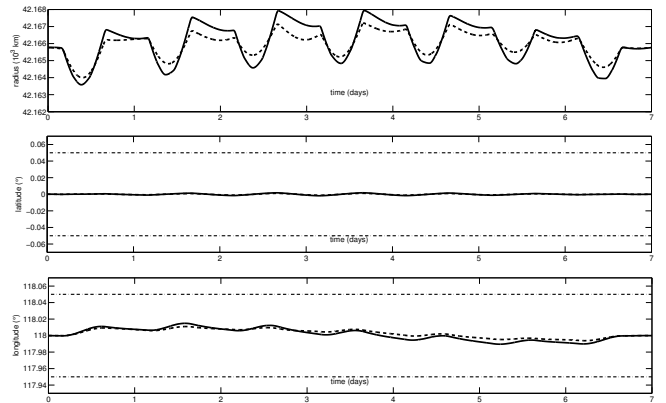

Fig. 3. Geographical position obtained solving Problem 5 (plain line) and geographical position used as initial guess (dashed line). $\mu = 1.10^4$.

Figure 6 shows that the equivalence schemes do not lead to identical results. The effect-based one leads to a trajectory that remain close to the center of the station keeping window whereas the consumption based one lead to a trajectory that goes close to the border of the window. Moreover, the consumption is very different in both cases: the control profile computed with the consumption-based equivalence requires more fuel that the control profile computed with the effect-base equivalence. The relative difference is $81,6\%$. Figure 7 presents a detailed view of
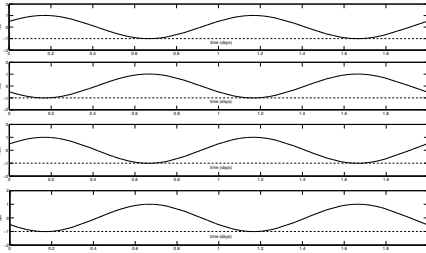
Fig. 4. Detailed view of the commutation $\lambda^t \bar{B}$ function for the solution of the TPBV problem for $\mu = 1.10^4$.
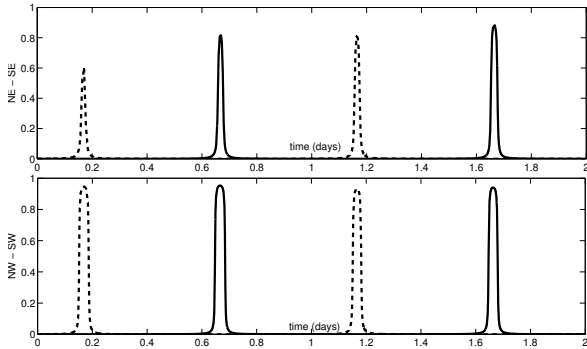

Fig. 5. Detailed view of the control profile for the TPBV problem 5. Up: plain line: North-Est, dashed line North-West. Down: plain line: South-Est, dashed line: South-West. $\mu = 1.10^4$.

the firing sequence for both equivalence schemes. Despite these differences, the two control profiles solve the station keeping problem, as can be seen in Figure 8.
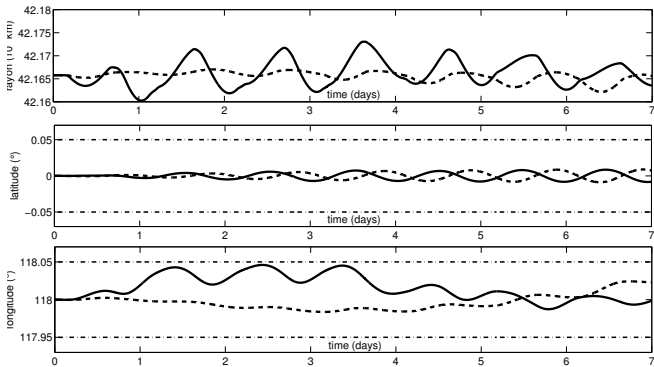

Fig. 6. Geographical parameters. Plain line: consumption-based equivalence scheme (for $\mu = 1.10^4$). Dashed line: effect-based equivalence scheme (for $\mu = 1.10^{12}$).

6. CONCLUSION

In this paper a decomposition of the overall station keeping optimal control problem under many operational constraints is used to take into account some difficult constraints inherent to the use of electric propulsion. Firstly, a classical optimal control problem is solved with state constraints using a precise indirect method initialized by a collocation based direct method. Secondly, two ways
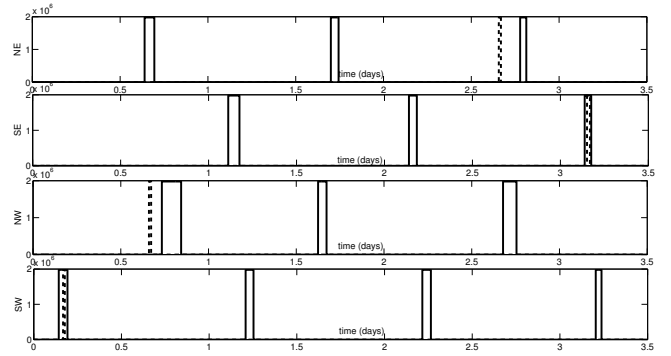

Fig. 7. Detailed view of the thrust profile after enforcing the operational constraints. Plain line: consumption-based equivalence ($\mu = 1.10^4$). Dashed line: effect-based equivalence ($\mu = 1.10^{12}$).
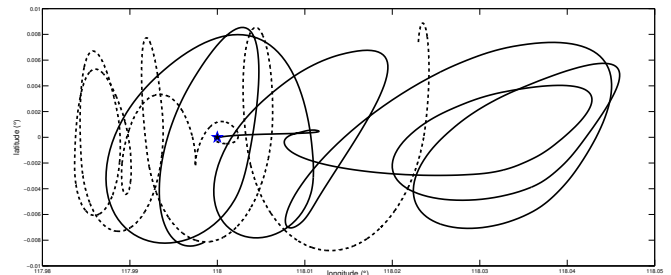

Fig. 8. Geographical latitude vs. longitude. Plain line: consumption-based equivalence scheme ($\mu = 1.10^4$). Dashed line: effect-based equivalence ($\mu = 1.10^{12}$).

of dealing with the thrusters operational constraints are proposed resulting in to different fuel consumption results. Despite the positive results presented in this paper, different issues remain open: the optimality of possible singular arcs has to be studied and alternative formulation of the PMP for state constraints optimal control problems (as in Hartl et al. (1995)) should be considered.

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