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Decentralized Control of Parallel Rigid Formations with Direction Constraints and Bearing Measurements

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Abstract— **In this paper we analyze the relationship between scalability, minimality and rigidity, and its application to cooperative control. As a case study, we address the problem of multi-agent formation control by proposing a distributed control strategy that stabilizes a formation described with bearing (direction) constraints, and that only requires bearing measurements and parallel rigidity of the interaction graph. We also consider the possibility of having different graphs modeling the interaction network in order to explicitly take into account the conceptual difference between sensing, communication, control, and parameters stored in the network. We then show how the information can be 'moved' from a graph to another making use of decentralized estimation, provided the parallel rigidity property. Finally we present simulative examples in order to show the validity of the theoretical analysis in some illustrative cases.**

I. INTRODUCTION

Decentralized accomplishment of a prescribed formation is a cardinal problem in multi-agent control, see, e.g,. [1], [2], [3], [4]. In absence of central sensing and processing units, every agents is only able to obtain relative measurements and communicate with a certain subset of the whole group. Therefore a formation control task is better defined in terms of relative constraints, i.e., desired values for those relative quantities processed by pairs of agents. In these cases, in order to uniquely define a formation, a straightforward solution is the specification of the desired relative quantities (e.g., bearing and distances) among *all* the possible agent pairs. Although successful, this solution violates *scalability*, since it requires each individual agent to store a $O(n)$ number of desired values – with n being the total number of agents – thus resulting in $O(n^2)$ for the whole group. Nevertheless, exploiting the concept of *rigidity theory*, it is still possible to guarantee uniqueness of the specified formation by only resorting to a total number $O(n)$ of constraints for the whole group, both in the distance and bearing cases.

Rigidity is a well-known and fundamental tool in the context of formation control based on distances [5], [6], [7]. *Parallel rigidity* [8], [9], that is, the bearing counterpart of 'distance rigidity', has also been recently introduced for controlling bearing-constrained formations. In particular the authors in [10] have presented a control strategy where the formation is specified as a parallel rigid set of desired bearings. However, the proposed controller was still requiring both bearing and distance measurements in order to be implemented. In principle, consensus-based approaches, such as those proposed in [2], could also be used in order to achieve a desired bearing formation, but in this case measurement of distances would be needed as well by the controller.

It is worth noting that relative bearings represent a highly significant piece of information; in fact, bearing measurements span a large spectrum of possible sensors model such as, e.g., any monocular camera system. It is therefore interesting to invest research efforts into the formalization and suitable control strategies of bearing formations, where both the desired formation constraints and actual measurements are made of only bearings. Recent works have addressed such instances of formation control, but usually considering only a limited number of agents as, e.g., in [11], [12] where only 3 agents are contemplated. In [13], [14] a 3D bearing-only formation controller is presented which works in the case of an unlimited number of agents and shares several similarities with the the controller presented in this work. Nevertheless, the approach presented in [13], [14] assumes the presence of a particular class of interaction network, an assumption that will be greatly relaxed in our setting.

It is common practice in multi-agent literature to make use of a *single* interaction graph in order to model the presence of constraints (i.e., desired values) between any two agents as well as the presence of inter-agent sensing/communication networks. Nevertheless, formation constraints, relative sensing, and inter-agent communication usually originate from different domains (e.g., the constraint pairs could be chosen a priori by a high-level supervisor, while the sensing and communication pairs may depend upon the particular range and field-of-view of the used technology).

In this paper we study the possibility of having a more refined model of the multi-agent network by allowing a multiple-graph representation. While a unique graph guarantees the matching of all the local quantities (e.g., desired and measured values), this fact does not hold anymore in the case of multiple graphs. In order to compensate for the absence of that 'a priori' matching, we will resort to a decentralized estimation algorithm that is able to 'translate' the quantities from one graph to another graph, owing to the rigidity property. In this regard, the presented work place itself in the class of approaches combining decentralized estimation with cooperative control, see, e.g., [15], [16].

To summarize, the main contributions of this work are the following: *i)* we propose a decentralized formation controller that uses both bearing-only measurements and constraints (thus, does not make use of distances), *ii)* we allow for a thorough modeling of the required interaction network, by explicitly taking into account the conceptual difference

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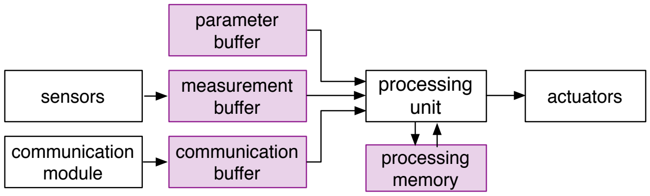

Fig. 1: Computational model of a generic agent.

between, sensing, communication, control, and parameters stored in the network, *iii)* by exploiting this modeling we analyze the connection between scalability, minimality and rigidity.

The paper is organized as follows. In Sec. II we define our agent and network model and analyze the interplay between scalability and parallel rigidity. We then introduce a bearing-only controller and prove its stability in Sec. III, while in Sec. IV we describe the multi-graph model and the associated rigidity-based estimation algorithm. In Sec. V we provide some numerical examples and we conclude the paper in Sec. VI.

II. SCALABILITY AND RIGIDITY IN BEARING-ONLY CONTROL

Consider a group of n agents and denote with $\boldsymbol{p}_i \in \mathbb{R}^2$ the position of the i-th agent, where $i \in \mathcal{V} = \{1, \dots, n\}$. Assume also presence of kinematic agents, i.e., $\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i$, where $\boldsymbol{u}_i \in \mathbb{R}^2$ is the control input of agent i.

Scalability is a cardinal concept in multi-agent literature and plays a pivotal role in our work as well. In order to illustrate how the concept of scalability applies to our setting, consider the computational model of a generic agent i depicted in Fig. 1. The main goal of an agent i is to implement a control algorithm that will steer its position towards a desired location, defined in terms of relative quantities w.r.t. neighboring agents. Since all the computations are performed by the agent internal processing unit, the paradigm of scalability requires that the amount of data and elementary operations needed by the control algorithm at any time t to be constant w.r.t. the total number of agents n. Therefore, seen from a global perspective, the multi-agent system can only process an amount $O(n)$ of data per unit of time.

For the sake of generality, let us assume that the data used by the control algorithm belongs to three distinct classes: (i) constant parameters, (ii) relative measurements, and (iii) communicated data. These are (conceptually) stored in the parameter, measurements, and communication buffers, respectively, as depicted in Fig. 1. Because of the reasons reported before, the buffer size must also be constant w.r.t. the total number of agents. By excluding typical parameters (e.g., control gains or other constants), the data used by an agent is always relative to other agents, that is, it represents relative quantities[1]. By denoting with $\mathcal{N}_i^s$ the set of agents whose quantities are needed by agent i, one can then define an *interacting* graph $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s)$, with $\mathcal{E}_s = \{(i,j) \in \mathcal{V} \times \mathcal{V} \,|\, j \in \mathcal{N}_i^s\}$, in order to represent all the 'interacting' agent pairs. In these terms, scalability requires $|\mathcal{N}_i^s|$ and $|\mathcal{E}_s|$ to be $O(1)$ and $O(n)$, respectively, with $|\mathcal{V}| = n$. Although this formulation is, as a matter of fact, standard in most multi-agent decentralized algorithms, it does not actually represent the most general situation. In fact, as it will be explained in the next sections, graph $\mathcal{G}_s$ can be actually thought as the union of four distinct subgraphs: a parameter graph, a measurement graph, a communication graph and a control graph, which are, in the general case, not coincident among themselves.

[1] In this setting we do not consider the interaction with an exogenous system, as, for example, the environment, or a human operator.

The following section will illustrate how these concepts apply to our particular case, and introduce the main relative quantity considered in this work as well as additional important properties related to scalability (minimality).

A. Relative bearings, parallel rigidity, and minimality

Given a generic vector of n distinct positions $\boldsymbol{q} = (\boldsymbol{q}_1^T \dots \boldsymbol{q}_n^T)^T \in \mathbb{R}^{2n}$ we denote with $\phi_{ij}(\boldsymbol{q}) = \phi(\boldsymbol{q}_i, \boldsymbol{q}_j)$ the bearing angle that an agent at $\boldsymbol{q}_i$ would measure w.r.t. another agent at $\boldsymbol{q}_j$. An equivalent representation is given by $\boldsymbol{\beta}_{ij}(\boldsymbol{q}) = \boldsymbol{\beta}(\boldsymbol{q}_i, \boldsymbol{q}_j) = (\cos\phi_{ij}(\boldsymbol{q}) \; \sin\phi_{ij}(\boldsymbol{q}))^T \in \mathbb{S}^1$, i.e., the unit vector pointing from $\boldsymbol{q}_i$ to $\boldsymbol{q}_j$ (bearing vector). Note that $\phi_{ji}(\boldsymbol{q}) \equiv \pi + \phi_{ij}(\boldsymbol{q}) \mod 2\pi$ and that $\boldsymbol{\beta}_{ij}(\boldsymbol{q}) = -\boldsymbol{\beta}_{ji}(\boldsymbol{q})$, for any $\boldsymbol{q}$ and i, j.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ we define the following set: $\mathcal{A}(\boldsymbol{q}, \mathcal{G}) = \{\phi_{ij}(\boldsymbol{q}) \,|\, (i,j) \in \mathcal{E},\, j > i\}$. We recall now some relevant facts concerning parallel rigidity, a fundamental property that allows to determine whether the set of bearing angles $\mathcal{A}(\boldsymbol{q}, \mathcal{G})$ contains the maximum possible information about the position vector $\boldsymbol{q}$ (see, e.g., [10] for a more detailed description about the topic).

Definition 1 (Point formation)**.** *A formation of points, or just* formation*, is a pair $(\mathcal{G}, \boldsymbol{q})$ consisting of a generic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ – where $\mathcal{V} = \{1, \dots n\}$ and $\mathcal{E} \subseteq \mathcal{T} = \{(i,j) \in \mathcal{V} \times \mathcal{V} \,|\, i \neq j\}$ – and a position vector $\boldsymbol{q} = (\boldsymbol{q}_1^T \dots \boldsymbol{q}_n^T)^T$ that assigns to every node $i \in \mathcal{V}$ the position $\boldsymbol{q}_i$. A formation is* degenerate *if the n positions in $\boldsymbol{q}$ are all aligned.*

This kind of 'graph-plus-configuration' structure is also called *framework* or *point-formation* in the literature, see, e.g. [8], [9] and references therein.

Definition 2 (Equivalent formations)**.** *A formation $(\mathcal{G}, \boldsymbol{r})$ is* equivalent *or* parallel *to $(\mathcal{G}, \boldsymbol{q})$ if $\mathcal{A}(\boldsymbol{q}, \mathcal{G}) = \mathcal{A}(\boldsymbol{r}, \mathcal{G})$, i.e., if*

$$(\boldsymbol{q}_i - \boldsymbol{q}_j)^{\perp} \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j) = 0 \quad \forall (i,j) \in \mathcal{E} \tag{1}$$

where the operator $^{\perp}$ rotates a vector by $\pi/2$ counterclockwise.

Definition 3 (Similar formations)**.** *The formation $(\mathcal{G}, \boldsymbol{r})$ is* similar *w.r.t. $(\mathcal{G}, \boldsymbol{q})$ if (1) is satisfied for any $(i,j) \in \mathcal{T}$, i.e., if $\boldsymbol{r}$ can be obtained from $\boldsymbol{q}$ by similarity (translation followed by dilation).*

Definition 4 (Parallel rigidity)**.** *A formation $(\mathcal{G}, \boldsymbol{q})$ is said to be* parallel rigid *if all the formations equivalent to $(\mathcal{G}, \boldsymbol{q})$ are similar to $(\mathcal{G}, \boldsymbol{q})$.*

Consider now a trajectory defined by the time-varying position vector $\boldsymbol{r}(t)$. The time-varying formation $(\mathcal{G}, \boldsymbol{r}(t))$ is equivalent to $(\mathcal{G}, \boldsymbol{q})$ if

$$(\boldsymbol{q}_i - \boldsymbol{q}_j)^{\perp} \cdot (\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)) = 0 \quad \forall (i,j) \in \mathcal{E},\ t \geq 0. \quad (2)$$

By differentiating (2) we obtain:

$$(\boldsymbol{q}_i - \boldsymbol{q}_j)^{\perp} \cdot (\dot{\boldsymbol{r}}_i(t) - \dot{\boldsymbol{r}}_j(t)) = 0 \quad (3)$$

which can be rewritten in matrix form as

$$R(\boldsymbol{q})\dot{\boldsymbol{r}} = \mathbf{0}. \quad (4)$$

Matrix $R(\boldsymbol{q}) \in \mathbb{R}^{|\mathcal{E}| \times 2|\mathcal{V}|}$ is called *bearing-constrained rigidity matrix*. A well know result in graph theory is the following:

Theorem 1. *If* $\operatorname{rank}[R(\boldsymbol{q})] = 2|\mathcal{V}| - 3$ *then the formation* $(\mathcal{G}, \boldsymbol{q})$ *is parallel rigid.*

Therefore a necessary condition for $(\mathcal{G}, \boldsymbol{q})$ to be parallel rigid is that $|\mathcal{E}| \geq 2|\mathcal{V}| - 3$.

Definition 5 (Generically Parallel Rigid Graph)**.** *If* $(\mathcal{G}, \boldsymbol{q})$ *is parallel rigid for any generic position vector*[2] $\boldsymbol{q}$ *then* $\mathcal{G}$ *is said to be* generically parallel rigid*, and* $(\mathcal{G}, \boldsymbol{q})$ *is a generically parallel rigid formation.*

The following theorem states that generic parallel rigidity is a topological property (e.g., see [8], [9]):

Theorem 2. *A graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is generically parallel rigid if and only if there exists a* $\mathcal{E}' \subseteq \mathcal{E}$ *such that:*

1) $|\mathcal{E}'| = 2n - 3$
2) $\forall \mathcal{E}'' \subseteq \mathcal{E}', \mathcal{E}'' \neq \emptyset$ *we have* $|\mathcal{E}''| \leq 2|\mathcal{V}_{\mathcal{E}''}| - 3$

where $\mathcal{V}_{\mathcal{E}''}$ *denotes the vertices adjacent to the edges in* $\mathcal{E}''$*.*

A parallel rigid formation is a formation where the bearings in $\mathcal{A}(\boldsymbol{q}, \mathcal{G})$ are sufficient to uniquely define all the other bearings $\phi_{ij}(\boldsymbol{q})\ \forall (i,j) \in \mathcal{T}/\mathcal{E}$, and, as a consequence, also the shape and orientation of $\boldsymbol{q}$ (but not its scale and translation). Rigidity has a twofold importance in multi-agent formation control. First, it can be used to check whether a subset of measured bearings is sufficient to reconstruct the value of any other needed (but not measured) bearing. Second, it can be used to check whether the stabilization of a subset of bearings to some desired values will result in the stabilization of all the remaining bearings of the formation to some uniquely defined values. It is therefore interesting to investigate whether the cardinality of these subsets can be scalable, i.e., $O(n)$. In this sense, one can exploit the fact that, due to Theorem 2, the minimal number of relative bearings needed to achieve rigidity is $2n-3$, so that a $O(n)$ number of measured/controlled quantities is sufficient for rigidity. This is formally stated in the following definition.

Definition 6 (Minimal Parallel Rigidity)**.** *Consider a formation* $(\mathcal{G}, \boldsymbol{q})$ *s.t.* $|\mathcal{E}| = 2|\mathcal{V}| - 3$

1) $(\mathcal{G}, \boldsymbol{q})$ *is minimally parallel rigid (m.p.r.) if* $\operatorname{rank}[R(\boldsymbol{q})] = 2n - 3$
2) $\mathcal{G}$ *is minimally and generically parallel rigid (*m.g.p.r.*) if it is also generically parallel rigid.*

[2]The position vector $\boldsymbol{q}$ is generic if its coordinates are not algebraically dependent.

Analogously to the definition of point formation in Definition 1, we now consider the dual case of *bearing formation* which will play an important role in the following developments.

Definition 7 (Bearing-formation)**.** *A formation of bearings, or simply a* bearing formation*, is a pair* $(\mathcal{G}, \boldsymbol{\alpha})$ *consisting of a generic graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *and a collection of* $|\mathcal{E}|$ *bearings* $\boldsymbol{\alpha} = \{\ldots \alpha_{ij} \ldots\}_{(i,j)\in\mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}$*, where the components of* $\boldsymbol{\alpha}$ *are listed in lexicographical order.*

A bearing formation $(\mathcal{G}, \boldsymbol{\alpha})$ *is said* feasible *if all the bearings contained in* $\boldsymbol{\alpha}$ *can simultaneously exist as* actual relative bearings $\phi_{ij}(\boldsymbol{q}^{\alpha})$ *for some configuration* $\boldsymbol{q}^{\alpha}$*, i.e., such that* $\phi_{ij}(\boldsymbol{q}^{\alpha}) = \alpha_{ij}\ \forall (i,j) \in \mathcal{E}$*. In this case,* $(\mathcal{G}, \boldsymbol{q}^{\alpha})$ *is called a* realization *of* $(\mathcal{G}, \boldsymbol{\alpha})$.

Within the class of feasible bearing formations, we define (minimally) parallel rigid *bearing formations those having only (minimally) parallel rigid realizations, and* degenerate *bearing formations those having at least a degenerate realization. This also implicitly defines the cases of (feasible)* non-rigid *and* non-degenerate *bearing formations.*

B. Bearing-only Control Problem

In order to represent the set of desired bearings in a scalable way, we assume that the multi-agent system distributively stores the desired bearings as a minimal and non-degenerate bearing formation $(\mathcal{G}_d, \boldsymbol{\phi}^d)$, where $\boldsymbol{\phi}^d = (\ldots \phi^d_{ij} \ldots)^T$, and $\mathcal{G}_d = (\mathcal{V}, \mathcal{E}_d)$ will be denoted from now on as *parameter graph*. Indicate also with $\mathcal{N}_i^d$ the set of neighbors of i in $\mathcal{G}_d$. In other words, agent i stores in its parameter buffer the set $\{\ldots \phi^d_{ij} \ldots\}_{j\in\mathcal{N}_i^d}$ which is $O(1)$ in cardinality w.r.t. $|\mathcal{V}|$. We also consider the alternative bearing representation by letting $\boldsymbol{\beta}^d_{ij} = (\cos\phi^d_{ij}\ \sin\phi^d_{ij})^T$.

By resorting to analogous arguments, we assume that the measurement of the current bearings are distributively taken as $\mathcal{A}(\boldsymbol{p}, \mathcal{G}_m)$ where the $\mathcal{G}_m = (\mathcal{V}, \mathcal{E}_m)$ will be denoted as *measure graph*. This implicitly defines the bearing formation $(\mathcal{G}_m, \boldsymbol{\phi}^m)$ where $\boldsymbol{\phi}^m = \operatorname{col}(\mathcal{A}(\boldsymbol{p}, \mathcal{G}_m))$ and "col" denotes the operator that stacks the elements of a set in a lexicographical order. Note that $(\mathcal{G}_m, \boldsymbol{\phi}^m)$ is feasible by definition. $\mathcal{G}_m$ is m.g.p.r. for the usual scalability issue (see Def. 6). Like before, we let $\mathcal{N}_i^m$ represent the set of neighbors of i in $\mathcal{G}_m$. In other words agent i stores in its measurement buffer the set $\{\ldots \phi^m_{ij}(\boldsymbol{p}) \ldots\}_{j\in\mathcal{N}_i^m}$ that is also $O(1)$ in cardinality w.r.t. $|\mathcal{V}|$.

Notice that, in general, it is $\mathcal{E}_d \neq \mathcal{E}_m$: considering this possibility constitutes a novelty and generalization w.r.t. the previous literature, and takes into account the fact that the desired bearings can be communicated to the agents at the beginning of the task (e.g., by a high-level planner/supervisor), while the measuring graph is necessarily a function of the actual state of the multi-agent system. Clearly, the case $\mathcal{E}_d = \mathcal{E}_m$ remains still possible in our framework.

As an additional degree of freedom, we also consider the possibility to have a distinct *control graph* $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$ (also assumed m.g.p.r. for the scalability of the control algorithm). This graph defines the set of relative bearings needed by the controller and compared with their corresponding desired values in order to produce the intended control action.

In literature, it is again typically assumed that $\mathcal{E}_c = \mathcal{E}_m$, as for example in [10] (where, however, also the relative distances are used in the control law). Nevertheless, in the general situation, a controllers may need a particular control graph $\mathcal{G}_c$ in order to implement its action, and this control graph may differ from the current measurement graph. In fact, in the following we will present a controller based on only bearings (no distances) measurements, and assuming that $\mathcal{G}_c$ belongs to a particular class of graphs.

From a theoretical standpoint, it is clear that a necessary condition for such a controller to be implementable is the possibility of "translating" $(\mathcal{G}_d, \phi^d)$ into a bearing formation over $\mathcal{G}_c$, i.e., $(\mathcal{G}_d, \phi^d) \to (\mathcal{G}_c, \phi^c)$, and, similarly, the possibility of translating the measurements $(\mathcal{G}_m, \phi^m) \to (\mathcal{G}_c, \phi^c)$ (owing to the parallel rigidity property). In order to perform such translations, we assume that the agents can communicate by means of a *communication graph* $\mathcal{G}_w = (\mathcal{V}, \mathcal{E}_w)$. We denote with $\mathcal{N}_i^w$ the set of communication neighbors of i and, obviously, $|\mathcal{N}_i^w|$ is $O(1)$ for every $i \in \mathcal{V}$. In other words at any time t agent i stores in its communication buffer a constant (small) number of values sent by every agent in $\mathcal{N}_i^w$. Clearly the union of all these graph constitutes the interaction graph, i.e., it holds that $\mathcal{E}_s = \mathcal{E}_d \cup \mathcal{E}_m \cup \mathcal{E}_c \cup \mathcal{E}_w$.

Problem 1 (Decentralized Bearing-only Control)**.** *Given a m.p.r. and non-degenerate desired bearing formation* $(\mathcal{G}_d, \boldsymbol{\phi}^d)$ *and m.g.p.r. measuring graph* $\mathcal{G}_m$*, design a control* $\boldsymbol{u}_i$*,* $\forall i \in \mathcal{V}$*, as a function of at most* $\{\phi_{ij}^m\}_{j \in \mathcal{N}_j^m}$ *and* $\{\phi_{ij}^d\}_{j \in \mathcal{N}_j^d}$*, such that*

$$\boldsymbol{\phi}^d - \mathrm{col}(\mathcal{A}(\boldsymbol{p}, \mathcal{G}_d)) \to \mathbf{0}. \tag{5}$$

III. STABILIZATION WITH COINCIDENT GRAPHS

In this section, and similarly to most previous works, we assume that all the graphs coincide with a particular control graph $\mathcal{G}_c$ defined in our case by

$$\mathcal{E}_c = \{(\iota, j) | j \in \mathcal{V}\backslash\{\iota\}\} \cup \{(\kappa, j) | j \in \mathcal{V}\backslash\{\iota, \kappa\}\}, \tag{6}$$

where $\iota, \kappa \in \mathcal{V}$ are called *center agents*. We will relax this assumption in Sec. IV.

First of all, for guaranteeing the well-posedness of the problem, we prove the following:

Proposition 1. *The graph* $\mathcal{G}_c$ *defined by* (6) *is m.g.p.r. for every choice of* ι, κ *with* $\iota \neq \kappa$*.*

Proof. We will show that $\mathcal{G}_c$ satisfies Theorem 2. Consider the following 3-set partition of $\mathcal{E}_c$: $\mathcal{E}_c^\iota = \{(\iota, j) | j \in \mathcal{V}\backslash\{\iota, \kappa\}\}$, $\mathcal{E}_c^\kappa = \{(\kappa, j) | j \in \mathcal{V}\backslash\{\iota, \kappa\}\}$, and $\{(\iota, \kappa)\}$. It is easy to see that $|\mathcal{E}_c| = |\mathcal{E}_c^\iota| + |\mathcal{E}_c^\kappa| + 1 = 2(n-2)+1 = 2n-3$.

Consider now any subset $\mathcal{E}_c'' \subset \mathcal{E}_c$, $\mathcal{E}_c \neq \emptyset$, which we also partition in 3 subsets: $A = \mathcal{E}_c'' \cap \mathcal{E}_c^\iota$, $B = \mathcal{E}_c'' \cap \mathcal{E}_c^\kappa$, and $C = \mathcal{E}_c'' \cap \{(\iota, \kappa)\}$. In order to satisfy Theorem 2 we need to show that

$$2(|V_A \cup V_B \cup V_C|) - 3 - (|A| + |B| + |C|) \geq 0, \tag{7}$$

where V_A, V_B, and V_C are the set of vertexes adjacent to the edges in A, B, and C respectively. If $A = B = \emptyset$ and $C = \{(\iota, \kappa)\}$ then the lhs of (7) reduces to $2 \cdot 2 - 3 - 1 = 0$, thus satisfying (7). Assume now, w.l.o.g., that $B \neq \emptyset$ and $|B| \geq |A|$. The worst case for the lhs of (7), i.e., the case in which the gap between $|V_A \cup V_B \cup V_C|$ and $|A| + |B| + |C|$ is maximum, arises when V_B contains all the vertexes of V_A different from ι. In this case $|V_A \cup V_B \cup V_C| = |B| + 1 + \sigma$, where $\sigma = 1$ if $|A| \geq 1$ or $|C| = 1$ (and $\sigma = 0$ otherwise), i.e., $\sigma = \mathrm{sign}(|A| + |C|)$. Evaluating the lhs of (7) in this worst case, we obtain

$$2(|B| + 1 + \sigma) - 3 - (|A| + |B| + |C|) = \tag{8}$$

$$|B| - |A| + 2\,\mathrm{sign}(|A| + |C|) - 1 - |C| = \tag{9}$$

$$|B| - |A| + \mathrm{sign}(|A| + |C|) - 1 \geq 0 \tag{10}$$

where *i)* the step (9)$\to$ (10) exploits the fact that $\mathrm{sign}(|A| + |C|) - |C| \geq 0$, and *ii)* (10) is based on the fact that if $|A| \geq 1$ then $\mathrm{sign}(|A| + |C|) - 1 = 0$, while if $|A| = 0$ then $|B| - 1 \geq 0$. Any other case different from the reported worst case would be more favorable to the fulfillment of condition (7).

The proof is concluded by noting that the case $A \neq \emptyset$ and $|A| \geq |B|$ can be treated in an analogous way. $\square$

Assuming that $\boldsymbol{p}_\iota \neq \boldsymbol{p}_\kappa$, we define the ratio between the inter-distance among agent i and ι and that among the 2 center agents as:

$$\gamma_{\iota\kappa i}(\boldsymbol{p}) = \frac{\|\boldsymbol{p}_i - \boldsymbol{p}_\iota\|}{\|\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa\|},$$

For the sake of brevity, we will also write $\gamma_i(\boldsymbol{p})$ or simply γ_i whenever the meaning is clear from the context.

Proposition 2. *If the positions* $\boldsymbol{p}$ *of the agents are not* all-aligned *then the* $\gamma_{\iota\kappa i}(\boldsymbol{p})$ *can be computed using only bearings.*

Proof. If the 3 agents ι, κ, and i are not aligned then, applying the law of sines to the corresponding triangle, we obtain:

$$\gamma_i(\boldsymbol{p}) = \frac{\|\sin(\phi_{i\iota} - \phi_{i\kappa})\|}{\|\sin(\phi_{\kappa i} - \phi_{\kappa\iota})\|} = \frac{\|\sin(\phi_{i\iota} - \phi_{i\kappa})\|}{\|\sin(\phi_{i\kappa} - \phi_{\kappa\iota})\|},$$

where we omitted the dependence on $\boldsymbol{p}$ in the bearings for brevity. If instead ι, κ, and i are aligned then, since the agents are not all-aligned, there exists at least an agent j not aligned with ι, κ, and i. Rewriting γ_i as $\frac{\|\boldsymbol{p}_i - \boldsymbol{p}_\iota\|}{\|\boldsymbol{p}_\iota - \boldsymbol{p}_j\|}\frac{\|\boldsymbol{p}_j - \boldsymbol{p}_\iota\|}{\|\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa\|}$ and noting that, since the two triples (ι, i, j) and (ι, κ, j) are not aligned, we can apply the law of sines twice obtaining:

$$\gamma_i(\boldsymbol{p}) = \frac{\|\sin(\phi_{i\iota} - \phi_{ij})\|}{\|\sin(\phi_{ij} - \phi_{j\iota})\|}\frac{\|\sin(\phi_{j\iota} - \phi_{j\kappa})\|}{\|\sin(\phi_{j\kappa} - \phi_{\kappa\iota})\|}. \qquad \square$$

Consider a minimal non-degenerate bearing formation $(\mathcal{G}, \boldsymbol{\alpha})$. By virtue of Proposition 2 we can (uniquely) define the quantity:

$$\gamma_{\iota\kappa i}(\mathcal{G}, \boldsymbol{\alpha}) = \frac{\|\boldsymbol{q}_i^\alpha - \boldsymbol{q}_\iota^\alpha\|}{\|\boldsymbol{q}_\iota^\alpha - \boldsymbol{q}_\kappa^\alpha\|},$$

being $(\mathcal{G}, \boldsymbol{q}^\alpha)$ any realization of $(\mathcal{G}, \boldsymbol{\alpha})$. This allows us to define:

$$\gamma^d_{\iota\kappa i} = \gamma_{\iota\kappa i}(\mathcal{G}_d, \boldsymbol{\phi}^d) = \gamma^d_i.$$

In the following we will make use of the bearing vector expression as a function of the agent positions:

$$\boldsymbol{\beta}_{ij}(\boldsymbol{p}) = \begin{pmatrix} \cos\phi_{ij}(\boldsymbol{p}) \\ \sin\phi_{ij}(\boldsymbol{p}) \end{pmatrix} = \frac{\boldsymbol{p}_j - \boldsymbol{p}_i}{\|\boldsymbol{p}_j - \boldsymbol{p}_i\|} \tag{11}$$

and of its derivative

$$\dot{\boldsymbol{\beta}}_{ij}(\boldsymbol{p}) = (I - \boldsymbol{\beta}_{ij}(\boldsymbol{p})\boldsymbol{\beta}^T_{ij}(\boldsymbol{p}))\frac{\dot{\boldsymbol{p}}_j - \dot{\boldsymbol{p}}_i}{\|\boldsymbol{p}_j - \boldsymbol{p}_i\|}. \tag{12}$$

A useful relation, easily proven using (11), is the following:

$$\gamma_{\iota\kappa i}(\boldsymbol{p})\boldsymbol{\beta}_{\iota i}(\boldsymbol{p}) - \boldsymbol{\beta}_{\iota\kappa}(\boldsymbol{p}) \propto \boldsymbol{\beta}_{\kappa i}(\boldsymbol{p}). \tag{13}$$

We are now ready to present the proposed bearing-only control law that constitutes one of the major contributions of this work.

Proposition 3. *Consider a control graph $\mathcal{G}_c$ defined by (6). If $\mathcal{G}_d = \mathcal{G}_m = \mathcal{G}_w = \mathcal{G}_c$ then, for any two positive gains k_1, k_2, the following control law solves Problem 1:*

$$\boldsymbol{u}_\iota = \boldsymbol{0} \tag{14}$$

$$\boldsymbol{u}_\kappa = k_1 \sin\left(\phi^d_{\kappa\iota} - \phi_{\kappa\iota}(\boldsymbol{p})\right)\boldsymbol{\beta}^\perp_{\kappa\iota}(\boldsymbol{p}) \tag{15}$$

$$\boldsymbol{u}_i = k_2\left(\gamma^d_i\boldsymbol{\beta}^d_{i\iota} - \gamma_i(\boldsymbol{p})\boldsymbol{\beta}_{i\iota}(\boldsymbol{p})\right) \quad \forall i \in \mathcal{V}\backslash\{\iota, \kappa\}. \tag{16}$$

Proof. In the following we omit the dependency from $\boldsymbol{p}$ for brevity. Firstly note that the inter-distance $\|\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa\|$ between the agents ι and κ is constant. In fact $\frac{d}{dt}(\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa)^T(\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa) = 2(\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa)^T(\dot{\boldsymbol{p}}_\iota - \dot{\boldsymbol{p}}_\kappa) = 2k_1\sin(\phi^d_{\kappa\iota} - \phi_{\kappa\iota})(\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa)^T\boldsymbol{\beta}^\perp_{\kappa\iota} \propto (\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa)^T(\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa)^\perp = \boldsymbol{0}$, where we applied (11).

We now prove that (5) holds by using the bearing vector dynamics in (12). The closed-loop dynamics of the bearing $\boldsymbol{\beta}_{\kappa\iota}$ with control (14) and (15) is $\dot{\boldsymbol{\beta}}_{\kappa\iota} = (I - \boldsymbol{\beta}_{\kappa\iota}\boldsymbol{\beta}^T_{\kappa\iota})\frac{\dot{\boldsymbol{p}}_\kappa - \dot{\boldsymbol{p}}_\iota}{\|\boldsymbol{p}_\kappa - \boldsymbol{p}_\iota\|} = \frac{k_1}{\|\boldsymbol{p}_\kappa - \boldsymbol{p}_\iota\|}\sin(\phi^d_{\kappa\iota} - \phi_{\kappa\iota})(I - \boldsymbol{\beta}_{\kappa\iota}\boldsymbol{\beta}^T_{\kappa\iota})\boldsymbol{\beta}^\perp_{\kappa\iota}$, so that:

$$\dot{\boldsymbol{\beta}}_{\kappa\iota} = -\frac{k_1}{\|\boldsymbol{p}_\kappa - \boldsymbol{p}_\iota\|}\sin(\phi^d_{\kappa\iota} - \phi_{\kappa\iota})\boldsymbol{\beta}^\perp_{\kappa\iota}. \tag{17}$$

Since $\|\boldsymbol{p}_\kappa - \boldsymbol{p}_\iota\| = \|\boldsymbol{p}_\kappa(0) - \boldsymbol{p}_\iota(0)\| = \text{const} > 0$, (17) implies that $\boldsymbol{\beta}_{\kappa\iota} \to \boldsymbol{\beta}^d_{\kappa\iota}$ for any initial condition but the zero-measure case $\phi_{\kappa\iota} \equiv \phi^d_{\kappa\iota} \mod \pi$.

We now consider $i \in \mathcal{V}\backslash\{\iota, \kappa\}$. The closed-loop dynamics of the vector $\boldsymbol{p}_i - \boldsymbol{p}_\iota$ with control (14,16) is $\dot{\boldsymbol{p}}_i - \dot{\boldsymbol{p}}_\iota = \frac{k_2}{\|\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa\|}(\|\boldsymbol{p}_\iota - \boldsymbol{p}_\kappa\|\gamma^d_i\boldsymbol{\beta}^d_{\iota i} - \|\boldsymbol{p}_i - \boldsymbol{p}_\iota\|\boldsymbol{\beta}_{\iota i})$ and then:

$$\frac{d}{dt}(\boldsymbol{p}_i - \boldsymbol{p}_\iota) = \bar{k}_i(\bar{\boldsymbol{p}}_i - (\boldsymbol{p}_i - \boldsymbol{p}_\iota)),$$

where $\bar{\boldsymbol{p}}_i = \|\boldsymbol{p}_\iota(0) - \boldsymbol{p}_\kappa(0)\|\gamma^d_i\boldsymbol{\beta}^d_{\iota i}$ is a constant non-zero vector directed as $\boldsymbol{\beta}^d_{\iota i}$ and $\bar{k}_i = \frac{k_2}{\|\boldsymbol{p}_\iota(0) - \boldsymbol{p}_\kappa(0)\|}$ is a positive constant gain. Therefore we can conclude that $\boldsymbol{p}_i - \boldsymbol{p}_\iota \to \bar{\boldsymbol{p}}_i$, which implies that $\boldsymbol{\beta}_{\iota i} \to \boldsymbol{\beta}^d_{\iota i}$, using (11).

Finally consider the vector $\boldsymbol{\beta}_{\kappa i} \propto \boldsymbol{p}_i - \boldsymbol{p}_\kappa = (\boldsymbol{p}_i - \boldsymbol{p}_\iota) - (\boldsymbol{p}_\kappa - \boldsymbol{p}_\iota)$ which, from the previous analysis, must converge to $\bar{\boldsymbol{p}}_i - \|\boldsymbol{p}_\kappa(0) - \boldsymbol{p}_\iota(0)\|\boldsymbol{\beta}^d_{\iota\kappa} = \|\boldsymbol{p}_\kappa(0) - \boldsymbol{p}_\iota(0)\|(\gamma^d_i\boldsymbol{\beta}^d_{\iota i} - \boldsymbol{\beta}^d_{\iota\kappa}) \propto \boldsymbol{\beta}^d_{\kappa i}$, where the proportionality derives from (13). Therefore $\boldsymbol{\beta}_{\kappa i} \to \boldsymbol{\beta}^d_{\kappa i}$, thus concluding the proof. □

From the proof of Proposition 3 we can conclude that, under the action of control law (14-16), agent ι remains stationary, agent κ rotates around ι, and any other agent moves following a straight path. The final location of the agents is completely determined by the desired bearing formation $(\mathcal{G}_d, \boldsymbol{\phi}^d)$, the initial position of agent ι, i.e., $\boldsymbol{p}_\iota(0)$, and the initial distance between agents ι and κ, i.e., $\|\boldsymbol{p}_\iota(0) - \boldsymbol{p}_\kappa(0)\|$, which, moreover, are constants of motion along the system trajectories. These properties avoid indefinite contraction, expansion or translation of the whole formation under the action of the controller.

IV. STABILIZATION WITH GENERAL GRAPHS

In order to implement (15-16), each agent $i \neq \iota$ needs to retrieve the two vectors $\gamma^d_i\boldsymbol{\beta}^d_{i\iota}$ and $\gamma_i(\boldsymbol{p})\boldsymbol{\beta}_{i\iota}(\boldsymbol{p})$. This holds also for $i = \kappa$, since the two scalar quantities employed in (15), $\phi^d_{\kappa\iota}$ and $\phi_{\kappa\iota}(\boldsymbol{p})$, are straightforwardly related to $\gamma^d_\kappa\boldsymbol{\beta}^d_{\kappa\iota}$ and $\gamma_\kappa(\boldsymbol{p})\boldsymbol{\beta}_{\kappa\iota}(\boldsymbol{p})$ (note that, in particular, $\gamma^d_\kappa = \gamma_\kappa(\boldsymbol{p}) \equiv 1$). If both $\mathcal{G}_d$, $\mathcal{G}_m$ $\mathcal{G}_w$ coincide with the particular $\mathcal{G}_c$ defined by (6), then each agent can easily compute the needed quantities from locally available information.

Now assume that $\mathcal{G}_d$ and $\mathcal{G}_m$ do not coincide with the particular $\mathcal{G}_c$ defined in (6). In order to address this situation, we will now present an estimator that can be exploited to obtain both $\gamma^d_i\boldsymbol{\beta}^d_{i\iota}$ and $\gamma_i(\boldsymbol{p})\boldsymbol{\beta}_{i\iota}(\boldsymbol{p})$.

A. Scale-free Position Estimator

Consider a minimally parallel rigid formation $(\mathcal{G}, \boldsymbol{q})$ and the following quantity to be estimated from $\mathcal{A}(\boldsymbol{q}, \mathcal{G})$:

$$\gamma_{\iota\kappa i}(\boldsymbol{q})\boldsymbol{\beta}_{ij}(\boldsymbol{q}) = \frac{\boldsymbol{q}_i - \boldsymbol{q}_\iota}{\|\boldsymbol{q}_\iota - \boldsymbol{q}_\kappa\|}, \tag{18}$$

i.e., the position of the agent i in a frame centered on $\boldsymbol{q}_\iota$ and suitably 'dilated' in order to have $\|\boldsymbol{q}_\iota - \boldsymbol{q}_\kappa\| = 1$. Let the estimate of (18) be $\boldsymbol{\xi}_i \in \mathbb{R}^2$, and consider the following quantities:

$$\boldsymbol{\xi}_{ij} = \boldsymbol{\xi}_j - \boldsymbol{\xi}_i, \quad \hat{\boldsymbol{\xi}}_{ij} = \frac{\boldsymbol{\xi}_{ij}}{\|\boldsymbol{\xi}_{ij}\|}, \quad \zeta_{ij} = \text{atan2}(\xi^y_{ij}, \xi^x_{ij}).$$

These represent the estimates of $\frac{\boldsymbol{q}_j - \boldsymbol{q}_i}{\|\boldsymbol{q}_\iota - \boldsymbol{q}_\kappa\|}$, $\boldsymbol{\beta}_{ij}(\boldsymbol{q})$, and $\phi_{ij}(\boldsymbol{q})$, respectively.

Since the actual 'real' quantities available for the estimation algorithm are only the bearing $\mathcal{A}(\boldsymbol{q}, \mathcal{G})$, we consider the following error driving the estimation update:

$$\boldsymbol{e}(\boldsymbol{\xi}, \boldsymbol{q}) = \text{col}(\mathcal{A}(\boldsymbol{\xi}, \mathcal{G})) - \text{col}(\mathcal{A}(\boldsymbol{q}, \mathcal{G})) \in \mathbb{R}^{|\mathcal{E}|},$$

where we recall that $\text{col}(\mathcal{A}(\boldsymbol{\xi}, \mathcal{G}))$ is the lexicographical-ordered stack of bearing estimates ζ_{ij}, for $(i,j) \in \mathcal{E}$.

It is possible to prove (see, e.g., [10]) that the Jacobian of $\text{col}(\mathcal{A}(\boldsymbol{\xi}, \mathcal{G}))$ is given by:

$$\nabla\text{col}(\mathcal{A}(\boldsymbol{\xi}, \mathcal{G})) = -D(\boldsymbol{\xi})^{-2}R(\boldsymbol{\xi}),$$

where $D(\boldsymbol{\xi}) \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$ is a diagonal matrix made of all the distances $\|\boldsymbol{\xi}_{ij}\|$ for every $(i,j) \in \mathcal{E}$, taken in the lexicographical order, and $R(\boldsymbol{\xi}) \in \mathbb{R}^{|\mathcal{E}|\times 2|\mathcal{V}|}$ is the bearing-constrained rigidity matrix computed at $\boldsymbol{\xi}$.

The objective of the estimation algorithm can be recast as the minimization of the scalar function

$$e(\boldsymbol{\xi},\boldsymbol{q}) = \frac{\boldsymbol{e}(\boldsymbol{\xi},\boldsymbol{q})^T\boldsymbol{e}(\boldsymbol{\xi},\boldsymbol{q}) + k_3\boldsymbol{\xi}_\iota^T\boldsymbol{\xi}_\iota + k_4(\boldsymbol{\xi}_\kappa^T\boldsymbol{\xi}_\kappa - 1)^2}{2}, \quad (19)$$

where the terms $k_3\boldsymbol{\xi}_\iota^T\boldsymbol{\xi}_\iota$ and $k_4(\|\boldsymbol{\xi}_\kappa\| - 1)^2$ account for the additional constraints: $\frac{\boldsymbol{q}_\iota - \boldsymbol{q}_\iota}{\|\boldsymbol{q}_\iota - \boldsymbol{q}_\kappa\|} = \boldsymbol{0}$ and $\frac{\|\boldsymbol{q}_\iota - \boldsymbol{q}_\kappa\|}{\|\boldsymbol{q}_\iota - \boldsymbol{q}_\kappa\|} = 1$, respectively. The positive gains k_3, k_4 can be used to finely tune the estimator behavior.

Minimization of (19) can be achieved by following the antigradient of $e(\boldsymbol{\xi},\boldsymbol{q})$, i.e., by choosing:

$$\dot{\boldsymbol{\xi}} = -(\nabla\boldsymbol{e})^T\boldsymbol{e} = \\ = R(\boldsymbol{\xi})^T D(\boldsymbol{\xi})^{-2}\boldsymbol{e} - k_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\xi}_\iota\\ \boldsymbol{0}\end{pmatrix} - k_4\begin{pmatrix}\boldsymbol{0}\\ (\boldsymbol{\xi}_\kappa^T\boldsymbol{\xi}_\kappa - 1)\hat{\boldsymbol{\xi}}_\kappa\\ \boldsymbol{0}\end{pmatrix}, \quad (20)$$

where the terms $\boldsymbol{\xi}_\iota$ and $(\boldsymbol{\xi}_\kappa^T\boldsymbol{\xi}_\kappa - 1)\hat{\boldsymbol{\xi}}_\kappa$ appear at the ι-th and κ-th entry pairs of $\dot{\boldsymbol{\xi}}$, respectively.

Expanding (20) for the ι-th, κ-th, and i-th agent, we obtain the following local estimation update rules:

$$\dot{\boldsymbol{\xi}}_\iota = \sum_{j\in\mathcal{N}_\iota}\frac{1}{\|\boldsymbol{\xi}_{\kappa j}\|}\hat{\boldsymbol{\xi}}_{\iota j}^\perp(\zeta_{\iota j} - \phi_{\iota j}) - k_4\boldsymbol{\xi}_\iota \quad (21a)$$

$$\dot{\boldsymbol{\xi}}_\kappa = \sum_{j\in\mathcal{N}_\kappa}\frac{1}{\|\boldsymbol{\xi}_{\kappa j}\|}\hat{\boldsymbol{\xi}}_{\kappa j}^\perp(\zeta_{\kappa j} - \phi_{\kappa j}) - k_4(\boldsymbol{\xi}_\kappa^T\boldsymbol{\xi}_\kappa - 1)\hat{\boldsymbol{\xi}}_\kappa \quad (21b)$$

$$\dot{\boldsymbol{\xi}}_i = \sum_{j\in\mathcal{N}_i}\frac{1}{\|\boldsymbol{\xi}_{ij}\|}\hat{\boldsymbol{\xi}}_{ij}^\perp(\zeta_{ij} - \phi_{ij}) \quad \forall i \in \mathcal{V}\backslash\{\iota,\kappa\}, \quad (21c)$$

where $\mathcal{N}_k = \{j \in \mathcal{V} \,|\, (k,j) \in \mathcal{E}\}$ $k \in \mathcal{V}$.

Thanks to the parallel rigidity of $\mathcal{G}$ there is only a 2d-point formation which is realizes the bearings $\mathcal{A}(\boldsymbol{q},\mathcal{G})$ and satisfies also the additional constraints on translation ($\boldsymbol{\xi}_\iota = \boldsymbol{0}$) and dilation ($\|\boldsymbol{\xi}_\iota - \boldsymbol{\xi}_\kappa\| = 0$). Therefore by following the antigradient law we ensure the local convergence of $\zeta_{ij} \to \phi_{ij}$ and $\hat{\boldsymbol{\xi}}_i \to \boldsymbol{\beta}_{ij}$ for any $(i,j) \in \mathcal{V}\times\mathcal{V}$.

The estimator (21) is intrinsically decentralized (and then scalable) since, in order to compute $\dot{\boldsymbol{\xi}}_i$, every agent i needs only to receive the current estimates $\{\boldsymbol{\xi}_j\}_{j\in\mathcal{N}_i}$ from its neighbors, as well as the locally available relative bearing quantities which are measured.

B. Estimation of the Control Quantities

We obtain a decentralized estimation of $\gamma_i^d\boldsymbol{\beta}_{i\iota}^d$ and $\gamma_i(\boldsymbol{p})\boldsymbol{\beta}_{i\iota}(\boldsymbol{p})$ by using in parallel two estimators of the form (21) whose estimates are denoted with a $\boldsymbol{\xi}^d$ and $\boldsymbol{\xi}^m$, respectively.

In order to implement these two parallel estimators the agents need to communicate their estimates to the neighbors in $\mathcal{G}_d$ and $\mathcal{G}_m$, respectively. For this reason the communication graph should at least meet the condition $\mathcal{E}_w = \mathcal{E}_d \cup \mathcal{E}_m$. This objective can be reached with any connected communication network by using some proper routing strategy. Notice that the pair of communicating links in $\mathcal{E}_w$ is still scalable, being $2O(n) = O(n)$.

Notice also the similarity between the control law presented in [10] and (21). Nevertheless, while in [10] the implementation of the control law requires to also measure the actual agent inter-distances, the implementation of the estimates (21) only requires relative bearings.

The estimates $\boldsymbol{\xi}^d$ and $\boldsymbol{\xi}^m$ can then be plugged in the control law (15-16) for implementing the proposed controller.

Fact 1. *Consider control graph $\mathcal{G}_c$ defined by* (6) *and any two positive gains k_1, k_2. The following controller can be used in place of (14-16) with any m.g.p.r. parameter graph $\mathcal{G}_d$ and measuring graph $\mathcal{G}_m$:*

$$\boldsymbol{u}_\iota = \boldsymbol{0} \quad (22)$$

$$\boldsymbol{u}_\kappa = k_1\sin\left(\zeta_{\kappa\iota}^d - \zeta_{\kappa\iota}^m\right)\hat{\boldsymbol{\xi}}_{\kappa\iota}^{m\perp} \quad (23)$$

$$\boldsymbol{u}_i = k_2(\boldsymbol{\xi}_{i\iota}^d - \boldsymbol{\xi}_{i\iota}^m) \quad \forall i \in \mathcal{V}\backslash\{\iota,\kappa\}. \quad (24)$$

From a practical standpoint, if the dynamics of the controlled system is slow w.r.t. the dynamics of the estimator, then the estimate will have an acceptable small error even if the bearings $\mathcal{A}$ are not constant. Note that is a standard assumption which has been used also in many other works, like [17], [16].

V. SIMULATIONS

We conducted a numerical study in order to validate the proposed approach. We describe here 2 significant set of simulations.

In the first simulation (reported in Fig. 2) 10 agents start from 10 distinct initial positions (denoted with red squares in all the plots) which are approximately arranged on a circle and are tasked to reach a desired bearing formation which also results in a circular formation (with no specified size and translation). The final locations of the agents are denoted with green circles in the plots.

In the first case the measure, parameter, and control graphs coincide, and then this case constitutes a sort of baseline to study the performances of the algorthm. Plot 2a shows the trajectories of the agents in this case. As expected, all the trajectories are straight lines except for the trajectory of agent ι, which is stationary in the middle, and κ, which follows an arc of circumference (and then keeps fixed its distance w.r.t agent ι). The measure graph (coincident with the control graph) is also shown by means of dashed lines connecting the final locations. Plot 2c shows the corresponding bearing formation error norm, which exponentially vanishes for all the edges of the parameter graph.

In the second case we have chosen randomly two different measure and parameter graphs, and therefore the control uses the estimates provided by the estimation algorithm presented in Sec. IV. The estimates are also initialized with random values. Plot 2b provides the agent trajectories, that are not perfectly straight anymore, due to the initial estimation errors and their evolution. Plot 2d shows the corresponding bearing formation error norm, whose trend is similar to the previous one but presents some intermediate oscillations. Finally Plots 2e and 2f show the norm of the estimation errors of the vectors composing $\boldsymbol{\xi}^m$ and $\boldsymbol{\xi}^d$, respectively. All the components of $\boldsymbol{\xi}^d$ monotonically vanish, thanks to the fact that the desired quantities are constant. The

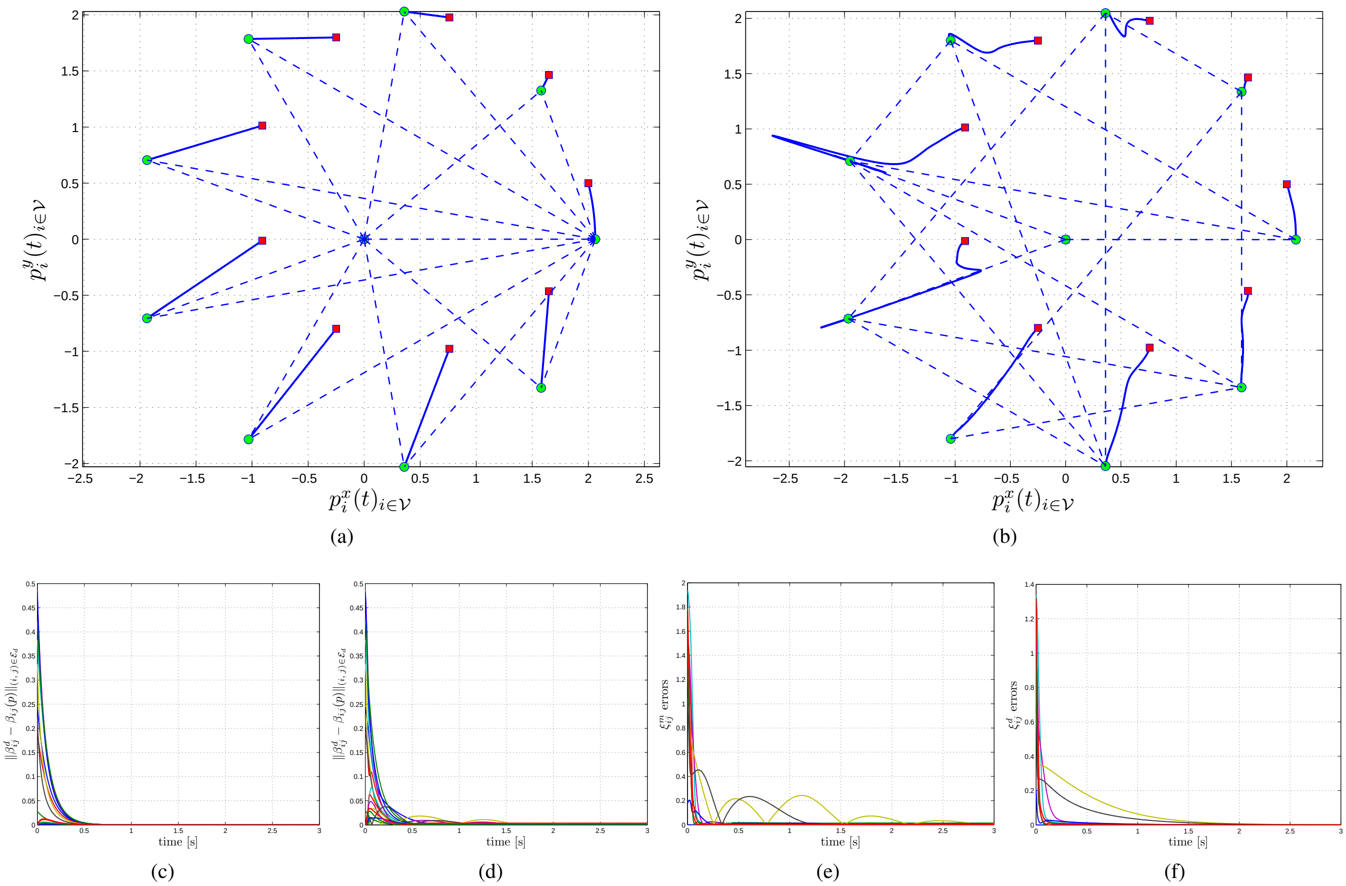

Fig. 2: First simulation: case of circular desired bearing formation. Red squares and green circles represent the initial and final positions, respectively. Solid lines represent the agent trajectories and dashed lines represent the measure graph used $\mathcal{G}_m$. First case (coincident graphs): (a): trajectories, (c): regulation error. Second case (non-coincident graphs) (b): trajectories, (d): regulation error norm, (e): measure estimation error norm, (f): parameter estimation error norm.

components of $\boldsymbol{\xi}^m$ present a more oscillating behavior, due to the fact that the actual bearings are changing during the agent motion. Nevertheless all the error eventually vanish with an exponential trend.

The second simulation, depicted in Fig. 3 presents the same pattern (control without and with estimation) applied to a different desired bearing formation, which recall a pseudo-lattice formation, (again, depicted with green circles)

The behavior of the trajectories, regulation and estimation errors is totally similar to the previous case. A more pronounced estimation error in this case is due to the fact that we set on purpose a much worse initial state for the estimators, in order to validate even more the robustness of the proposed method. Nevertheless also in this case the application of the method allows to successfully achieve the desired bearing formation, despite the minimality of the information available to the agents.

VI. CONCLUSIONS

In this work we investigated the relationship between scalability, minimality and rigidity, and its application to cooperative control. In particular, we proposed a distributed control strategy that stabilizes a formation described with bearing constraints, and that only requires bearing measurements (instead of bearing plus distance). We also considered the possibility of having different graphs modeling the interaction network in order to explicitly take into account the conceptual difference between sensing, communication, control, and parameters stored in the network.

Due to the relevant role of rigidity in formation control, it would be interesting to study a control strategy which ensures the parallel rigidity maintenance during motion, despite of sensing and communication constraints, while still allowing for a time-varying topology, as done in [7] for the distance-based rigidity case. The extension of this framework to different spaces and kind of relative quantities, as well as time-varying graphs, also constitutes a promising future development of this work.

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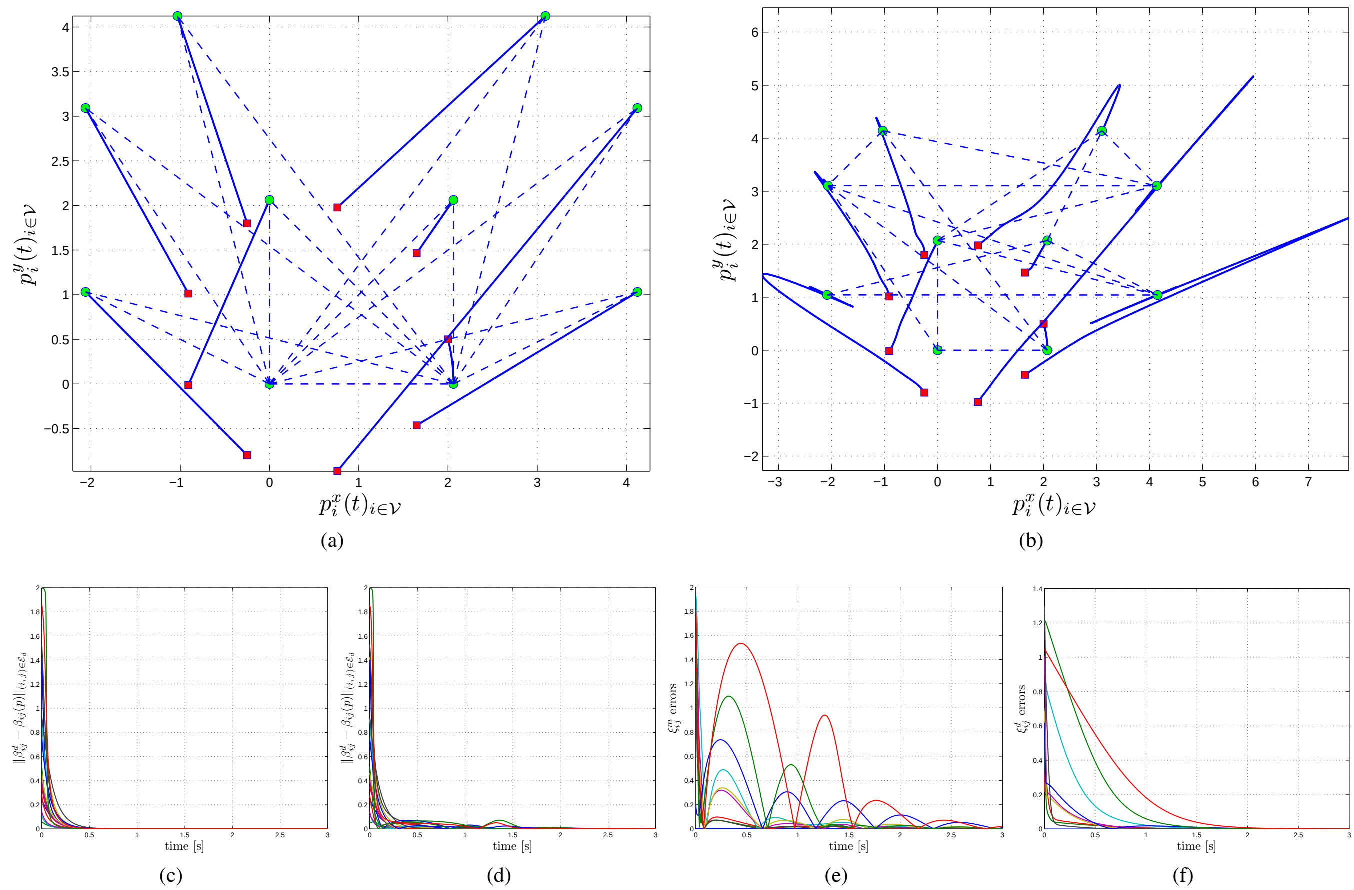

Fig. 3: Second simulation: case of pseudo-lattice desired bearing formation. Red squares and green circles represent the initial and final positions, respectively. Solid lines represent the agent trajectories and dashed lines represent the measure graph used $\mathcal{G}_m$. First case (coincident graphs): (a): trajectories, (c): regulation error. Second case (non-coincident graphs) (b): trajectories, (d): regulation error norm, (e): measure estimation error norm, (f): parameter estimation error norm.

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