Part I

Inference in first-order logic

A brief history of reasoning

450B.C.	Stoics	propositional logic, inference (maybe)
322B.C.	Aristotle	"syllogisms" (inference rules), quantifiers
1565	Cardano	probability theory (propositional logic + uncertainty)
1847	Boole	propositional logic (again)
1879	Frege	first-order logic
1922	Wittgenstein	proof by truth tables
1930	Gödel	∃ complete algorithm for FOL :-)
1930	Herbrand	complete algorithm for FOL (*reduce to propositional*)
1931	Gödel	¬∃ complete algorithm for arithmetic :-(
1960	Davis/Putnam	"practical" algorithm for propositional logic
1965	Robinson	"practical" algorithm for FOL — *resolution*

Outline

Contents

1 Reduction to propositional inference

Universal instantiation (UI)

Definition 1. Inference rule: Every instantiation of a universally quantified sentence is entailed by it.

$$\frac{\forall v\ \alpha}{\textsc{Subst}(\{v/g\},\alpha)}$$

for any *variable* v and *ground term* g (term without variable)

Example 2. $\forall x\ King(x) \land Greedy(x) \Rightarrow Evil(x)$ yields
$King(John) \land Greedy(John) \Rightarrow Evil(John) \quad [g = John]$ $King(Richard) \land Greedy(Richard) \Rightarrow Evil(Richard) \quad [g = Richard]$ $King(Father(John)) \land Greedy(Father(John)) \Rightarrow Evil(Father(John)) \quad [g = Father(John)]$

Existential instantiation (EI)

Definition 3. Inference rule: For any sentence α, variable v, and constant symbol k *that does not appear elsewhere in the knowledge base*:

$$\frac{\exists v\ \alpha}{\textsc{Subst}(\{v/k\},\alpha)}$$

Example 4. $\exists x\ Crown(x) \land OnHead(x, John)$ yields
$Crown(C_1) \land OnHead(C_1, John) \quad [k = C_1]$
provided C_1 is a new constant symbol, called a *Skolem constant*.

EI and UI

UI can be applied several times to *add* new sentences; the new KB is logically equivalent to the previous one.

$KB = \{\forall x\ P(x)\}$ $newKB = \{\forall x\ P(x), P(Richard)\}$...

EI can be applied once to *replace* the existential sentence; the new KB is *not* equivalent to the previous one!

$KB = \{\exists x\ P(x)\}$ $newKB = \{P(SkolemConstant)\}$...

Reduction to propositional inference

Example 5. Suppose the KB contains just the following:

$\forall x\ King(x) \land Greedy(x) \Rightarrow Evil(x)$ $King(John)$ $Greedy(John)$ $Brother(Richard, John)$

Instantiating the universal sentence in *all possible* ways, we have:

$King(John) \land Greedy(John) \implies Evil(John)$ $King(Richard) \land Greedy(Richard) \implies Evil(Richard)$ $King(John)$ $Greedy(John)$ $Brother(Richard, John)$

The new KB is *propositionalised*: proposition symbols are $King(John),\ Greedy(John),\ Evil(John), King(Richard) \cdots$

Reduction contd.

A sentence α is entailed by an FOL KB if and only if it is entailed by a *finite* subset of the propositional KB.

For $n = 0$ **to** $maxDepth$ create a $KB_{prop,n}$ by reduction with depth-m terms $(m = 1 \ldots n)$ **if** α_{prop} is entailed by $KB_{prop,n}$ (i.e. $KB_{prop,n} \models \alpha_{prop}$) **then** STOP.

If $KB \models \alpha$ then the algorithm stops but if $KB \not\models \alpha$, the algorithm does not stop. If KB contains a function then the terms can have infinite depth ($maxDepth = \infty$): $Father(Father(Father(.....Father(John)....)))$

NO. Entailement in FOL is semi-decidable Turing(1936) Church(1936)

Problems with propositionalization

Propositionalization seems to generate lots of irrelevant sentences.

Example 6. From $KB = \{$
$\forall x\ King(x) \land Greedy(x) \implies Evil(x), King(John), \forall y\ Greedy(y), Brother(Richard, John)\}$
it seems obvious that $Evil(John)$, but propositionalization produces lots of facts such as $Greedy(Richard)$ that are irrelevant.

With p k-ary predicates and n constants, there are $p \cdot n^k$ instantiations. With function symbols, it gets much much worse!

2 Unification and lifting

Inference rules for FOL

We can get the inference immediately on the FOL KB.

We use *rules* from propositional logic that are *lifted* like the *Generalised Modus Ponens*. Then we can *update* FC, BC and Resolution for FOL :-)

The problem is the *instantiation of the variables*. We need some new operators:

1. *Substitution* (SUBST)
2. *Unification* (UNIFY)

to define the inference rules and to choose "clever" instantiations

Unification

Definition 7. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Unification

Definition 8. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 9.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	
$Knows(John, x)$	$Knows(y, Bill)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, Elizabeth)$	

Unification

Definition 10. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 11.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Bill)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, Elizabeth)$	

Unification

Definition 12. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 13.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Bill)$	$\{x/Bill, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, Elizabeth)$	

Unification

Definition 14. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 15.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Bill)$	$\{x/Bill, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, Elizabeth)$	

Unification

Definition 16. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 17.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Bill)$	$\{x/Bill, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, Elizabeth)$	$fail$

Unification

Definition 18. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 19.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Bill)$	$\{x/Bill, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, Elizabeth)$	$fail$

Standardizing apart eliminates overlap of variables (renaming of variables) $Knows(z_{17}, Elizabeth)$

Unification

Definition 20. Let p, q be two sentences of FOL, the result of the *Unification* is:

$$\text{UNIFY}(p, q) = \theta$$

where θ is a substitution on variables of α, β such that:

$$\text{SUBST}(\theta, p) = \text{SUBST}(\theta, q)$$

Example 21.

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	$\{x/Jane\}$
$Knows(John, x)$	$Knows(y, Bill)$	$\{x/Bill, y/John\}$
$Knows(John, x)$	$Knows(y, Mother(y))$	$\{y/John, x/Mother(John)\}$
$Knows(John, x)$	$Knows(x, Elizabeth)$	$fail$

Standardizing apart eliminates overlap of variables (renaming of variables) $Knows(z_{17}, Elizabeth)$ $Knows(John, x)$ | $Knows(z_{17}, Elizab$

Generalized Modus Ponens (GMP)

Definition 22.

$$\frac{p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}',\ \ (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q)}{\text{SUBST}(\theta, q)}$$

where $\theta = \text{UNIFY}(p_i, p_i')$ for all i.
Variables are universally quantified.

Example 23.

$$\frac{King(John),\ \ Greedy(y)\ \ (King(x) \land Greedy(x) \Rightarrow Evil(x))}{\text{SUBST}(\theta, Evil(x))}$$

What is θ? What is $\text{SUBST}(\theta, Evil(x))$?

Generalized Modus Ponens (GMP)

Definition 24.

$$\frac{p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}',\ \ (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q)}{\text{SUBST}(\theta, q)}$$

where $\theta = \text{UNIFY}(p_i, p_i')$ for all i.
Variables are universally quantified.

Example 25.

$$\frac{King(John),\ \ Greedy(y)\ \ (King(x) \land Greedy(x) \Rightarrow Evil(x))}{\text{SUBST}(\theta, Evil(x))}$$

What is θ? What is $\text{SUBST}(\theta, Evil(x))$? $\theta = \text{UNIFY}(King(John), King(x)) = \text{UNIFY}(Greedy(y), Greedy(x))$ $\theta = \{x/John, y/John\}$

Generalized Modus Ponens (GMP)

Definition 26.

$$\frac{p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}',\ \ (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q)}{\text{SUBST}(\theta, q)}$$

where $\theta = \text{UNIFY}(p_i, p_i')$ for all i.
Variables are universally quantified.

Example 27.

$$\frac{King(John),\ \ Greedy(y)\ \ (King(x) \land Greedy(x) \Rightarrow Evil(x))}{\textsc{Subst}(\theta, Evil(x))}$$

What is θ? What is $\textsc{Subst}(\theta, Evil(x))$? $\theta = \textsc{Unify}(King(John), King(x)) = \textsc{Unify}(Greedy(y), Greedy(x))$ $\theta = \{x/John, y/John\}$ $\textsc{Subst}(\theta, Evil(x)) = Evil(John)$

GMP is sound

GMP is sound. We have:

$$p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}'\ \ (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q) \models \textsc{Subst}(\theta, q)$$

provided $\theta = \textsc{Unify}(p_i, p_i')$ for all i.

1. for a sentence p (with variables universally quantified), we have for all θ:

$$p \models \textsc{Subst}(\theta, p)$$

2. because of 1 and UI, from $p_1{}',\ \ p_2{}',\ \ldots,\ p_n{}'$ we infer

$$\textsc{Subst}(\theta, p_1') \land \cdots \land \textsc{Subst}(\theta, p_n{}')$$

3. because of 1 and UI, from $p_1 \land p_2 \land \ldots \land p_n \Rightarrow q$ we infer

$$\textsc{Subst}(\theta, p_1) \land \cdots \land \textsc{Subst}(\theta, p_n) \Rightarrow \textsc{Subst}(\theta, q)$$

4. If $\theta = \textsc{Unify}(p_i, p_i')$ for all i then from 2 and 3 we infer $\textsc{Subst}(\theta, q)$ $\square$

3 Knowledge base: an example

Knowledge base: an example

Example 28. "One says that a person who gives good lectures about FOL to students is a good teacher. This group of people, studying at the ANU, have very good lectures about Logic and all of those lectures are given by Yannick who is a person." We must prove that "Yannick is a good teacher"

Knowledge base: an example

Example 29. "...a person who gives good lectures about FOL to students is a good teacher...":

Knowledge base: an example

Example 30. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \land Gives(x, y, z) \land GoodLecturesFOL(y) \land Students(z) \Rightarrow GoodTeacher(x)$

Knowledge base: an example

Example 31. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \land Gives(x, y, z) \land GoodLecturesFOL(y) \land Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic":

Knowledge base: an example

Example 32. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \land Gives(x, y, z) \land GoodLecturesFOL(y) \land Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People, x) \land GoodLecturesLogic(x)$

Knowledge base: an example

Example 33. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...":

Knowledge base: an example

Example 34. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People,x) \Rightarrow Gives(Yannick,x,People)$

Knowledge base: an example

Example 35. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People,x) \Rightarrow Gives(Yannick,x,People)$
"..This group of people, studying at the ANU":

Knowledge base: an example

Example 36. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People,x) \Rightarrow Gives(Yannick,x,People)$
"..This group of people, studying at the ANU": $Study(People,ANU)$

Knowledge base: an example

Example 37. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People,x) \Rightarrow Gives(Yannick,x,People)$
"..This group of people, studying at the ANU": $Study(People,ANU)$
"..Yannick who is a person..":

Knowledge base: an example

Example 38. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People, x) \Rightarrow Gives(Yannick, x, People)$
"..This group of people, studying at the ANU": $Study(People, ANU)$
"..Yannick who is a person..": $Person(Yannick)$

Knowledge base: an example

Example 39. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x, y, z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People, x) \wedge GoodLecturesLogic(x)$

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"..This group of people, studying at the ANU": $Study(People, ANU)$
"..Yannick who is a person..": $Person(Yannick)$
People that study at the ANU, are students

Knowledge base: an example

Example 40. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x, y, z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People, x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People, x) \Rightarrow Gives(Yannick, x, People)$
"..This group of people, studying at the ANU": $Study(People, ANU)$
"..Yannick who is a person..": $Person(Yannick)$
People that study at the ANU, are students $\forall x\ Study(x, ANU) \Rightarrow Students(x)$

Knowledge base: an example

Example 41. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x, y, z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People, x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People, x) \Rightarrow Gives(Yannick, x, People)$
"..This group of people, studying at the ANU": $Study(People, ANU)$
"..Yannick who is a person..": $Person(Yannick)$
People that study at the ANU, are students $\forall x\ Study(x, ANU) \Rightarrow Students(x)$
If the lectures about Logic are good, the lectures about FOL are good:

Knowledge base: an example

Example 42. "...a person who gives good lectures about FOL to students is a good teacher...": $\forall x \forall y \forall z\ Person(x) \wedge Gives(x, y, z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$

"This group of people.... have very good lectures about Logic": $\exists x\ Have(People, x) \wedge GoodLecturesLogic(x)$

"...all of their lectures are given by Yannick...": $\forall x\ GoodLecturesLogic(x) \wedge Have(People, x) \Rightarrow Gives(Yannick, x, People)$
"..This group of people, studying at the ANU": $Study(People, ANU)$
"..Yannick who is a person..": $Person(Yannick)$
People that study at the ANU, are students $\forall x\ Study(x, ANU) \Rightarrow Students(x)$
If the lectures about Logic are good, the lectures about FOL are good: $\forall x\ GoodLecturesLogic(x) \Rightarrow GoodLecturesFOL(x)$

4 Forward chaining

Restriction: Definite clauses

Definition 43. As said previously, a *Horn clause* is

- a predicate $P(..)$, or
- something like $P_1(..) \cdots P_n(..) \Rightarrow C(..)$
- $\equiv \neg P_1(..) \vee \cdots \vee \neg P_n(..) \vee C(..)$

($P_i(..)$ is a *premise* and $C(..)$ the *conclusion*)

The conclusion $C(..)$ can simply be "True" (i.e. no positive literal). $\neg P_1(..) \vee \cdots \vee \neg P_n(..)$ is also a Horn clause.

Definition 44. A *definite clause* is a Horn Clause with *exactly* one positive literal. $\neg P_1(..) \vee \cdots \vee \neg P_n(..) \vee C(..)$.

FC applies GMP and only works with Definite Clauses. The presented KB is a set of Definite Clauses.

Forward chaining: algorithm

```
function FOL-FC-ASK(KB, α) returns a substitution or false
  repeat until new is empty
     new ← { }
     for each sentence r in KB do
        (p1 ∧ ... ∧ pn ⇒ q) ← STANDARDIZE-APART(r)
        for each θ such that (p1 ∧ ... ∧ pn)θ = (p'1 ∧ ... ∧ p'n)θ
                    for some p'1, ..., p'n in KB
           q' ← SUBST(θ, q)
           if q' is not a renaming of a sentence already in KB or new then do
              add q' to new
              φ ← UNIFY(q', α)
              if φ is not fail then return φ
     add new to KB
  return false
```

Example

Example 45. S1: $\forall x \forall y \forall z\ Person(x) \wedge Gives(x,y,z) \wedge GoodLecturesFOL(y) \wedge Students(z) \Rightarrow GoodTeacher(x)$ S2: $\exists x\ Have(People,x) \wedge GoodLecturesLogic(x)$ We replace S2 thanks to EI, we introduce a new symbol $M1$ S2: $Have(People,M1) \wedge GoodLecturesLogic(M1)$

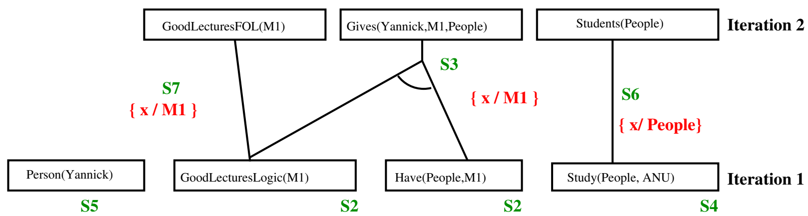
S3: $\forall x\ GoodLecturesLogic(x) \wedge Have(People,x) \Rightarrow Gives(Yannick,x,People)$
S4: $Study(People,ANU)$
S5: $Person(Yannick)$
S6: $\forall x\ Study(x,ANU) \Rightarrow Students(x)$
S7: $\forall x\ GoodLecturesLogic(x) \Rightarrow GoodLecturesFOL(x)$

Forward chaining: example

Example 46.

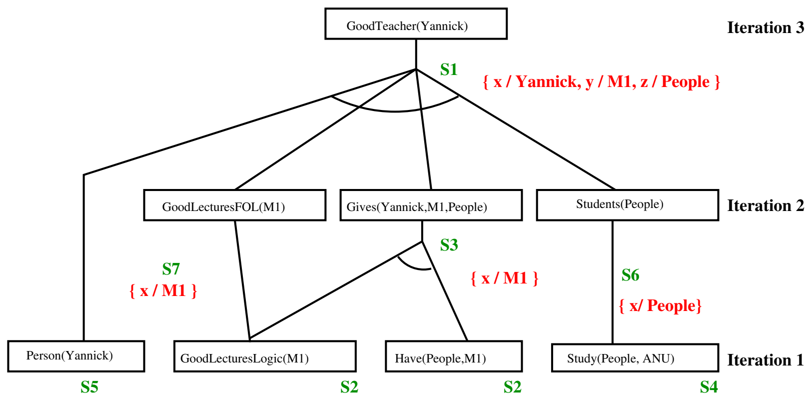
Forward chaining: example

Example 47.



Forward chaining: example

Example 48.



Properties of forward chaining

FC is *sound* for KB with Definite Clauses. (Use of GMP)

FC is *complete* for KB with Definite Clauses. (Brute force)

FC always terminates for KB = Datalog (no functions). FC *may not terminate* for KB with Definite Clauses.

Even on Definite Clauses, the problem is *semidecidable*.

Efficiency of FC

The algorithm is brute-force. We can optimise.

- no need to match a rule on iteration k if a premise wasn't added on iteration $k - 1$
- match each rule whose premise contains a newly added literal

Algorithm RETE (managment of a working memory...)