Periodic Memory State-feedback Controller: New formulation, analysis and design results

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***Abstract*—This paper proposes a unified setup for robust stability and performance analysis and synthesis for periodic polytopic discrete-time systems. Relying on a general formulation for state-feedback periodic memory controllers and a new time-lifting, new sufficient LMI conditions for the existence of robust stability certificates and $\mathcal{H}_2$ guaranteed cost control laws are derived. Comparisons of the efficiency of different controller structures illustrate these developments on a numerical example.**

***Index Terms*—Linear periodic systems, LMIs, State feedback with memory, $\mathcal{H}_2$ guaranteed cost**

I. Introduction

It is widely recognized that extending Linear Time-Invariant (LTI) robust synthesis tools to Linear Time-Varying (LTV) models is a very challenging and promising research field [13]. For instance, robust analysis and control of periodic systems has known a renewed interest, mainly due to the variety and the originality of the possible applications [1], [19], [4], [25]. For recent work on this class and some background for the present work, we refer the reader to [3] and the references therein. Convex synthesis methods have been proposed recently, that yield such periodic controllers when the uncertain periodic discrete-time model is assumed to be affected by polytopic uncertainties [7], [11], [10]. In particular, in [10], a further attempt was made to significantly outperform previous results by introducing a new class of controllers. The usual family of periodic static control laws is extended by incorporating memorized past states of the plant. Different control structures, characterized by their specific use of the state memory, have then been proposed (see [10], [21], [23], [8] and [22]).

The first objective of this paper is to give a general setup for robust state-feedback periodic memory controllers design unifying all these previous results. This new framework is used to highlight features of the different control structures characterized by a sequence of insightful parameters for the designer. This sequence of parameters may be viewed as a way to manage additional degrees of freedom and complexity of the designed control laws. In addition, we propose new robust LMI conditions for stability and performance analysis and synthesis of periodic discrete time systems affected by polytopic uncertainties under this general formulation. Technically, these conditions are obtained by applying classical results on a specific time-invariant reformulation. Time-lifting procedure has therefore been generalized in the context of memory periodic model and a new descriptor-like lifting has been introduced in this paper for the first time. Even though descriptor-like lifting is briefly introduced in [10] for the autonomous case, the authors there derived LMI synthesis conditions of a specific periodic memory controller by relying on the classical time-lifting and involved matrix manipulation. It is hard to deal with general periodic memory controller synthesis in the same direction, and this difficulty can be circumvented successfully by a rigorous treatment of the descriptor-like lifting.

Finally, the general duality theory of LTI systems developed in [14] has been revisited and applied to this particular lifted representation.

In the last section, numerical examples illustrate these results and confirm that adding some degrees of freedom to the control law contributes to effectively decrease conservatism. Furthermore, it is shown that some particular configuration of the memory allows to drastically reduce the conservatism of the synthesis conditions.

We use the following notations in this paper. The symbols $\mathbf{1}_n$ and $\mathbf{0}_{m\times n}$ stand for the identity and zero matrices of dimensions $n\times n$ and $m\times n$, respectively. When the dimensions are clear from the context, they are omitted. The set of symmetric matrices and positive-definite symmetric matrices of size l are denoted by $\mathbb{S}^l$ and $\mathbb{S}^l_+$ respectively. For a real square matrix A, we define $\mathrm{He}\left\{A\right\} = A + A^T$. Let also $\mathrm{Sq}\left\{A\right\} = AA^T$. The operator 'diag' builds block-diagonal matrix from input arguments. The convex hull of the collection of L elements $A^{[1]}, \cdots, A^{[L]}$ is denoted by $\mathrm{co}\left\{A^{[1]}, \cdots, A^{[L]}\right\}$. σ stands for the shift operator forward in time: $\sigma x_{Nq} = x_{N(q+1)}$ The standard operator of modular arithmetic is referred as 'mod'. The Kronecker product is denoted by $\otimes$.

II. A general formulation for state-feedback memory controllers

A. Preliminaries

Throughout this paper, rather than considering each instant time, periods are treated globally via liftings. To this end, every time instant is expressed as $\tau + Nq + k$ with $0 \leq k \leq N-1$ such that $q \in \mathbb{N}$ characterizes the considered period which starts at $\tau + Nq$ and ends at $\tau + Nq + N - 1 = \tau + N(q+1)-1$. As it has already been pointed out in [12], conservatism of robust analysis results may depend on the choice of τ.

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However, without loss of generality, this paper focuses on the special case where $\tau = 0$. Some guidelines about the influence of the choice of $\tau \in \{0, \cdots, N-1\}$ are provided. Future works will investigate this issue in more details.

Consider the linear discrete-time N-periodic system $\Sigma(\theta)$ described by

$$\Sigma(\theta): \begin{bmatrix} x_{Nq+k+1} \\ z_{Nq+k} \end{bmatrix} = \underbrace{\begin{bmatrix} A_k(\theta) & B_k(\theta) & Q_k(\theta) \\ C_k(\theta) & D_k(\theta) & R_k(\theta) \end{bmatrix}}_{M_k(\theta)} \begin{bmatrix} x_{Nq+k} \\ w_{Nq+k} \\ u_{Nq+k} \end{bmatrix} \quad (1)$$

with $M_{k+N}(\theta) = M_k(\theta)$ and where[1] $x_{Nq+k} \in \mathbb{R}^n$, $w_{Nq+k} \in \mathbb{R}^m$, $u_{Nq+k} \in \mathbb{R}^{m_u}$ and $z_{Nq+k} \in \mathbb{R}^p$. In addition, the model $\Sigma(\theta)$ is subject to polytopic uncertainties gathered in θ such that for all θ in the uncertain domain Θ defined as the unit simplex:

$$\begin{bmatrix} M_0(\theta) \\ \vdots \\ M_{N-1}(\theta) \end{bmatrix} \in \text{co} \left\{ \begin{bmatrix} M_0^{[1]} \\ \vdots \\ M_{N-1}^{[1]} \end{bmatrix}, \cdots, \begin{bmatrix} M_0^{[L]} \\ \vdots \\ M_{N-1}^{[L]} \end{bmatrix} \right\} \quad (2)$$

In the nominal case (i.e when there is no uncertainty in the system matrices), it is known that a static periodic state-feedback, described by $u_{Nq+k} = K_k x_{Nq+k}$, allows to arbitrarily shape the dynamics of the closed-loop system within the reachability domain via pole assignment [3]. This is not true anymore when the system is affected by uncertainties. This is the reason why, this basic control law was enriched in [10] by letting the controlled input u_{Nq+k} depend not only on the current state x_{Nq+k}, but also on the state history. In line with these works, this paper is concerned with the design of controllers described by

$$u_{Nq+k} = K_{k,0} x_{Nq+k} + K_{k,1} x_{Nq+k-1} + \cdots + K_{k,\alpha_k} x_{Nq+k-\alpha_k} \quad (3)$$

The sequence $\{\alpha_k\}_{k=0}^{N-1}$ is composed of the numbers of past states required to evaluate u_{Nq+k} for every $0 \leq k \leq N-1$. It characterizes the control structure. Among every possible choice allowed by this general structure, the three following cases characterized by $\kappa \in \mathbb{N}$ are particularly remarkable:

- $\alpha_k = 0$: Classical memoryless Periodic State-Feedback Controller (PSFC);
- $\alpha_k = k + \kappa$: Periodic Full Memory Controller (PFMC) of order κ;
- $\alpha_k = \kappa$: Periodic FIR Controller (PFIRC) of constant order κ.

Referring to this new terminology, [7], [10] and [21] deal respectively with the following structures: PSFC (or PFIRC of order 0), PFMC of order 0 and PFIRC of order $N-1$. Thus, the controller (3) can be regarded as a unifying way for combining existing results on discrete-time periodic state-feedback controllers.

Remark. Changing τ, the starting point of the considered period, while keeping the same control structure gives rise to different controllers. Reversely, changing τ imposes to change the structure to keep the same controller. As an example, consider a 2-periodic control law with structure $\{\alpha_0, \alpha_1\} = \{0, 2\}$ for $\tau = 0$. In a new time frame starting at $\tau = 1$, the same controller has structure $\{\alpha_0, \alpha_1\} = \{2, 0\}$.

[1] Although only signals of constant dimensions are considered in this paper, every result may easily be extended to allow n, m, m_u and p to vary along the period.

B. The special case of PFMCs

If α_k refers to the number of past states required to evaluate u_{Nq+k} and is constant all along the period for the cases PSFC and PFIRC, its significance is richer when dealing with the PFMC structure. As an illustration, consider a 3-periodic PFMC of order $\kappa = 1$ corresponding to the sequence $\{\alpha_0, \alpha_1, \alpha_2\} = \{1, 2, 3\}$. Control inputs can be written down over the period q as:

$$\begin{cases} u_{3q} = & & \begin{bmatrix} K_{0,0} & K_{0,1} \end{bmatrix} \eta_q \\ u_{3q+1} = & \begin{bmatrix} K_{1,0} \end{bmatrix} \beta_{3q+1} + & \begin{bmatrix} K_{1,1} & K_{1,2} \end{bmatrix} \eta_q \\ u_{3q+2} = & \begin{bmatrix} K_{2,0} & K_{2,1} \end{bmatrix} \beta_{3q+2} + & \begin{bmatrix} K_{2,2} & K_{2,3} \end{bmatrix} \eta_q \end{cases} \quad (4)$$

with

$$\beta_{3q+1} = \begin{bmatrix} x_{3q+1} \end{bmatrix}, \ \beta_{3q+2} = \begin{bmatrix} x_{3q+2} \\ x_{3q+1} \end{bmatrix}, \ \eta_q = \begin{bmatrix} x_{3q} \\ x_{3q-1} \end{bmatrix} \quad (5)$$

Thus, state memory required to compute u_{3q+k} is composed of the vector η_q, of constant size and made of $l = \kappa + 1$ states, and of β_{3q+k} which dimension is varying over the period. More precisely, β_{3q+k} incorporates, at each step of the period, the current state of the system, x_{3q+k}. On the other side, η_q contains the state memory of the system prior to the period. Note that, when considering the general class of controllers (3), different sequences α_k lead to the same definition of β_{3q+k} and η_q. As an example, the 3-periodic controller characterized by the sequence $\{\alpha_0, \alpha_1, \alpha_2\} = \{0, 2, 0\}$ can be written down as (4) with $K_{0,1} = K_{2,1} = K_{2,2} = K_{2,3} = \mathbf{0}$. In fact, PFMCs offer the largest number of degrees of freedom for a given knowledge about past states of the plant defined by the order κ. The name of this class of controller originates from this observation.

Consequently, every control structure (even PSFC or PFIRC) can be worked out as a particular structured PFMC of order $\kappa = l - 1$ where l is defined by

$$l = \max_{k \in [0, \cdots, N-1]} \alpha_k - k + 1 \geq 1 \quad (6)$$

Therefore, instead of (3), PFMCs of order κ formalized as:

$$u_{Nq+k} = \sum_{j=0}^{\kappa+k} K_{k,j} x_{Nq+k-j} = \sum_{j=0}^{l-1+k} K_{k,j} x_{Nq+k-j} \quad (7)$$

are particularly considered and all robust synthesis results will be formulated for this particular class of controllers.

C. Problems statement

The periodic autonomous system with memory $\Sigma_{cl}(\theta)$ arising from the closed-loop of (1) with (7) is described by

$$\Sigma_{cl}(\theta): \begin{bmatrix} x_{Nq+k+1} \\ z_{Nq+k} \end{bmatrix} = \sum_{j=0}^{l-1+k} \begin{bmatrix} A_{k,j}(\theta) \\ C_{k,j}(\theta) \end{bmatrix} x_{Nq+k-j} + \begin{bmatrix} B_k(\theta) \\ D_k(\theta) \end{bmatrix} w_{Nq+k} \quad (8)$$

where $A_{k,j}$ and $C_{k,j}$ are given by

$$\begin{bmatrix} A_{k,j}(\theta) \\ C_{k,j}(\theta) \end{bmatrix} = \begin{cases} \begin{bmatrix} A_k(\theta) + Q_k(\theta)K_{k,0} \\ C_k(\theta) + R_k(\theta)K_{k,0} \end{bmatrix}, \ (j=0) \\ \begin{bmatrix} Q_k(\theta)K_{k,j} \\ R_k(\theta)K_{k,j} \end{bmatrix}, \ (1 \le j \le l-1+k) \end{cases} \tag{9}$$

Remark. The dynamics of $\Sigma_{cl}(\theta)$ can be more easily visualized by rewriting (8) using a formulation similar to (4) and requiring the definition of the vectors η_q and β_q. It will be seen in the following that the properties of these vectors have some consequence for the robust analysis of $\Sigma_{cl}(\theta)$.

This paper is first concerned with robust stability and $\mathcal{H}_2$ performance analysis of $\Sigma_{cl}(\theta)$.

Problem 1 (Robust Analysis)**.** *Provide a certificate assessing the robust stability of $\Sigma_{cl}(\theta)$. If it exists, find an $\mathcal{H}_2$ guaranteed cost $\gamma_{\rm g} \ge \gamma_{\rm wc}$ where $\gamma_{\rm wc}$ is defined as the squared worst-case $\mathcal{H}_2$ norms of $\Sigma_{cl}(\theta)$:*

$$\gamma_{\rm wc} = \max_{\theta \in \Theta} \|\Sigma_{cl}(\theta)\|_2^2 \tag{10}$$

Remark. In the sequel, time-invariant reformulations relying on time lifting are intensively used. This requires to specify τ. As it has already been pointed out by [12], this specification may have a large influence on the conservatism of subsequent robust analysis results. However, this very difficult issue is only considered in the numerical experiments for illustration. Theoretical developments on this topic are left for further investigations.

In a second stage, the corresponding synthesis problem is considered:

Problem 2 (Robust Synthesis)**.** *Find a controller* (3) *characterized by a given sequence $\{\alpha_k\}_{k=0}^{N-1}$ which robustly stabilizes $\Sigma_{cl}(\theta)$ and minimizes a guaranteed upper bound of $\gamma_{\rm wc}$.*

Remark. In the remaining of the paper, notations are sometimes simplified by omitting the dependency of matrices with respect to the uncertain vector θ.

III. Robust analysis

It is well-known that time-invariant reformulations may offer a suitable way to analyze periodic systems [2], [3]. Indeed, the analysis condition stated for the time-invariant model can be utilized for the periodic one as this transformation preserves stability and input/output relationship. Among procedures provided in [2], [3], time-lifting is probably the most classical one. Consider model (1) where $u_{Nq+k} = 0$. The time-invariant reformulation of (1), called the lifted system at time τ, is a state-sampled representation of (1) with state vector $\eta_q^\tau = x_{Nq+\tau}$ and with augmented input $\hat{w}_q^\tau = [w_{Nq+\tau}^T \ \cdots \ w_{N(q+1)+\tau-1}^T]^T$ and output $\hat{z}_q = [z_{Nq+\tau}^T \ \cdots \ z_{N(q+1)+\tau-1}^T]^T$. Details of the corresponding state-space matrices are omitted for space reasons and may be found in [2], [3]. In the sequel, except for the numerical section, we consider lifted systems at time $\tau = 0$ where the dependence upon τ is omitted.

In the context of periodic models with memory this reformulation can be obtained by recasting $\Sigma_{cl}(\theta)$ as a memoryless model with time-varying dimensions. However, when dealing with uncertain polytopic periodic models, usual time-liftings destroy this underlying geometry preventing to express simple LMI conditions for robust analysis. To overcome this difficulty, this paper proposes a new lifting procedure based on a peculiar descriptor-like formulation. The next subsection is dedicated to the presentation of the general strategy utilized to get tractable LMI conditions for robust stability.

A. A general strategy for robust analysis via time-liftings

In order to make this discussion clearer and with lighter notations, consider the memoryless periodic model (1) for $N = 2$ and where the control input $\hat{u}_q$ has been removed for simplicity. The lifted system is given by:

$$\begin{bmatrix} \eta_{q+1} \\ \hline \hat{z}_q \end{bmatrix} = \left[\begin{array}{c|cc} A_1(\theta)A_0(\theta) & B_1(\theta) & A_1(\theta)B_0(\theta) \\ \hline C_1(\theta)A_0(\theta) & D_1(\theta) & C_1(\theta)B_0(\theta) \\ C_0(\theta) & \mathbf{0} & D_0(\theta) \end{array}\right] \begin{bmatrix} \eta_q \\ \hline \hat{w}_q \end{bmatrix} \tag{11}$$

where[2]

$$\hat{w}_q = \begin{bmatrix} w_{2q+1}^T \ w_{2q}^T \end{bmatrix}^T \in \mathbb{R}^{2m}, \hat{z}_q = \begin{bmatrix} z_{2q+1}^T \ z_{2q}^T \end{bmatrix}^T \in \mathbb{R}^{2p} \tag{12}$$

and $\eta_q = x_{Nq}$.

The widely used formulation (11) makes this model suitable for establishing nominal design and analysis results [24]. Unfortunately, as this example clearly shows, the polytopic dependence of the matrices of (11) is lost through this particular lifting procedure.

To address this problem, the same periodic model can be alternatively formulated as:

$$\left[\begin{array}{ccc|cc|cc} -\mathbf{1} & A_1(\theta) & \mathbf{0} & B_1(\theta) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & A_0(\theta) & \mathbf{0} & B_0(\theta) & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & -\sigma\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C_1(\theta) & \mathbf{0} & D_1(\theta) & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_0(\theta) & \mathbf{0} & D_0(\theta) & \mathbf{0} & -\mathbf{1} \end{array}\right] \begin{bmatrix} \hat{x}_q \\ \hline \hat{w}_q \\ \hline \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{13}$$

where $\hat{x}_q^T = [x_{2q+2}^T, x_{2q+1}^T, x_{2q}^T]$ and $\sigma x_{2q} = x_{2(q+1)} = x_{2q+2}$ as previously defined in the notations paragraph of the introduction. Even though the polytopic structure of the original periodic uncertain model is preserved, the unconventional structure of (13) makes its analysis tricky. In the following, this new reformulation is coined as *descriptor lifting* while the procedure leading to (11) is referred to as monodromy lifting since the so-called monodromy matrix appears explicitly in the LTI representation (matrix $A_1(\theta)A_0(\theta)$ in (11)).

The strategy proposed in this paper can then be regarded as a way to take advantage of both time-invariant reformulations in order to derive tractable analysis conditions for the periodic model with memory $\Sigma_{cl}(\theta)$. The strategy is the following:

1) Obtain an equivalent state-space formulation of $\Sigma_{cl}(\theta)$ via the monodromy lifting.
2) Establish a first set of robust analysis conditions.

[2]The convention is such that $\hat{w}_q$ stacks w_q's from the bottom to the top when going forward in time along the period although the opposite way is sometimes used in the literature.

$$\Psi = \Phi_{N,0} \quad , \quad \mathfrak{B} = \begin{bmatrix} \bar{B}_{N-1} & \Phi_{N,N-1}\bar{B}_{N-2} & \cdots & \Phi_{N,1}\bar{B}_0 \end{bmatrix}$$
$$\mathfrak{C} = \begin{bmatrix} \bar{C}_{N-1}\Phi_{N-1,0} \\ \vdots \\ \bar{C}_1\Phi_{1,0} \\ \bar{C}_0 \end{bmatrix} \quad , \quad \mathfrak{D} = \begin{bmatrix} \bar{D}_{N-1} & \bar{C}_{N-1}\bar{B}_{N-2} & \cdots & \bar{C}_{N-1}\Phi_{N-1,2}\bar{B}_1 & \bar{C}_{N-1}\Phi_{N-1,1}\bar{B}_0 \\ \mathbf{0} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \bar{C}_2\bar{B}_1 & \bar{C}_2\Phi_{2,1}\bar{B}_0 \\ \vdots & & \ddots & \bar{D}_1 & \bar{C}_1\bar{B}_0 \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \bar{D}_0 \end{bmatrix} \tag{16}$$

3) Derive the new descriptor lifting and exhibit the linear map existing between η_q and $\hat{x}_q$ which, for the previous 2-periodic example, corresponds to

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \hat{x}_q. \tag{14}$$

4) Using this correspondence, derive tractable LMI conditions of robust analysis for $\Sigma_{cl}(\theta)$.

The approach, outlined above, is now detailed for the solution of the robust stability analysis problem in 1 while the case of $\mathcal{H}_2$ performance will be dealt with more concisely.

B. Robust analysis using monodromy lifting

By recasting $\Sigma_{cl}(\theta)$ as a memoryless periodic model, the monodromy lifting procedure recalled in [3] can be readily applied to arrive at the following proposition.

Proposition III.1 (Monodromy lifting)**.** *The polytopic periodic model $\Sigma_{cl}(\theta)$ given by* (8) *can always be lifted to the following time-invariant formulation, denoted by $\Gamma_m(\theta)$:*

$$\Gamma_m(\theta) : \begin{bmatrix} \eta_{q+1} \\ \hat{z}_q \end{bmatrix} = \begin{bmatrix} \Psi(\theta) & \mathfrak{B}(\theta) \\ \mathfrak{C}(\theta) & \mathfrak{D}(\theta) \end{bmatrix} \begin{bmatrix} \eta_q \\ \hat{w}_q \end{bmatrix} \text{ with } \eta_q \in \mathbb{R}^{nl} \tag{15}$$

where Ψ, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ are polynomial functions of θ and are defined by (16) *where the transition matrix $\Phi_{k,j}$ associated to $\bar{A}$ is such that*

$$\Phi_{k,j} = \begin{cases} \mathbf{1} & ,(k=j) \\ \bar{A}_{k-1}\bar{A}_{k-2}\cdots\bar{A}_j & ,(k>j) \end{cases} \tag{17}$$

and

$$\bar{A}_k = \begin{cases} \begin{bmatrix} \grave{A}_k \\ \mathbf{1}_{n(l+k)} \end{bmatrix}, & (0 \le k \le N-2) \\ \begin{bmatrix} \grave{A}_{N-1} \\ [\mathbf{1}_{n(l-1)} \;\; \mathbf{0}] \end{bmatrix}, & (k = N-1) \end{cases} \tag{18}$$

$$\begin{aligned} \grave{A}_k &= \begin{bmatrix} A_{k,0} & \cdots & A_{k,l+k-1} \end{bmatrix}, \; \bar{B}_k = \begin{bmatrix} B_k \\ \mathbf{0} \end{bmatrix} \\ \bar{C}_k &= \begin{bmatrix} C_{k,0} & \cdots & C_{k,l+k-1} \end{bmatrix}, \; \bar{D}_k = D_k \end{aligned} \tag{19}$$

Proof of proposition III.1 is provided in Appendix A.

Remark. Ψ may be regarded as the generalization of the monodromy matrix for periodic models with memory Σ_{cl}.

The time-invariant structure of $\Gamma_m(\theta)$ allows to readily apply well-known stability and worst case $\mathcal{H}_2$ theorems to analyze the periodic model $\Sigma_{cl}(\theta)$.

Theorem III.1 (Stability using Γ_m)**.** *The polytopic periodic model $\Sigma_{cl}(\theta)$ is robustly stable* if and only if *the following condition holds, $\forall\theta \in \Theta$:*

$$\begin{aligned} \exists P(\theta) \in \mathbb{S}_+^{nl} : \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix}^T \begin{bmatrix} P(\theta) & \mathbf{0} \\ \mathbf{0} & -P(\theta) \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} < 0 \\ \text{s.t. } \begin{bmatrix} -\mathbf{1} & \Psi(\theta) \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = \mathbf{0} \end{aligned} \tag{20}$$

Theorem III.2 (Worst case $\mathcal{H}_2$ cost using Γ_m)**.** *The worst case $\mathcal{H}_2$ analysis problem is* equivalent *to the following optimization problem:*

$$\gamma_{\mathrm{wc}} = \max_{\theta\in\Theta} \min_{P(\theta)\in\mathbb{S}_+^{nl},\; Z(\theta)\in\mathbb{S}^{mN}} \gamma \quad \text{s.t.} \tag{21}$$

$$\Psi^T(\theta)P(\theta)\Psi(\theta) - P(\theta) + \mathfrak{C}^T(\theta)\mathfrak{C}(\theta) \prec 0 \tag{22}$$

$$\mathfrak{B}^T(\theta)P(\theta)\mathfrak{B}(\theta) + \mathfrak{D}^T(\theta)\mathfrak{D}(\theta) - Z(\theta) \prec 0 \tag{23}$$

$$\mathrm{trace}(Z(\theta)) < \gamma \tag{24}$$

Unfortunately, these results are not useful to derive tractable LMI conditions of robust stability and worst case $\mathcal{H}_2$ analysis due to the non polytopic dependence of the matrices of Γ_m. It is therefore necessary to resort to a different lifted LTI model of $\Sigma_{cl}(\theta)$.

C. The new descriptor lifting

As the proof of Prop III.1 makes it clear, the polytopic nature of Σ_{cl} is destroyed by the monodromy lifting procedure when dropping some the states of Σ_{cl}. For this reason, we propose a new time-lifting procedure which mainly relies on the concatenated extended state vector $\hat{x}_q$. The origins of this new lifting can be traced back to [17]. In this paper, it is formally expressed for the first time for periodic models with memory Σ_{cl}. The relationships between Γ_m and the descriptor lifted model are also clarified.

Proposition III.2 (Descriptor lifting)**.** *$\Gamma_e(\theta)$, given by* (25)*, defines a polytopic lifted time-invariant formulation of the polytopic uncertain periodic model $\Sigma_{cl}(\theta)$ described by* (8).

$$\Gamma_e(\theta) : \begin{bmatrix} \begin{bmatrix} \mathcal{E}(\theta) \; \mathcal{A}(\theta) \end{bmatrix} & \mathcal{B}(\theta) & \mathbf{0} \\ \begin{bmatrix} \mathbf{1}_{nl} \; \mathbf{0} \end{bmatrix} - \sigma \begin{bmatrix} \mathbf{0} \; \mathbf{1}_{nl} \end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{bmatrix} \mathcal{C}_1(\theta) \; \mathcal{C}_2(\theta) \end{bmatrix} & \mathcal{D}(\theta) & -\mathbf{1}_{pN} \end{bmatrix} \begin{bmatrix} \hat{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{25}$$

where $\mathcal{E} \in \mathbb{R}^{nN\times nN}$, $\mathcal{A} \in \mathbb{R}^{nN\times nl}$, $\mathcal{B} \in \mathbb{R}^{nN\times mN}$,

$\mathcal{C}_1 \in \mathbb{R}^{pN\times nN}$, $\mathcal{C}_2 \in \mathbb{R}^{pN\times nl}$, $\mathcal{D} \in \mathbb{R}^{pN\times mN}$ *are*

$$\mathcal{E} = \begin{bmatrix} A_{N-1,N-1} & A_{N-1,N} & \cdots & A_{N-1,l+N-2} \\ A_{N-2,N-2} & A_{N-2,N-1} & \cdots & A_{N-2,l+N-3} \\ \vdots & & & \vdots \\ A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ A_{0,0} & A_{0,1} & \cdots & A_{0,l-1} \end{bmatrix}$$

$$\mathcal{A} = \begin{bmatrix} -\boldsymbol{I}_n & A_{N-1,0} & A_{N-1,1} & \cdots & A_{N-1,N-2} \\ \boldsymbol{0}_n & -\boldsymbol{I}_n & A_{N-2,0} & \cdots & A_{N-2,N-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\boldsymbol{I}_n & A_{1,0} \\ \boldsymbol{0}_n & \cdots & \cdots & \boldsymbol{0}_n & -\boldsymbol{I}_n \end{bmatrix} \tag{26}$$

$$\mathcal{C}_1 = \begin{bmatrix} \boldsymbol{0}_{p\times n} & C_{N-1,0} & \cdots & \cdots & C_{N-1,N-2} \\ \vdots & \ddots & C_{N-2,0} & \cdots & C_{N-2,N-3} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & C_{1,0} \\ \boldsymbol{0}_{p\times n} & \cdots & \cdots & \cdots & \boldsymbol{0}_{p\times n} \end{bmatrix} \tag{27}$$

$$\mathcal{C}_2 = \begin{bmatrix} C_{N-1,N-1} & C_{N-1,N} & \cdots & C_{N-1,l+N-2} \\ C_{N-2,N-2} & C_{N-2,N-1} & \cdots & C_{N-2,l+N-3} \\ \vdots & & & \vdots \\ C_{1,1} & C_{1,2} & \cdots & C_{1,l} \\ C_{0,0} & C_{0,1} & \cdots & C_{0,l-1} \end{bmatrix}$$

$$\begin{array}{c} \mathcal{B} = \mathrm{diag}\{B_{N-1}, \cdots, B_0\} \\ \mathcal{D} = \mathrm{diag}\{D_{N-1}, \cdots, D_0\}. \end{array} \tag{28}$$

By construction, both state vectors η_q and $\hat{x}_q$ corresponding respectively to Γ_m and Γ_e are composed of some of the internal variables involved in the dynamical equation of Σ_{cl} such that

$$\eta_q^T = [x_{Nq}^T, \cdots, x_{Nq-l+1}^T],\ \hat{x}_q^T = [x_{Nq+N}^T, \cdots, x_{Nq-l+1}^T]. \tag{29}$$

Relying on this observation, the following constant linear mapping between η_q and $\hat{x}_q$ may be easily deduced.

$$\begin{bmatrix} \eta_{q+1} \\ \eta_q \end{bmatrix} = T\hat{x}_q \text{ with } T = \begin{bmatrix} \begin{bmatrix} \boldsymbol{1}_{nl} & \boldsymbol{0}_{nl\times nN} \end{bmatrix} \\ \begin{bmatrix} \boldsymbol{0}_{nl\times nN} & \boldsymbol{1}_{nl} \end{bmatrix} \end{bmatrix} \in \mathbb{R}^{2nl\times n(N+l)}. \tag{30}$$

Remark. From the dimensions of T, it appears that the monodromy representation Γ_m derives from the descriptor representation Γ_e by contraction when $l < N$ and by expansion when $l > N$.

This transformation allows to derive new tractable robust analysis conditions for problem 1.

D. New robust analysis conditions

For clarity reasons, the robust stability case is treated first by carefully following the strategy given by subsection III-A. Then, as the guaranteed cost problem for $\mathcal{H}_2$ worst-case analysis can be tackled along the same lines, only the final formulation of the LMI conditions will be given for conciseness reasons.

1) Robust stability conditions: A new robust stability analysis condition for the lifted descriptor system Γ_e (and therefore for Σ_{cl}) may be easily obtained by applying the constant linear mapping defined by (30) to the condition (20).

Theorem III.3 (Robust stability via Γ_e)**.** *The polytopic uncertain periodic model $\Sigma_{cl}(\theta)$ is robustly stable* if and only if *the following condition holds, $\forall\ \theta \in \Theta$:*

$$\begin{array}{c} \exists P(\theta) \in \mathbb{S}_+^{nl}, \exists \mathcal{F}(\theta) \in \mathbb{R}^{n(N+l)\times nN} : \\ -\mathcal{P}(P(\theta)) + \mathrm{He}\{\mathcal{F}(\theta) \begin{bmatrix} \mathcal{E}(\theta) & \mathcal{A}(\theta) \end{bmatrix}\} \prec 0 \end{array} \tag{31}$$

where the operator $\mathcal{P}(P) \in \mathbb{R}^{n(l+N)}$ is defined as follows:

$$\mathcal{P}(P) = \begin{bmatrix} -P & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0}_{nN} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0}_{nN} & \boldsymbol{0} \\ \boldsymbol{0} & P \end{bmatrix} \tag{32}$$

Proof: Since T is a full rank matrix, the use of (30) allows to rewrite the condition (20) as

$$\hat{x}_q^T T^T \begin{bmatrix} P(\theta) & \mathbf{0} \\ \mathbf{0} & -P(\theta) \end{bmatrix} T\hat{x}_q < 0 \text{ s.t. } \begin{bmatrix} \mathcal{E}(\theta)\, \mathcal{A}(\theta) \end{bmatrix} \hat{x}_q = \mathbf{0} \tag{33}$$

Noticing that $\mathcal{P}(P) = T^T \mathrm{diag}\{-P, P\} T$ and applying elimination Lemma [20] leads to (31) without conservatism. ■

For $N = 1$ and $\alpha_0 = 0$, these robust stability analysis conditions reduce to the well-known extended LMIs for LTI system analysis [15], [16]. Therefore, the same strategy can be employed to derive tractable LMI robust analysis conditions: Condition (31) may be relaxed by enforcing $\mathcal{F}(\theta)$ to be independent of the uncertain vector θ and choosing a polytopic parameter-dependent Lyapunov function $P(\theta) = \sum_{i=1}^{L} \theta_i P_i$. Finally, applying the usual vertexization trick, the following sufficient LMI robust stability conditions may be expressed.

Theorem III.4 (Robust Stability Analysis)**.** *The polytopic periodic model $\Sigma_{cl}(\theta)$ is robustly stable* if *there exist L matrices $P^{[i]} \in \mathbb{S}_+^{nl}$ and a matrix $\mathcal{F} \in \mathbb{R}^{n(N+l)\times nN}$ such that*

$$\forall i \in [1, \cdots, L],\ -\mathcal{P}(P^{[i]}) + \mathrm{He}\{\mathcal{F} \begin{bmatrix} \mathcal{E}^{[i]} & \mathcal{A}^{[i]} \end{bmatrix}\} \prec 0 \tag{34}$$

where the operator $\mathcal{P}(P^{[i]})$ is defined by (32)*.*

Proof: Summing the conditions as follows $\sum_{i=1}^{L} \theta_i(34)$ implies (31) with $P(\theta) = \sum_{i=1}^{L} \theta_i P^{[i]}$, $\mathcal{E}(\theta) = \sum_{i=1}^{L} \theta_i \mathcal{E}^{[i]}$, $\mathcal{A}(\theta) = \sum_{i=1}^{L} \theta_i \mathcal{A}^{[i]}$. ■

The condition (34) is exactly the dual condition of the one proposed in [10]. When coming to the synthesis problem 2 in the sequel, this relation of duality will be explicitly presented and generalized to the descriptor representation.

2) Guaranteed cost for worst-case $\mathcal{H}_2$ analysis: Following roughly the same line, the following theorem can be stated.

Theorem III.5 (Guaranteed cost for worst-case $\mathcal{H}_2$ Analysis)**.** *The solution γ_{g} of the following semidefinite program is a guaranteed cost of the worst case $\mathcal{H}_2$ cost, i.e. $\gamma_{\mathrm{wc}} \leq \gamma_{\mathrm{g}}$.*

$$\gamma_{\mathrm{g}} = \min_{P^{[i]},\ \mathcal{F}_1,\ \mathcal{F}_2,\ Z^{[i]}} \gamma \tag{35}$$

$$\Gamma_e^d(\theta): \quad \left[\begin{array}{cc|c} \begin{bmatrix}\breve{\mathcal{A}}^T(\theta) & \breve{\mathcal{E}}^T(\theta)\end{bmatrix} & \begin{bmatrix}\breve{\mathcal{C}}_2^T(\theta) & \breve{\mathcal{C}}_1^T(\theta)\end{bmatrix} & \mathbf{0} \\ \sigma^{-1}\begin{bmatrix}\mathbf{1}_{nl} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{1}_{nl}\end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^{-1}\begin{bmatrix}\mathbf{1}_{p(l-1)} & \mathbf{0}\end{bmatrix} - \begin{bmatrix}\mathbf{0} & \mathbf{1}_{p(l-1)}\end{bmatrix} & \mathbf{0} \\ \hline \begin{bmatrix}\mathbf{0} & \breve{\mathcal{B}}^T(\theta)\end{bmatrix} & \begin{bmatrix}\mathbf{0} & \breve{\mathcal{D}}^T(\theta)\end{bmatrix} & \mathbf{1}_{mN} \end{array}\right] \left[\begin{array}{c} \begin{bmatrix}\hat{x}_q^d \\ \breve{w}_q^d\end{bmatrix} \\ \hline \breve{z}_q^d \end{array}\right] = \mathbf{0} \tag{40}$$

such that, for $i \in [1, \cdots, L]$,

$$\begin{aligned} -\mathcal{P}(P^{[i]}) + \mathrm{Sq}\left\{\begin{bmatrix}\mathcal{C}_1^{[i]} & \mathcal{C}_2^{[i]}\end{bmatrix}^T\right\} \\ +\mathrm{He}\left\{\mathcal{F}_1\begin{bmatrix}\mathcal{E}^{[i]} & \mathcal{A}^{[i]}\end{bmatrix}\right\} \prec 0 \end{aligned} \tag{36}$$

$$\begin{aligned} \mathcal{Z}(P^{[i]}, Z^{[i]}) + \mathrm{Sq}\left\{\begin{bmatrix}(\mathcal{C}_1^{[i]})^T \\ (\mathcal{D}^{[i]})^T\end{bmatrix}\right\} \\ +\mathrm{He}\left\{\mathcal{F}_2\begin{bmatrix}\mathcal{E}^{[i]} & \mathcal{B}^{[i]}\end{bmatrix}\right\} \prec 0 \end{aligned} \tag{37}$$

$$\mathrm{trace}(Z^{[i]}) < \gamma \tag{38}$$

where $\mathcal{Z}(P, Z) \in \mathbb{S}^{(n+m)N}$ is given by

$$\mathcal{Z}(P, Z) = \begin{bmatrix} \begin{bmatrix}\mathbf{1}_{nN} \\ \mathbf{0}_{nl\times nN}\end{bmatrix}^T \begin{bmatrix}P & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{nN}\end{bmatrix} \begin{bmatrix}\mathbf{1}_{nN} \\ \mathbf{0}_{nl\times nN}\end{bmatrix} & \mathbf{0} \\ \mathbf{0} & -Z \end{bmatrix} \tag{39}$$

and $P^{[i]} \in \mathbb{S}_+^{nl}$, $\mathcal{F}_1 \in \mathbb{R}^{n(N+l)\times nN}$, $\mathcal{F}_2$, $Z^{[i]} \in \mathbb{S}^{mN}$

A detailed proof is given in appendix III.5.

As will be seen in the next section for synthesis purpose, a dual SDP of the program presented in theorem III.5 has to be derived. In the classical LTI case, robust state-feedback controller synthesis with reduced conservatism is one of the striking achievement of the results arising from the extended LMI formulation [5], [6]. In order to make analysis conditions applicable for such a purpose, it is known that system duality is a key issue and that the structure of additional variables contained in $\mathcal{F}$ or $\mathcal{F}_1$ and $\mathcal{F}_2$ has to be restricted somehow. In the context of this paper, one has to face two issues:

- how to construct the dual model Σ_{cl}^d of the periodic model with memory Σ_{cl} ?
- how to define the additional matrix variables to ensure that the designed control laws have the desired structure ?

The next subsections deal with the former question while the next section is dedicated to an extended discussion providing a way to tackle the latter issue and to the derivation of new synthesis conditions.

E. System duality for memory periodic models

As system duality is in general well-known in the time-invariant framework, this subsection first presents a dual version of Γ_e, denoted by Γ_e^d, as an intermediate step. To this end, $l \geq 1$, given by (6), is decomposed as $l = bN + r$ such that $1 \leq r \leq N$ and $b \in \mathbb{N}$.

Proposition III.3 (Dual of Γ_e)**.** *A dual model of $\Gamma_e(\theta)$, given by* (25)*, is defined by* (40) *with $\hat{x}_q^d \in \mathbb{R}^{n(N+l)+p(l-1)}$, $\breve{w}_q^d \in \mathbb{R}^{pN}$ and $\breve{z}_q^d \in \mathbb{R}^{mN}$. All matrices involved in* (40) *are polytopic functions of θ with $\breve{\mathcal{B}} = \begin{bmatrix}\mathcal{V}(B) \\ \mathbf{0}_{n\times mN}\end{bmatrix}$ and $\breve{\mathcal{D}} = \mathcal{V}(D)$, referring to the operator $\mathcal{V}(\cdot)$, defined by*

$$\mathcal{V}(X) = \left\{\begin{array}{ll} \mathrm{diag}\{X_{N-2}, \cdots, X_0\} & (r = 1) \\ \mathrm{diag}\{X_{N-r-1}, \cdots, X_0, X_{N-1}, \cdots, X_{N-r+1}\} & (r > 1) \end{array}\right. \tag{41}$$

In addition, $\breve{\mathcal{A}} \in \mathbb{R}^{nl\times nN}$, $\breve{\mathcal{E}} \in \mathbb{R}^{nN\times nN}$, $\breve{\mathcal{C}}_2 \in \mathbb{R}^{p(l-1)\times nN}$ and $\breve{\mathcal{C}}_1 \in \mathbb{R}^{pN\times nN}$ are defined by

$$\begin{aligned} \begin{bmatrix}\breve{\mathcal{A}}^T & \breve{\mathcal{E}}^T\end{bmatrix} &= \begin{bmatrix}\mathscr{A}_{b+1}^T & \cdots & \mathscr{A}_1^T & \begin{bmatrix}\mathbf{0} \\ \mathscr{A}_{0,0}^T\end{bmatrix}\end{bmatrix} \\ \begin{bmatrix}\breve{\mathcal{C}}_2^T & \breve{\mathcal{C}}_1^T\end{bmatrix} &= \begin{bmatrix}\mathscr{C}_{b+1}^T & \cdots & \mathscr{C}_1^T & \begin{bmatrix}\mathbf{0} \\ \mathscr{C}_{0,0}^T\end{bmatrix}\end{bmatrix} \end{aligned} \tag{42}$$

with

$$\left[\begin{array}{c} \mathscr{A}_0 \\ \hline \mathscr{C}_0 \end{array}\right] = \left[\begin{array}{cc} \mathbf{0} & \mathscr{A}_{0,0} \\ \mathbf{0}_{n(N-r)} & \mathbf{0} \\ \hline \mathbf{0} & \mathscr{C}_{0,0} \\ \mathbf{0}_{p(N-r+1)\times n(N-r+1)} & \mathbf{0} \end{array}\right] \tag{43}$$

$$\mathscr{A}_{0,0} = \begin{bmatrix} -\mathbf{1}_n & A_{N-1,0} & \cdots & A_{N-1,r-2} \\ \mathbf{0}_n & -\mathbf{1}_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{N-r+1,0} \\ \mathbf{0}_n & \cdots & \mathbf{0}_n & -\mathbf{1}_n \end{bmatrix} \tag{44}$$

$$\mathscr{C}_{0,0} = \begin{bmatrix} C_{N-1,0} & \cdots & \cdots & C_{N-1,r-2} \\ \mathbf{0}_{p\times n} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{p\times n} & \cdots & \mathbf{0}_{p\times n} & C_{N-r+1,0} \end{bmatrix} \tag{45}$$

$$\mathscr{A}_1 = \begin{bmatrix} A_{N-1,r-1} \cdots & A_{N-1,N-2} & A_{N-1,N-1} \cdots & A_{N-1,N+r-2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N-r,0} & \vdots & \vdots & \vdots \\ -\mathbf{1}_n \quad \ddots & \vdots & \vdots & \vdots \\ \ddots & A_{1,0} & \vdots & \vdots \\ \mathbf{0} & -\mathbf{1}_n & A_{0,0} & \cdots \quad A_{0,r-1} \end{bmatrix} \tag{46}$$

$$\mathscr{C}_1 = \begin{bmatrix} C_{N-1,r-1} \cdots & C_{N-1,N-2} & C_{N-1,N-1} \cdots & C_{N-1,N+r-2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{N-r,0} & \vdots & \vdots & \vdots \\ \mathbf{0}_{p\times n} \quad \ddots & \vdots & \vdots & \vdots \\ \vdots \quad \ddots & C_{1,0} & \vdots & \vdots \\ \mathbf{0}_{p\times n} \quad \cdots & \mathbf{0}_{p\times n} & C_{0,0} & \cdots \quad C_{0,r-1} \end{bmatrix} \tag{47}$$

and with $(2 \leq j \leq b+1)$

$$
\begin{aligned}
\mathscr{A}_j &= \begin{bmatrix} A_{N-1,(j-1)N+r-1} & \cdots & A_{N-1,jN+r-2} \\ \vdots & & \vdots \\ A_{0,(j-2)N+r} & \cdots & A_{0,(j-1)N+r-1} \end{bmatrix} \\
\mathscr{C}_j &= \begin{bmatrix} C_{N-1,(j-1)N+r-1} & \cdots & C_{N-1,jN+r-2} \\ \vdots & & \vdots \\ C_{0,(j-2)N+r} & \cdots & C_{0,(j-1)N+r-1} \end{bmatrix}
\end{aligned} \tag{48}
$$

The proof is given in the appendix C.

By reversing the lifting procedure, it is possible to build a periodic model with memory, Σ_{cl}^d, from Γ_e^d . Thus, Γ_e^d can be considered as the descriptor lifted model of Σ_{cl}^d which itself can be regarded as the dual of Σ_{cl}. Decomposing signals of Γ_e^d as

$$
\begin{aligned}
\hat{x}_q^d &= \begin{bmatrix} x^d_{Nq+l-r-1} \\ \cdots \\ x^d_{N(q-1)-r} \\ w^d_{Nq+l-r-1} \\ \vdots \\ w^d_{Nq-r+1} \end{bmatrix}, \\
\breve{w}_q^d &= \begin{bmatrix} w^d_{Nq-r} \\ \vdots \\ w^d_{N(q-1)-r+1} \end{bmatrix}, \; \breve{z}_q^d = \begin{bmatrix} z^d_{Nq-r} \\ \vdots \\ z^d_{N(q-1)-r+1} \end{bmatrix}
\end{aligned} \tag{49}
$$

we get the next theorem.

Theorem III.6. *A dual model of* Σ_{cl}*, given by* (8)*, is described by*

$$
\Sigma_{cl}^d : \begin{cases} x^d_{Nq-k-r-1} = \displaystyle\sum_{j=0}^{l-1+k} \Big(A^T_{\vartheta(j,k+r),j} x^d_{Nq-k-r+j} \\ \qquad\qquad + C^T_{\vartheta(j,k+r),j} w^d_{Nq-k-r+j} \Big) \\ z^d_{Nq-k-r} = -B^T_{\vartheta(0,k+r)} x^d_{Nq-k-r} \\ \qquad\qquad -D^T_{\vartheta(0,k+r)} w^d_{Nq-k-r} \end{cases} \tag{50}
$$

for $0 \leq k \leq N-1$ *and with* $\vartheta(j,k) = j-k \ (\mathrm{mod} N)$.

To our best knowledge, this result extending system duality for periodic systems with memory is new.

F. Robust analysis condition for the dual model

Derivation of LMI tractable analysis conditions for Σ_{cl}^d can be carried out following roughly the same lines as previously. Yet, two important differences have to be pointed out: First, time goes backward for Σ_{cl}^d. Second, dynamic equation of the model Σ_{cl}^d involves past inputs. The implications of such differences are brought to light in the proof given in the appendix D.

The dual set of robust stability and worst case $\mathcal{H}_2$ analysis conditions follows.

Theorem III.7 (Stability Analysis - Dual)**.** *The polytopic periodic model* $\Sigma_{cl}(\theta)$ *is robustly stable* if *there exist* L *matrices* $X^{[i]} \in \mathbb{S}_+^{nl}$ *and a matrix* $\mathcal{F} \in \mathbb{R}^{nN \times n(N+l)}$ *such that*

$$
\forall i \in [1, \cdots, L], \quad \mathcal{P}(X^{[i]}) + \mathrm{He}\left\{ \begin{bmatrix} \breve{\mathcal{A}}^{[i]} \\ \breve{\mathcal{E}}^{[i]} \end{bmatrix} \mathcal{F} \right\} \prec 0 \tag{51}
$$

where the operator $\mathcal{P}(X)$ *is defined by* (32).

Theorem III.8 (Guaranteed cost for worst-case $\mathcal{H}_2$ Analysis - Dual)**.** *The solution* γ_{g}^d *of the following semidefinite program is a guaranteed cost of the worst case* $\mathcal{H}_2$ *cost, i.e.* $\gamma_{\mathrm{wc}} \leq \gamma_{\mathrm{g}}^d$.

$$
\gamma_{\mathrm{g}}^d = \min_{X^{[i]},\ \mathcal{F}_1,\ \mathcal{F}_2,\ Z^{[i]}} \gamma \tag{52}
$$

such that, for $i \in [1, \cdots, L]$,

$$
\begin{bmatrix} \mathbf{1}_{n(N+l)+p(l-1)} \\ \mathbf{0} \end{bmatrix}^T \mathcal{P}^d(X^{[i]}, \mathbf{0}) \begin{bmatrix} \mathbf{1}_{n(N+l)+p(l-1)} \\ \mathbf{0} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} \mathbf{0}_{n(l-1)\times mN} \\ \breve{\mathcal{B}}^{[i]} \\ \mathbf{0}_{p(l-1)\times mN} \end{bmatrix} \right\} + \mathrm{He}\left\{ \begin{bmatrix} \breve{\mathcal{A}}^{[i]} \\ \breve{\mathcal{E}}^{[i]} \\ \breve{\mathcal{C}}_2^{[i]} \end{bmatrix} \mathcal{F}_1 \right\} \prec 0 \tag{53}
$$

$$
\mathcal{Z}^d(X^{[i]}, Z^{[i]}) + \mathrm{Sq}\left\{ \begin{bmatrix} \begin{bmatrix} \mathbf{0}_{nN\times n} & \mathbf{1}_{nN} \end{bmatrix} \breve{\mathcal{B}}^{[i]} \\ \breve{\mathcal{D}}^{[i]} \end{bmatrix} \right\} + \mathrm{He}\left\{ \begin{bmatrix} \breve{\mathcal{E}}^{[i]} \\ \breve{\mathcal{C}}_1^{[i]} \end{bmatrix} \mathcal{F}_2 \right\} \prec 0 \tag{54}
$$

$$
\mathrm{trace}(Z^{[i]}) < \gamma \tag{55}
$$

where operators $\mathcal{P}^d(X,Z)$ *and* $\mathcal{Z}^d(X,Z)$ *are defined as follows for* $X \in \mathbb{S}^{nl+p(l-1)}$ *and* $Z \in \mathbb{S}^{pN}$

$$
\mathcal{P}^d(X,Z) = (T^d)^T \begin{bmatrix} X & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -Z \end{bmatrix} T^d \tag{56}
$$

$$
\begin{aligned}
\mathcal{Z}^d(X,Z) = &\begin{bmatrix} \mathbf{0}_{nl\times nN} & \mathbf{0} \\ \mathbf{1}_{nN} & \mathbf{0} \\ \mathbf{0}_{p(l-1)\times nN} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{pN} \end{bmatrix}^T (T^d)^T \\
&\begin{bmatrix} \mathbf{0}_{nl+p(l-1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -Z \end{bmatrix} T^d \begin{bmatrix} \mathbf{0}_{nl\times nN} & \mathbf{0} \\ \mathbf{1}_{nN} & \mathbf{0} \\ \mathbf{0}_{p(l-1)\times nN} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{pN} \end{bmatrix}
\end{aligned} \tag{57}
$$

$$
T^d = \begin{bmatrix} \begin{bmatrix} \mathbf{1}_{nl} & \mathbf{0}_{nl\times nN} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \mathbf{1}_{p(l-1)} & \mathbf{0}_{p(l-1)\times pN} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0}_{nl\times nN} & \mathbf{1}_{nl} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} \mathbf{0}_{p(l-1)\times pN} & \mathbf{1}_{p(l-1)} \end{bmatrix} \\ \mathbf{0} & \begin{bmatrix} \mathbf{0}_{pN\times p(l-1)} & \mathbf{1}_{pN} \end{bmatrix} \end{bmatrix} \tag{58}
$$

Remark. γ_g and γ_g^d are guaranteed costs for the worst-case $\mathcal{H}_2$ analysis problem, obtained by solving different SDP relaxations of the original NP-hard problem. In the general case, these primal and dual stability guaranteed costs are not equal their relative ordering is not known *a priori*. A systematic quantitative evaluation of the quality of these upper bounds, relying on exactness verification, is possible via SDP duality [10] as shown in the papers [18] and [9].

IV. Robust synthesis

A. New robust synthesis conditions

To move on to the problem of memory controller synthesis, open-loop matrices of Σ_{cl} are reintroduced in the analysis conditions using (9). To this end, the following decomposition is introduced

$$\begin{bmatrix} \breve{\mathcal{A}} \\ \breve{\mathcal{E}} \\ \hline \breve{\mathcal{C}}_2 \\ \breve{\mathcal{C}}_1 \end{bmatrix} = \begin{bmatrix} \breve{\mathcal{A}}_{op} \\ \breve{\mathcal{E}}_{op} \\ \hline \mathbf{0}_{p(l-1)\times nN} \\ \breve{\mathcal{C}}_{1op} \end{bmatrix} + \begin{bmatrix} \breve{\mathcal{Q}}_{2op} \\ \breve{\mathcal{Q}}_{1op} \\ \hline \breve{\mathcal{R}}_{2op} \\ \breve{\mathcal{R}}_{1op} \end{bmatrix} \breve{\mathcal{K}} \tag{59}$$

where $\breve{\mathcal{A}}_{op} \in \mathbb{R}^{nl\times nN}$, $\breve{\mathcal{E}}_{op} \in \mathbb{R}^{nN\times nN}$, $\breve{\mathcal{Q}}_{1op} \in \mathbb{R}^{nN\times m_u(N+l-1)}$, $\breve{\mathcal{Q}}_{2op} \in \mathbb{R}^{nl\times m_u(N+l-1)}$, $\breve{\mathcal{C}}_{1op} \in \mathbb{R}^{pN\times nN}$, $\breve{\mathcal{R}}_{1op} \in \mathbb{R}^{pN\times m_u(N+l-1)}$ and $\breve{\mathcal{R}}_{2op} \in \mathbb{R}^{p(l-1)\times m_u(N+l-1)}$ are defined by

$$\begin{aligned} \breve{\mathcal{A}}_{op} &= \begin{bmatrix} \mathbf{0} & \mathbf{0}_{n(l-1)\times n(N-1)} \\ A_{N-r} & \mathbf{0} \end{bmatrix} \\ \breve{\mathcal{E}}_{op} &= -\mathbf{1}_{nN} + \begin{bmatrix} \mathbf{0} & \mathcal{V}(A) \\ \mathbf{0}_n & \mathbf{0} \end{bmatrix} \\ \breve{\mathcal{C}}_{1op} &= \mathrm{diag}\{C_{N-r}, \mathcal{V}(C)\} \end{aligned} \tag{60}$$

$$\begin{aligned} \breve{\mathcal{R}}_{1op} &= \begin{bmatrix} \mathbf{0}_{pN\times m_u(l-1)} & \mathrm{diag}\{R_{N-r}, \mathcal{V}(R)\} \end{bmatrix} \\ \breve{\mathcal{Q}}_{1op} &= \begin{bmatrix} \mathbf{0} & \mathcal{V}(Q) \\ \mathbf{0}_{n\times m_u l} & \mathbf{0} \end{bmatrix} \end{aligned} \tag{61}$$

$$\begin{aligned} \breve{\mathcal{R}}_{2op} = [&\mathrm{diag}\{\mathbf{1}_b \otimes \mathrm{diag}\{R_{N-1}, \cdots, R_0\}, R_{N-1} \\ &, \cdots, R_{N-r+1}\}, \mathbf{0}_{p(l-1)\times m_u N}] \end{aligned} \tag{62}$$

$$\begin{aligned} \breve{\mathcal{Q}}_{2op} = [&\mathrm{diag}\{\mathbf{1}_b \otimes \mathrm{diag}\{Q_{N-1}, \cdots, Q_0\}, Q_{N-1} \\ &, \cdots, Q_{N-r}\}, \mathbf{0}_{nl\times m_u(N-1)}] \end{aligned} \tag{63}$$

where the operator $\mathcal{V}$ is defined by 41.
According to this definition, entries of $\breve{\mathcal{K}}$ are the gains $K_{k,j}$ of the periodic controller (7):

$$\breve{\mathcal{K}} = \begin{bmatrix} & \mathcal{K}_{b+1} \\ & \vdots \\ & \mathcal{K}_1 \\ \mathbf{0}_{m_u r\times n(N-r+1)} & \mathcal{K}_{0,0} \end{bmatrix} \tag{64}$$

with $\mathcal{K}_{0,0} \in \mathbb{R}^{m_u(r-1)\times n(r-1)}$ and, for $j = \{1, \cdots, b+1\}$, $\mathcal{K}_j \in \mathbb{R}^{m_u N\times nN}$ are described by

$$\begin{aligned} \mathcal{K}_{0,0} &= \begin{bmatrix} K_{N-1,0} & \cdots & \cdots & K_{N-1,r-2} \\ \mathbf{0}_{m_u\times n} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0}_{m_u\times n} & \cdots & \mathbf{0}_{m_u\times n} & K_{N-r+1,0} \end{bmatrix} \\ \mathcal{K}_j &= \begin{bmatrix} K_{N-1,(j-1)N+r-1} & \cdots & K_{N-1,jN+r-2} \\ \vdots & & \vdots \\ K_{0,(j-2)N+r} & \cdots & K_{0,(j-1)N+r-1} \end{bmatrix} \end{aligned} \tag{65}$$

$$\mathcal{K}_1 = \begin{bmatrix} K_{N-1,r-1} \cdots & K_{N-1,N-2} & K_{N-1,N-1} \cdots & K_{N-1,N+r-2} \\ \vdots & \vdots & \vdots & \vdots \\ K_{N-r,0} & \vdots & \vdots & \vdots \\ \mathbf{0}_{m_u\times n} \quad \ddots & \vdots & \vdots & \vdots \\ \vdots \quad \ddots & K_{1,0} & \vdots & \vdots \\ \mathbf{0}_{m_u\times n} \quad \cdots & \mathbf{0}_{m_u\times n} & K_{0,0} \quad \cdots & K_{0,r-1} \end{bmatrix}. \tag{66}$$

Introduction of these new expressions in the analysis conditions gives rise to bilinear terms destroying the convexity of the related optimization problem. The usual approach is to use the dual conditions on which an invertible linearizing changes of variable is performed after restricting the structure of $\mathcal{F}$ [10]. When dealing with the lifted model of a periodic system, special care is required when constraining the structure of $\mathcal{F}$. As noticed in [3], such a reformulation leads to a *particular* class of time-invariant system. As clearly shown by the form of $\breve{\mathcal{K}}$, the related synthesis problem amounts to solve a particular *structured* controller synthesis problem for which there is no general convex solution. Likewise what is proposed in the reference [5], for the special case of LTI discrete-time systems, it is always possible to impose a particular structure to the slack variables $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ while keeping some of their advantages in terms of conservatism. Considering, for instance, the robust stabilization problem, choosing the slack variable such that $\mathcal{F} = \begin{bmatrix} \mathbf{0}_{nN\times nl} & \mathcal{G} \end{bmatrix}$ allows to define the invertible change of variables $\breve{\mathcal{Y}} = \breve{\mathcal{K}}\mathcal{G}$. Indeed, if such an $\mathcal{F}$ satisfies (51) then $\mathcal{G} \in \mathbb{R}^{nN\times nN}$ is always invertible and $\breve{\mathcal{K}}$ is retrieved via $\breve{\mathcal{K}} = \breve{\mathcal{Y}}\mathcal{G}^{-1}$. In addition, it is important to stress that $\breve{\mathcal{Y}}\mathcal{G}^{-1}$ must comply with the structure of $\breve{\mathcal{K}}$. This issue is now discussed for the general case and for the special case of PFMCs via constraints imposed on the structure of $\mathcal{G}$.

1) General case: Remember that for any arbitrary sequence $\{\alpha_k\}_{k=0}^{N-1}$, the control law (3) may always be reformulated as a structured PFMC of sufficient order like (7). This particular structure is preserved if $\mathcal{G}$ is chosen to be a block-diagonal matrix as in [21].

$$\mathcal{G} = \mathrm{diag}\{G_{N-1}, \cdots, G_0\} \quad \text{with} \quad G_j \in \mathbb{R}^{nN\times nN} \tag{67}$$

In addition, imposing that $\breve{\mathcal{Y}}$ comply with the structure of $\breve{\mathcal{K}}$ allows to derive the following sufficient robust stabilization condition.

Theorem IV.1 (Robust stabilization)**.** *If there exist L matrices $X^{[i]} \in \mathbb{S}_+^{nl}$, a block-diagonal matrix $\mathcal{G} \in \mathbb{R}^{nN\times nN}$ and a matrix $\breve{\mathcal{Y}} \in \mathbb{R}^{m_u(N+l-1)\times nN}$ such that the following condition holds*

$$\begin{aligned} &\forall i \in [1, \cdots, L],\ \mathcal{P}(X^{[i]}) \\ &+\mathrm{He}\left\{ \left(\begin{bmatrix} \breve{\mathcal{A}}_{op}^{[i]} \\ \breve{\mathcal{E}}_{op}^{[i]} \end{bmatrix} \mathcal{G} + \begin{bmatrix} \breve{\mathcal{Q}}_{2op}^{[i]} \\ \breve{\mathcal{Q}}_{1op}^{[i]} \end{bmatrix} \breve{\mathcal{Y}} \right) \begin{bmatrix} \mathbf{0}_{nN\times nl} & \mathbf{1}_{nN} \end{bmatrix} \right\} \prec 0 \end{aligned} \tag{68}$$

then the controller (69) *robustly stabilizes the polytopic uncertain periodic model $\Sigma_{cl}(\theta)$.*

$$\breve{\mathcal{K}} = \breve{\mathcal{Y}}\mathcal{G}^{-1} \tag{69}$$

$$\begin{bmatrix}\mathbf{1}_{n(N+l)+p(l-1)}\\ \mathbf{0}\end{bmatrix}^T \mathcal{P}^d(X^{[i]},\mathbf{0}) \begin{bmatrix}\mathbf{1}_{n(N+l)+p(l-1)}\\ \mathbf{0}\end{bmatrix} + \mathrm{Sq}\left\{\begin{bmatrix}\mathbf{0}_{n(l-1)\times mN}\\ \breve{\mathcal{B}}^{[i]}\\ \mathbf{0}_{p(l-1)\times mN}\end{bmatrix}\right\} + \mathrm{He}\left\{\left(\begin{bmatrix}\breve{\mathcal{A}}^{[i]}_{op}\\ \breve{\mathcal{E}}^{[i]}_{op}\\ \mathbf{0}_{p(l-1)\times nN}\end{bmatrix}\mathcal{G} + \begin{bmatrix}\breve{\mathcal{Q}}^{[i]}_{2op}\\ \breve{\mathcal{Q}}^{[i]}_{1op}\\ \breve{\mathcal{R}}^{[i]}_{2op}\end{bmatrix}\breve{\mathcal{Y}}\right)\begin{bmatrix}\mathbf{0}_{nN\times nl} & \mathbf{1}_{nN} & \mathbf{0}_{nN\times p(l-1)}\end{bmatrix}\right\} \prec 0 \tag{71}$$

$$\mathcal{Z}^d(X^{[i]},Z^{[i]}) + \mathrm{Sq}\left\{\begin{bmatrix}\begin{bmatrix}\mathbf{0}_{nN\times n} & \mathbf{1}_{nN}\end{bmatrix}\breve{\mathcal{B}}^{[i]}\\ \breve{\mathcal{D}}^{[i]}\end{bmatrix}\right\} + \mathrm{He}\left\{\left(\begin{bmatrix}\breve{\mathcal{E}}^{[i]}_{op}\\ \breve{\mathcal{C}}^{[i]}_{1op}\end{bmatrix}\mathcal{G} + \begin{bmatrix}\breve{\mathcal{Q}}^{[i]}_{1op}\\ \breve{\mathcal{R}}^{[i]}_{1op}\end{bmatrix}\breve{\mathcal{Y}}\right)\begin{bmatrix}\mathbf{1}_{nN} & \mathbf{0}_{nN\times pN}\end{bmatrix}\right\} \prec 0 \tag{72}$$

Remark. Note that when adding degrees of freedom to the control law, the computational effort is increased. Indeed, any addition of a $K_{k,j}$ term modifies $\breve{\mathcal{K}}$ and therefore leads to a matrix $\breve{\mathcal{Y}}$ involving more variables.

Imposing $\mathcal{F}_1 = \begin{bmatrix}\mathbf{0}_{nN\times nl} & \mathcal{G} & \mathbf{0}_{nN\times p(l-1)}\end{bmatrix}$ and $\mathcal{F}_2 = \begin{bmatrix}\mathcal{G} & \mathbf{0}_{nN\times pN}\end{bmatrix}$ with $\mathcal{G}$ of the form given by (67) leads to the counterpart of Theorem IV.1 for guaranteed $\mathcal{H}_2$ synthesis of periodic state-feedback control laws with memory.

Theorem IV.2 (Guaranteed $\mathcal{H}_2$ synthesis)**.** *If the following semidefinite program has a solution:*

$$\gamma^s_{\mathrm{g}} = \min_{X^{[i]},\ \mathcal{G},\ \breve{\mathcal{Y}},\ Z^{[i]}} \gamma \tag{70}$$

such that, $\forall i \in [1,\cdots,L]$*,* (71), (72) *and*

$$\mathrm{trace}(Z^{[i]}) < \gamma \tag{73}$$

hold. Then the controller $\breve{\mathcal{K}} = \mathcal{Y}\mathcal{G}^{-1}$ robustly stabilizes the polytopic uncertain periodic model $\Sigma_{cl}(\theta)$ and γ^s_{g} is a guaranteed closed-loop cost such that $\gamma^s_{\mathrm{g}} \geq \gamma^d_{\mathrm{g}} \geq \gamma_{\mathrm{wc}}$.

Proof: Reconstruction of closed-loop matrices allows to recover the analysis condition with $\mathcal{F}_1 = \begin{bmatrix}\mathbf{0}_{nN\times nl} & \mathcal{G} & \mathbf{0}_{nN\times p(l-1)}\end{bmatrix}$ and $\mathcal{F}_2 = \begin{bmatrix}\mathcal{G} & \mathbf{0}_{nN\times pN}\end{bmatrix}$. This proves closed-loop stability and performance. In addition, restrictions on the structure of $\mathcal{F}_1$ and $\mathcal{F}_2$ imply that $\gamma^d_{\mathrm{g}} \leq \gamma^s_{\mathrm{g}}$. ■

Obviously, these synthesis results have been obtained at the cost of introducing some extra conservatism in the analysis conditions.

2) The special case of PFMCs: Consider now the case of a PFMC of order κ given by (7). $\breve{\mathcal{K}}$ described by (64) is now an upper trapezoidal matrix like $\breve{\mathcal{Y}}$ which inherits its structure from $\breve{\mathcal{K}}$. Note that every other controllers corresponding to structured PFMC of the *same* order κ gives rise to a more sparse $\breve{\mathcal{Y}}$ which might lead to more conservative synthesis conditions.

In addition, for (unstructured) PFMC, it is possible to multiply $\breve{\mathcal{K}}$ from the right by $\mathcal{G}$ being upper triangular while preserving the structure of $\breve{\mathcal{K}}$. This was first proposed in [10] where the special case of PFMC of order 0 was considered.

Theorem IV.3 (Robust synthesis via PFMC)**.** *In the case of PFMC controllers, synthesis theorems IV.1 and IV.2 hold with $\mathcal{G}$ upper triangular:*

$$\mathcal{G} = \begin{bmatrix} G_{N-1,0} & \cdots & \cdots & G_{N-1,N-1}\\ \mathbf{0}_n & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ \mathbf{0}_n & \cdots & \mathbf{0}_n & G_{0,0}\end{bmatrix} \in \mathbb{R}^{nN\times nN} \tag{74}$$

with $G_{k,j} \in \mathbb{R}^{nN\times nN}$. Furthermore, γ^{sm}_{g} is such that $\gamma^s_{\mathrm{g}} \geq \gamma^{sm}_{\mathrm{g}} \geq \gamma^d_{\mathrm{g}} \geq \gamma_{\mathrm{wc}}$.

This last observation suggests that PFMCs controllers benefit from a particular status as it is illustrated by these particularly appropriate synthesis conditions.

Remark. As noticed in [10], the restriction of $\mathcal{G}$ to a block-diagonal structure is reasonable as it could be regarded as a natural extension of the LTI-case results [5], [6]. However, in our case with memory and even in the nominal case, the resulting LMI conditions are only sufficient conditions. A major achievement of [10] was to prove that, for the nominal condition in the particular case of PFMC of order 0, there is no conservatism if $\mathcal{G}$ is upper-triangular. It was conjectured that this result is strongly related to causality issues. Generalization of this non conservatism proof to the case of PFMC of order greater than 0 is a topic for future research.

B. Some hints about the design strategy

This subsection aims at giving some guidelines for the choice of $\{\alpha_k\}_{k=0}^{N-1}$ which appears to be a trade-off between the complexity of the control law and the reduction of the conservatism of the corresponding synthesis condition.

From the implementation point of view, the PFIRC represents an interesting choice because of the simplicity of its structure. In this paper synthesis conditions for this controller has been derived by regarding it as a structured PFMC of the same order κ. Consequently, the (unstructured) PFMC of the *same* order κ decreases the conservatism of the same synthesis theorems. Nonetheless, for this last control structure Th. IV.3 will be applied such that $\mathcal{G}$ will be modified to comply with a less sparse triangular structure. To summarize, among every PFIRC and PFMC of the same order κ, the following hierarchy holds with respect to the conservatism of the $\mathcal{H}_2$ guaranteed cost.

PFMC of order κ from Th. IV.3	$\leq$	PFMC of order κ from Th. IV.2	$\leq$	PFIRC of order κ from Th. IV.2

V. Numerical experiments

This section is organized in two different parts: The first one is dedicated to the design of four different robust periodic controllers and the analysis of their relative conservatism. The second part gives some insight about the choice of the starting point τ when applying robust analysis conditions to one controller picked out of the four.

TABLE I
$\mathcal{H}_2$ SYNTHESIS AND ANALYSIS RESULTS FOR DIFFERENT CHOICES OF α_k

Controller		Synthesis				Analysis	
id.	$\{\alpha_0, \alpha_1, \alpha_2\}$	Theorem	$\sqrt{\gamma_g^s}$	Nb. lines	Nb. var.	$\sqrt{\gamma_g}$ (Th. III.5)	$\sqrt{\gamma_g^d}$ (Th. III.8)
1	$\{1, 2, 3\}$	Th. IV.3	8.4175	92	133	3.7897	3.8804
2	$\{3, 3, 3\}$	Th. IV.2	13.3717	156	335	4.1835	3.8567
3	$\{3, 4, 5\}$	Th. IV.2	12.2624	156	341	4.2654	3.9285
4	$\{3, 4, 5\}$	Th. IV.3	4.5505	156	353	3.5125	3.3568

A. Enriching the control law decrease the guaranteed costs

Efficiency of the proposed analysis and synthesis results is evaluated by using the following 3-periodic model borrowed from [11] and reemployed in [10]

$$\begin{aligned} A_0 &= \begin{bmatrix} -3-\theta_1 & 2 \\ -3 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} -1-\theta_1 & 2 \\ 0.5 & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1-\theta_1 & 2 \\ 2.5 & 3 \end{bmatrix}, B_0 = Q_0 = \begin{bmatrix} 1 \\ \theta_2 \end{bmatrix}, \\ B_1 &= Q_1 = \begin{bmatrix} 1 \\ -0.3\theta_2 - 0.2 \end{bmatrix}, \\ B_2 &= Q_2 = \begin{bmatrix} 0.5(\theta_2+1) \\ 1 \end{bmatrix}, C_0 = C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ D_0 &= D_1 = D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, R_0 = R_1 = R_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \end{aligned} \tag{75}$$

The two real uncertainties $\theta_{1,2}$ are time invariant and satisfy $|\theta_1| \leq 0.6$ and $0 \leq \theta_2 \leq 1$. From this definition, it clearly appears that this example can be modeled as a polytopic-type uncertain system (1), where the number of vertices is $L = 4$. Our goal here is to design robust state-feedback controllers that minimize an upper bound of the worst-case $\mathcal{H}_2$ norm of the closed-loop systems.

Periodic controllers corresponding to different choices of α_k are designed relying on Th. IV.2 and, when it is possible, on Th. IV.3. Upper bounds of robust $\mathcal{H}_2$ performance of every resulting closed-loop system are computed by means of the related primal and dual analysis theorems. Results are gathered in Table I where each controller is identified by a different number. The computational effort at the synthesis step is reported as well in terms of number of lines and number of decision variables.

The experiments illustrate that for a given synthesis theorem, the addition of degrees of freedom to the control law may be effective for reducing the conservatism of the synthesis condition although it increases the computational cost. Indeed, referring to their capacity of lowering $\sqrt{\gamma_g^s}$, the controllers designed using Th. IV.3 can be ordered as $1 < 4$ and the ones obtained by means of Th. IV.2 verify $2 < 3$. Nevertheless, this remark does not hold anymore when different synthesis theorems are employed. The value $\sqrt{\gamma_g^s}$ obtained with controller 2 is larger than the one with 1 although its control structure is richer. Furthermore, from the controller 3 to 4, $\sqrt{\gamma_g^s}$ is decreased by 63 percent although the control structure remains the same and the computation effort is almost identical. These two observations bring to light how crucial is the structure of $\mathcal{G}$.

When designing a state-feedback minimizing the upper bound $\sqrt{\gamma_g^s}$, it is, of course, intended to design a control law that decreases $\sqrt{\gamma_{wc}}$. In order to estimate this last value, two different guaranteed costs are provided, namely $\sqrt{\gamma_g}$ and $\sqrt{\gamma_g^d}$ which are different in general and closer to $\sqrt{\gamma_{wc}}$ since $\sqrt{\gamma_g^d} \leq \sqrt{\gamma_g^s}$. In most of the cases, ordering relationships between control laws based on $\sqrt{\gamma_g^s}$ or on $\min\{\sqrt{\gamma_g}, \sqrt{\gamma_g^d}\}$ are consistent with each other. Table I provides an example of this fact as controller id.4 is better than id.1, for both synthesis and analysis guaranteed costs. Controllers id.2 and id.3 give a counter example to this conjecture since id.3 leads to a smaller value of $\sqrt{\gamma_g^s}$ than id.2 while the analysis step seems to reverse this ordering. However, even if it is hard, in general to conclude about the genuine hierarchy between the worst-case costs $\sqrt{\gamma_{wc}}$ obtained by different sufficient conditions, the reduction of the upper bound $\sqrt{\gamma_g^s}$ appears to be a good indicator for the reduction of conservatism.

B. Impact of time-shifting

Analysis and synthesis have been obtained via the intensive use of time lifting. To this end, the definition of extended input/output vectors has implicitly required to define the starting point of the considered period. Indeed, in the most general case, the extended input vector $\hat{w}_q$ is parameterized by τ such that $\hat{w}_q^\tau = \begin{bmatrix} w_{qN+N-1+\tau}^T & \cdots & w_{qN+\tau}^T \end{bmatrix}^T$. So far, this paper was concerned by the special case where $\tau = 0$. Nevertheless, cyclic re-indexation allows to consider different situations. If the periodic model Σ_{cl}^τ is derived from Σ_{cl} by shifting its matrices of τ forward in time, i.e. $A_{k,j}^\tau = A_{k+\tau(\mathrm{mod} N),j}$, $B_k^\tau = B_{k+\tau(\mathrm{mod} N)}$, etc, then applying previous theorems to Σ_{cl}^τ is equivalent to consider Σ_{cl} with $\tau \neq 0$. The impact of this time-shifting on the conservatism of the robust analysis conditions is now analyzed on the same example with controller # 1 computed for $\tau = 0$

TABLE II
IMPACT OF TIME-SHIFTING ON THE ANALYSIS RESULTS

	$\mathcal{H}_2$	
τ	$\sqrt{\gamma_g}$ (Th. III.5)	$\sqrt{\gamma_g^d}$ (Th. III.8)
0	3.7897	3.8804
1	3.9275	3.8295
2	3.7693	3.8403

Results, shown in Table II, confirm that in the general case, modifying τ may affect the conservatism of the computed upper bounds. It is also interesting to remark that the improvement is not consistent since $\tau = 1$ is better than $\tau = 0$ for $\sqrt{\gamma_g^d}$ but not for $\sqrt{\gamma_g}$. Thus, τ gives a way to refine the analysis result by retaining only the minimum of primal and dual costs over all different starting points. Obviously, instead of systematic computation, a rationale leading to the optimal value of τ is desirable although probably hard to define.

VI. Conclusion

In this paper, we presented new LMI sufficient conditions for robust stability analysis and robust state-feedback design for polytopic uncertain periodic systems with memory. The flexibility of the proposed approach allows the user to freely add degrees-of-freedom to the control law which appears to effectively decrease conservatism of the results. Interestingly, numerical examples have shown that for a particular structure of controllers, the efficiency of the design theorem can be significantly enhanced by relaxing the matrix of slack-variables. As noticed in [21], providing deeper guidelines for the choice of α_ks and for the definition of the starting point of the considered period remain challenging topics and will be the objective of future works.

Appendix

A. Proofs of Prop. III.1 (monodromy lifting of Σ_{cl})

In order to build the monodromy lifting of Prop. III.1, the system with memory $\Sigma_{cl}(\theta)$ is first recast as a memoryless periodic model.

Proposition A.1. *Σ_{cl} given by (8) is equivalent to the following model:*

$$\begin{bmatrix} \bar{x}_{qN+k+1} \\ z_{qN+k} \end{bmatrix} = \begin{bmatrix} \bar{A}_k & \bar{B}_k \\ \bar{C}_k & \bar{D}_k \end{bmatrix} \begin{bmatrix} \bar{x}_{qN+k} \\ w_{qN+k} \end{bmatrix} \tag{76}$$

with $\bar{x}^T_{Nq+k} = \begin{bmatrix} x^T_{Nq+k} & \cdots & x^T_{Nq-l+1} \end{bmatrix} \in \mathbb{R}^{n(k+l)}$.

Proof: The periodic model Σ_{cl} can be rewritten under the following form:

$$\begin{bmatrix} x_{Nq+k+1} \\ z_{Nq+k} \end{bmatrix} = \begin{bmatrix} \grave{A}_k \\ \bar{C}_k \end{bmatrix} \bar{x}_{Nq+k} + \begin{bmatrix} B_k \\ D_k \end{bmatrix} w_{Nq+k} \tag{77}$$

where $\bar{x}_{Nq+k}$ is given above and $\grave{A}_k$ and $\bar{C}_k$ are defined by (19). Thus, $\bar{x}_{Nq+k}$ concatenates the states history required to evaluate x_{Nq+k+1} and z_{Nq+k}. The current state is incorporated to this history in such a way that the dimension of $\bar{x}_{Nq+k}$ grows at each instant time along the period while being partially reset for each new period:

$$\begin{aligned} \bar{x}_{Nq+k+1} &= \begin{bmatrix} x_{Nq+k+1} \\ \bar{x}_{Nq+k} \end{bmatrix}, \ (0 \le k \le N-2) \\ \bar{x}_{N(q+1)} &= \begin{bmatrix} x_{N(q+1)} \\ \begin{bmatrix} \mathbf{1}_{n(l-1)} & \mathbf{0} \end{bmatrix} \bar{x}_{N(q+1)-1} \end{bmatrix} \end{aligned} \tag{78}$$

From (77) and (78), the memoryless model (76) can be easily obtained. ■

Using this reformulation, the monodromy lifting procedure recalled in [3] can be readily applied which leads to Γ_m defined by (15) with all matrices given by (16) and (17).

B. Proof of Th. III.5 ($\mathcal{H}_2$ worst-case analysis)

Proof: Applying Schur complement to (36) and (37) and operating the convex combination of the obtained conditions followed by another Schur complement leads to the following conditions, $\forall\ \theta \in \Theta$:

$$\begin{aligned} -\mathcal{P}(P(\theta)) + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{C}_1(\theta) \, \mathcal{C}_2(\theta) \end{bmatrix}^T \right\} \\ + \mathrm{He}\left\{ \mathcal{F}_1 \begin{bmatrix} \mathcal{E}(\theta) \, \mathcal{A}(\theta) \end{bmatrix} \right\} \prec 0 \end{aligned} \tag{79}$$

$$\begin{aligned} \mathcal{Z}(P(\theta), Z(\theta)) + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{C}_1^T(\theta) \\ \mathcal{D}^T(\theta) \end{bmatrix} \right\} \\ + \mathrm{He}\left\{ \mathcal{F}_2 \begin{bmatrix} \mathcal{E}(\theta) & \mathcal{B}(\theta) \end{bmatrix} \right\} \prec 0 \end{aligned} \tag{80}$$

Invoking the elimination Lemma and using the definitions of $\mathcal{P}$ and $\mathcal{Z}$ given by (32) and (39) allows to rewrite equivalently these conditions as (dependency upon θ is dropped for conciseness reasons):

$$\begin{aligned} \hat{x}_q^T \left(T^T \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & -P \end{bmatrix} T + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix}^T \right\} \right) \hat{x}_q < 0 \\ \text{s.t.} \ \begin{bmatrix} \mathcal{E} & \mathcal{A} \end{bmatrix} \hat{x}_q = \mathbf{0} \end{aligned} \tag{81}$$

$$\begin{aligned} \begin{bmatrix} \begin{bmatrix} \mathbf{1}_{nN} \ \mathbf{0} \end{bmatrix} \hat{x}_q \\ \hat{w}_q \end{bmatrix}^T \left(\begin{bmatrix} \begin{bmatrix} \mathbf{1}_{nN} \\ \mathbf{0} \end{bmatrix}^T T^T \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{nl} \end{bmatrix} T \begin{bmatrix} \mathbf{1}_{nN} \\ \mathbf{0} \end{bmatrix} & \mathbf{0} \\ \mathbf{0} & -Z \end{bmatrix} \right. \\ \left. + \mathrm{Sq}\left\{ \begin{bmatrix} \mathcal{C}_1^T \\ \mathcal{D}^T \end{bmatrix} \right\} \right) \begin{bmatrix} \begin{bmatrix} \mathbf{1}_{nN} \ \mathbf{0} \end{bmatrix} \hat{x}_q \\ \hat{w}_q \end{bmatrix} < 0 \\ \text{s.t.} \ \begin{bmatrix} \mathcal{E} \ \mathcal{B} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathbf{1}_{nN} \ \mathbf{0} \end{bmatrix} \hat{x}_q \\ \hat{w}_q \end{bmatrix} = \mathbf{0} \end{aligned} \tag{82}$$

After tedious but straightforward algebraic manipulations, we get the following conditions:

$$\begin{aligned} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix}^T \begin{bmatrix} P & & & \\ & -P & & \\ & & \mathbf{0} & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} < 0 \\ \text{s.t.} \ \begin{bmatrix} -\mathbf{1} & \Psi & \mathfrak{B} & \mathbf{0} \\ \mathbf{0} & \mathfrak{C} & \mathfrak{D} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \end{aligned} \tag{83}$$

$$\begin{aligned} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix}^T \begin{bmatrix} P & & & \\ & \mathbf{0} & & \\ & & -Z & \\ & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} < 0 \\ \text{s.t.} \ \begin{bmatrix} -\mathbf{1} & \Psi & \mathfrak{B} & \mathbf{0} \\ \mathbf{0} & \mathfrak{C} & \mathfrak{D} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta_{q+1} \\ \eta_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \end{aligned} \tag{84}$$

Using elimination lemma leads to (22) and (23). The inequality $\gamma_{\mathrm{g}} \geq \gamma_{wc}$ comes from the enforcement for $\mathcal{F}_1$, $\mathcal{F}_2$ to be independent of the uncertain parameter θ ■

C. Dual of descriptor model

Proof: To derive the dual of Γ_e despite its peculiar structure, the key idea is to partition $\hat{x}_q \in \mathbb{R}^{n(N+l)}$ in sub-vectors of size nN giving rise to a time-invariant polynomial model for which duality is well-known [14]. Nonetheless, since l is not a multiple of N in the general case, a vector $\xi_q \in \mathbb{R}^{n(N-r)}$

defined by $\xi_q = \sigma \begin{bmatrix} \mathbf{0} & \mathbf{1}_{n(N-r)} & \mathbf{0}_{n(N-r)\times nl} \end{bmatrix} \hat{x}_q$ is appended forward to $\hat{x}_q$ without changing the model. Proceeding this way allows to equivalently rewrite (25) as

$$\begin{bmatrix} \mathbf{0} & \begin{bmatrix}\mathcal{E} \ \mathcal{A}\end{bmatrix} & \mathcal{B} & \mathbf{0} \\ \mathbf{1}_{n(N-r)} & -\sigma \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{n(N-r)} \ \mathbf{0}_{n(N-r)\times nl}\end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix}\mathbf{1}_{nl} \ \mathbf{0}\end{bmatrix} - \sigma \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{nl}\end{bmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix}\mathcal{C}_1 \ \mathcal{C}_2\end{bmatrix} & \mathcal{D} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \xi_q \\ \hat{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{85}$$

where the obtained extended state vector can be divided in $b+2$ terms of equal size, denoted by $\tilde{x}_q \in \mathbb{R}^{nN}$ and linked with each others by the shift operator σ. Indeed, (85) has the following form

$$\left[\begin{array}{ccccc|cc} \mathscr{A}_0 & \cdots & \cdots & \mathscr{A}_{b+1} & & \mathcal{B} & \mathbf{0} \\ \mathbf{1}_{nN} & -\sigma\mathbf{1}_{nN} & & & & \mathbf{0} & \mathbf{0} \\ & \ddots & \ddots & & & \vdots & \vdots \\ & & \mathbf{1}_{nN} & -\sigma\mathbf{1}_{nN} & & \mathbf{0} & \mathbf{0} \\ \mathscr{C}_0 & \cdots & \cdots & \mathscr{C}_{b+1} & & \mathcal{D} & -1 \end{array}\right] \left[\begin{array}{c} \tilde{x}_q \\ \vdots \\ \tilde{x}_{q-b-1} \\ \hline \hat{w}_q \\ \hat{z}_q \end{array}\right] = \mathbf{0} \tag{86}$$

where $\begin{bmatrix} \mathbf{0} & \mathcal{E} & \mathcal{A} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0} & \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix}$ have been split in $b+2$ sub-matrices defined by (43), (46), (47), (48). From (86), it comes that (25) is equivalent to

$$\begin{bmatrix} \sum_{j=0}^{b+1} \mathscr{A}_j \sigma^{-j} & \mathcal{B} & \mathbf{0} \\ \sum_{j=0}^{b+1} \mathscr{C}_j \sigma^{-j} & \mathcal{D} & -\mathbf{1}_{pN} \end{bmatrix} \begin{bmatrix} \tilde{x}_q \\ \hat{w}_q \\ \hat{z}_q \end{bmatrix} = \mathbf{0} \tag{87}$$

which is a suitable form for applying the well-known theory of duality for discrete-time time-invariant models, see e.g. [14]. The dual version of (87) is given by

$$\begin{bmatrix} \sum_{j=0}^{b+1} \mathscr{A}_j^T \sigma^{j} & \sum_{j=0}^{b+1} \mathscr{C}_j^T \sigma^{j} & \mathbf{0} \\ \mathcal{B}^T & \mathcal{D}^T & \mathbf{1} \end{bmatrix} \begin{bmatrix} \tilde{x}_q^d \\ \hat{w}_q^d \\ \hat{z}_q^d \end{bmatrix} = \mathbf{0} \tag{88}$$

Then, as Γ_e has been artificially enlarged to arrive at (87) by incorporating ξ_q, (88) is now reduced. To this end, consider first the following change of variables:

$$\begin{bmatrix} \tilde{x}_{q+b+1}^d \\ \vdots \\ \tilde{x}_q^d \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{nr}\end{bmatrix} \breve{x}_{q+b+1}^d \\ \breve{x}_{q+b}^d \\ \vdots \\ \breve{x}_q^d \\ \begin{bmatrix}\mathbf{1}_{n(N-r)} \ \mathbf{0}\end{bmatrix} \breve{x}_{q-1}^d \end{bmatrix},$$
$$\begin{bmatrix} \hat{w}_{q+b+1}^d \\ \vdots \\ \hat{w}_q^d \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{p(r-1)}\end{bmatrix} \breve{w}_{q+b+1}^d \\ \breve{w}_{q+b}^d \\ \vdots \\ \breve{w}_q^d \\ \begin{bmatrix}\mathbf{1}_{p(N-r+1)} \ \mathbf{0}\end{bmatrix} \breve{w}_{q-1}^d \end{bmatrix} \tag{89}$$

where $\breve{x}_q^d \in \mathbb{R}^{nN}$ and $\breve{w}_q^d \in \mathbb{R}^{pN}$ share respectively the same size as $\tilde{x}_q^d$ and $\hat{w}_q^d$. From these definitions, it comes that $\breve{x}_{q-1}^d$ and $\breve{w}_{q-1}^d$ are not involved in the state equation because of the structure of $\mathscr{A}_0^T$ and $\mathscr{C}_0^T$. Consequently, they can be dropped since $\breve{x}_{q-1}^d$ goes to zero when $\breve{x}_q^d$ does (the time goes backward for the dual model).

Following the same rule for $\breve{z}_q^d$, the output equation for the period q and $q+1$

$$\begin{bmatrix} \mathcal{B}^T & \\ & \mathcal{B}^T \end{bmatrix} \begin{bmatrix} \tilde{x}_{q+1}^d \\ \tilde{x}_q^d \end{bmatrix} + \begin{bmatrix} \mathcal{D}^T & \\ & \mathcal{D}^T \end{bmatrix} \begin{bmatrix} \hat{w}_{q+1}^d \\ \hat{w}_q^d \end{bmatrix} + \begin{bmatrix} \hat{z}_{q+1}^d \\ \hat{z}_q^d \end{bmatrix} = \mathbf{0} \tag{90}$$

may be rewritten as

$$\begin{aligned} &\begin{bmatrix} \mathcal{B}^T & \\ & \mathcal{B}^T \end{bmatrix} \begin{bmatrix} \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{nr}\end{bmatrix} \breve{x}_{q+1}^d \\ \breve{x}_q^d \\ \begin{bmatrix}\mathbf{1}_{n(N-r)} \ \mathbf{0}\end{bmatrix} \breve{x}_{q-1}^d \end{bmatrix} \\ &+ \begin{bmatrix} \mathcal{D}^T & \\ & \mathcal{D}^T \end{bmatrix} \begin{bmatrix} \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{p(r-1)}\end{bmatrix} \breve{w}_{q+1}^d \\ \breve{w}_q^d \\ \begin{bmatrix}\mathbf{1}_{p(N-r+1)} \ \mathbf{0}\end{bmatrix} \breve{w}_{q-1}^d \end{bmatrix} \\ &+ \begin{bmatrix} \begin{bmatrix}\mathbf{0} \ \mathbf{1}_{p(r-1)}\end{bmatrix} \breve{z}_{q+1}^d \\ \breve{z}_q^d \\ \begin{bmatrix}\mathbf{1}_{p(N-r+1)} \ \mathbf{0}\end{bmatrix} \breve{z}_{q-1}^d \end{bmatrix} = \mathbf{0} \end{aligned} \tag{91}$$

which can be reduced to

$$\breve{\mathcal{B}}^T \begin{bmatrix} \begin{bmatrix}\mathbf{0} & \mathbf{1}_n\end{bmatrix} \breve{x}_{q+1}^d \\ \breve{x}_q^d \end{bmatrix} + \breve{\mathcal{D}}^T \breve{w}_q^d + \breve{z}_q^d = \mathbf{0} \tag{92}$$

with $\breve{\mathcal{B}}$ and $\breve{\mathcal{D}}$ defined in Proposition III.3.

Then, expression (40) follows if $\hat{x}_q^d \in \mathbb{R}^{n(N+l)+p(l-1)}$ stacks states and past inputs:

$$\begin{aligned} \hat{x}_q^{dT} = \Big[& \left\{ \begin{bmatrix}\mathbf{0} & 1_{nr}\end{bmatrix} \breve{x}_{q+b+1}^d \right\}^T \quad \breve{x}_{q+b}^{dT} \quad \cdots \quad \breve{x}_q^{dT} \\ & \left\{ \begin{bmatrix}\mathbf{0} & 1_{p(r-1)}\end{bmatrix} \breve{w}_{q+b+1}^d \right\}^T \quad \breve{w}_{q+b}^{dT} \quad \cdots \quad \breve{w}_{q+1}^{dT} \Big] \end{aligned} \tag{93}$$

and defining $\breve{\mathcal{A}} \in \mathbb{R}^{nl\times nN}$, $\breve{\mathcal{E}} \in \mathbb{R}^{nN\times nN}$, $\breve{\mathcal{C}}_2 \in \mathbb{R}^{p(l-1)\times nN}$ and $\breve{\mathcal{C}}_1 \in \mathbb{R}^{pN\times nN}$ as in (42). ■

D. Proofs for dual analysis conditions

To derive analysis conditions for Σ_{cl}^d, we use the same strategy as the one used for the primal model Σ_{cl}.

1) Monodromy lifting: In order to recast Σ_{cl}^d into a memoryless periodic model, not only past states but also past inputs have to be memorized along the period. To this end, the following time varying vector $\bar{v}_{Nq-k-r}^d$ described is introduced

$$\bar{v}_{Nq-k-r}^d = \begin{bmatrix} \bar{x}_{Nq-k-r}^d \\ \bar{w}_{Nq-k-r+1}^d \end{bmatrix}$$
$$\bar{x}_{Nq-k-r}^d = \begin{bmatrix} x_{Nq+l-r-1}^d \\ \vdots \\ x_{Nq-k-r}^d \end{bmatrix}, \ \bar{w}_{Nq-k-r+1}^d = \begin{bmatrix} w_{Nq+l-r-1}^d \\ \vdots \\ w_{Nq-k-r+1}^d \end{bmatrix} \tag{94}$$

From this definition and keeping in mind the proof of Prop. III.1, it is easy to check that the following proposition holds.

Proposition A.2 (Memoryless periodic representation of Σ_{cl}^d). *The following model is equivalent to Σ_{cl}^d:*

$$\begin{bmatrix} \bar{v}_{qN-k-r-1}^d \\ z_{qN-k-r}^d \end{bmatrix} = \begin{bmatrix} \bar{A}_k^d & \bar{C}_k^d \\ \bar{B}_k^d & \bar{D}_k^d \end{bmatrix} \begin{bmatrix} \bar{v}_{qN-k-r}^d \\ w_{qN-k-r}^d \end{bmatrix} \tag{95}$$

with

$$\bar{A}_k^d = \begin{cases} \begin{bmatrix} \mathbf{1}_{n(l+k)} & \mathbf{0} \\ \grave{A}_k^d & \grave{C}_k^d \\ \mathbf{0} & \mathbf{1}_{p(l+k-1)} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ (0 \le k \le N-2) \\ \begin{bmatrix} \begin{bmatrix}\mathbf{0} & \mathbf{1}_{n(l-1)}\end{bmatrix} & \mathbf{0} \\ \grave{A}_{N-1}^d & \grave{C}_{N-1}^d \\ \mathbf{0} & \begin{bmatrix}\mathbf{0} & \mathbf{1}_{p(l-2)}\end{bmatrix} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ (k = N-1) \end{cases}$$

$$\bar{C}_k^d = \begin{cases} \begin{bmatrix} \mathbf{0}_{n(l+k)\times p} \\ C^T_{\vartheta(0,k),0} \\ \mathbf{0}_{p(l+k-1)\times p} \\ \mathbf{1}_p \end{bmatrix}, \ (0 \le k \le N-2) \\ \begin{bmatrix} \mathbf{0}_{n(l-1)\times p} \\ C^T_{\vartheta(0,k),0} \\ \mathbf{0}_{p(l-2)\times p} \\ \mathbf{1}_p \end{bmatrix}, \ (k = N-1) \end{cases} \tag{96}$$

$$\begin{aligned} \bar{B}_k^d &= \begin{bmatrix} \begin{bmatrix}\mathbf{0}_{m\times n(l+k-1)} & -B^T_{\vartheta(0,k)}\end{bmatrix} & \mathbf{0}_{m\times p(l+k-1)} \end{bmatrix} \\ \bar{D}_k^d &= -D^T_{\vartheta(0,k+r)} \end{aligned} \tag{97}$$

$$\begin{aligned} \grave{A}_k^d &= \begin{bmatrix} A^T_{\vartheta(l+k-1,k+r),l+k-1} \cdots A^T_{\vartheta(1,k+r),1} \ A^T_{\vartheta(0,k+r),0} \end{bmatrix} \\ \grave{C}_k^d &= \begin{bmatrix} C^T_{\vartheta(l+k-1,k+r),l+k-1} & \cdots & C^T_{\vartheta(1,k+r),1} \end{bmatrix} \end{aligned} \tag{98}$$

Remember that time goes backward, then monodromy lifting may be readily applied to this memoryless model, which leads to Γ_m^d.

$$\rho_q^{dT} = \begin{bmatrix} \bar{x}_{Nq-r}^{dT} & \bar{w}_{Nq-r+1}^{dT} \end{bmatrix} = \begin{bmatrix} \eta_q^{dT} & \phi_q^{dT} \end{bmatrix} \tag{99}$$

Proposition A.3 (Monodromy lifting of Σ_{cl}^d)**.** *The periodic model Σ_{cl}^d, given by* (50)*, can be lifted to the following time-invariant formulation, denoted by Γ_m^d:*

$$\Gamma_m^d: \ \begin{bmatrix} \rho_{q-1}^d \\ \breve{z}_q^d \end{bmatrix} = \begin{bmatrix} \breve{\Psi}^T & \breve{\mathfrak{C}}^T \\ \breve{\mathfrak{B}}^T & \breve{\mathfrak{D}}^T \end{bmatrix} \begin{bmatrix} \rho_q^d \\ \breve{w}_q^d \end{bmatrix} \tag{100}$$

where $\rho_q^d \in \mathbb{R}^{nl+p(l-1)}$ and $\breve{\Psi}$, $\breve{\mathfrak{B}}$, $\breve{\mathfrak{C}}$ and $\breve{\mathfrak{D}}$ are non linear functions of matrices of Σ_{cl}^d.

Once again, Γ_e^d and Γ_m^d share the same input/output vectors but differ by their internal representation. The following constant linear map relates the signals involved in the representation of Γ_e^d and Γ_m^d.

$$\begin{bmatrix} \rho_q^d \\ \rho_{q-1}^d \\ \breve{w}_q^d \end{bmatrix} = T^d \begin{bmatrix} \hat{x}_q^d \\ \breve{w}_q^d \end{bmatrix} \tag{101}$$

where $T^d \in \mathbb{R}^{2((n+p)l-p)\times((n+p)(l+N)-p)}$ is defined by (58). It should be pointed out that, in contrast with the corresponding relationship in the primal context, the inputs vector is now involved. This is directly related to the fact that the history of the inputs plays a role in the dynamics of Γ_e^d although it was not the case for Γ_e.

An outright application of time-invariant theory to Γ_m^d is now possible. For conciseness reasons, only robust stability is studied here.

Theorem A.1 (Robust stability via Γ_m^d)**.** *The polytopic periodic model Σ_{cl}^d is robustly stable if and only if the following condition holds, $\forall\theta \in \Theta$:*

$$\exists P(\theta) \in \mathbb{S}_+^{nl}: \ \breve{\Psi}_{00}^T(\theta)P(\theta)\breve{\Psi}_{00}(\theta) - P(\theta) \prec 0 \tag{102}$$

with

$$\breve{\Psi}_{00}^T(\theta) = \begin{bmatrix} \mathbf{1}_{nl} \\ \mathbf{0}_{p(l-1)\times nl} \end{bmatrix}^T \breve{\Psi}^T(\theta) \begin{bmatrix} \mathbf{1}_{nl} \\ \mathbf{0}_{p(l-1)\times nl} \end{bmatrix} \tag{103}$$

Proof: Vector ρ_q^d is partially composed of $\bar{w}_{Nq-r+1}^d$, representing history of inputs required to evaluate x_{Nq-r-1}^d. Thus, for the next period, $\bar{w}_{N(q-1)-r+1}^d$ uses entries of $\bar{w}_{Nq-r+1}^d$ and $\breve{w}_q^d$ to update this history. From (49) and (94), we get

$$\bar{w}_{N(q-1)-r+1}^d = \begin{bmatrix} \mathbf{0}_{p(l-1)\times pN} & \mathbf{1}_{p(l-1)} \end{bmatrix} \begin{bmatrix} \bar{w}_{Nq-r+1}^d \\ \breve{w}_q^d \end{bmatrix} \tag{104}$$

Using notations of (94), the dynamic equation of Γ_m^d, given by (100) can be described by:

$$\begin{bmatrix} \bar{x}_{N(q-1)-r}^d \\ \bar{w}_{N(q-1)-r+1}^d \end{bmatrix} = \breve{\Psi}^T \begin{bmatrix} \bar{x}_{Nq-r}^d \\ \bar{w}_{Nq-r+1}^d \end{bmatrix} + \breve{\mathfrak{C}}^T \breve{w}_q^d \tag{105}$$

Equation (104) exhibits the structure of $\breve{\Psi}^T$:

$$\breve{\Psi}^T = \begin{bmatrix} \breve{\Psi}_{00}^T & \breve{\Psi}_{01}^T \\ \mathbf{0}_{p(l-1)\times nl} & \breve{\Psi}_{11}^T \end{bmatrix} \tag{106}$$

where

$$\begin{aligned} \breve{\Psi}_{11}^T &= \begin{bmatrix} \mathbf{0}_{p(l-1)\times pN} & \mathbf{1}_{p(l-1)} \end{bmatrix} \begin{bmatrix} \mathbf{1}_{p(l-1)} \\ \mathbf{0}_{pN\times p(l-1)} \end{bmatrix} \\ &= \begin{cases} \mathbf{0}_{p(l-1)} & (l \le N+1) \\ \begin{bmatrix} & \mathbf{1}_{p(l-1-N)} \\ \mathbf{0}_{pN} & \end{bmatrix} & (l > N+1) \end{cases} \end{aligned} \tag{107}$$

Since, $\breve{\Psi}_{11}^T$ is always Schur stable, the model Γ_m^d is stable if and only if $\breve{\Psi}_{00}^T$ is Schur stable. ∎

All the key ingredients have been now obtained in order to establish dual analysis theorems III.7 and III.8 for Σ_{cl}^d.

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