## Part I
# First-order logic

**Outline**

## Contents

## 1 First-order logic, why?

**Pros and cons of propositional logic**

- Propositional logic is *declarative*: pieces of syntax correspond to facts
- Propositional logic allows partial/disjunctive/negated information
  (unlike most data structures and databases)
- Propositional logic is *compositional*: meaning of $a \wedge b$ is derived from meaning of $a$ and of $b$
- Meaning in propositional logic is *context-independent* (unlike natural language, where meaning depends on context)

Propositional logic has very limited expressive power (unlike natural language)

**Expressiveness of propositional logic**

*Example* 1. I want to declare:

*Every friend of my sisters has a blue car*

In the natural language, I need one simple sentence. What about in propositional logic?

I need symbols!! Lots of them!! Because I have a big family and my sisters are very friendly.

- Sister_1_Friend_1_Has_Blue_Car
- Sister_1_Friend_2_Has_Blue_Car
- ...
- Sister_4_Friend_23_Has_Blue_Car

**First-order logic**

Whereas propositional logic assumes world contains *facts*, first-order logic (like natural language) assumes the world contains

- *Objects*: people, houses, numbers, theories, colors, cricket games, centuries . . . and me, and cars!!
- *Relations*: red, round, bogus, prime, multistoried . . ., brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . . and blue!!
- *Functions*: father of, third inning of, one more than, end of . . . and friend of, sister of!!

## 2 Syntax and semantics

### Syntax of First-Order Logic

1. *Constants*: $KingJohn, 2, ANU, Yannick \cdots$
2. *Predicate*: $Sister, > \cdots$
3. *Functions*: $Sqrt, FriendOf \cdots$
4. *Variables*: $x$, $y$, $a$, $b \cdots$
5. *Connectives*: $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$
6. *Equality*: $=$
7. *Quantifiers*: $\forall\ \exists$

### Syntax of First-Order Logic

A *term* represents an object in FOL. Its syntax is:

- a constant, or
- a variable, or
- a function of terms $function(term_1, \cdots, term_n)$

An *atomic sentence* represents an elementary relation between terms. Its syntax is:

- a predicate $predicate(term_1, \cdots, term_n)$
- an equality of terms $term_1 = term_2$

*Example* 2. $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$
$carOf(friendOf(oneSisterOf(Yannick))) = colorOf(Ocean)$

### Syntax of First-Order Logic

*Complex sentences* are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

*Example* 3. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$ $>(1,2) \vee \leq(1,2)$ $>(1,2) \wedge \neg >(1,2)$ $Sister(Marie, Yannick) \Rightarrow CarColor(FriendOf(Marie), blue)$

### Truth in first-order logic

Sentences are true with respect to a *model* and an *interpretation*.

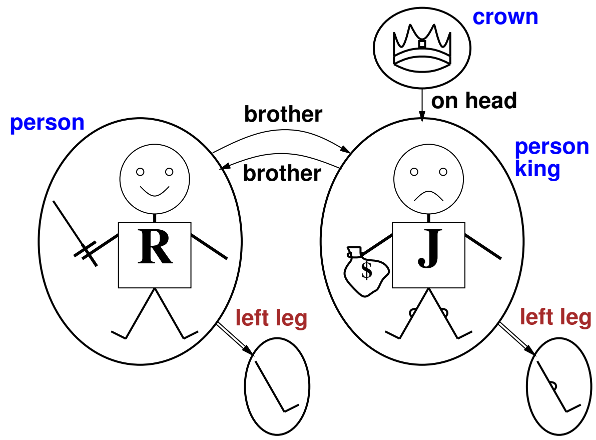
Model contains `objects` (*domain elements*) and `relations` among them.

Interpretation specifies referents for

- *constant symbols* $\rightarrow$ `objects`
- *predicate symbols* $\rightarrow$ `relations`
- *function symbols* $\rightarrow$ `functional relations`

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true iff the `objects` referred to by $term_1, \ldots, term_n$ are in the `relation` referred to by $predicate$.

### Models for FOL: Example




*Example* 4.

### Models for FOL: Example

*Example* 5. Consider the interpretation in which

- *Richard* → `Richard the lionheart`
- *John* → `the evil King John`
- *Brother* → `the brotherhood relation`

Under this interpretation, $Brother(Richard, John)$ is true just in case `Richard the Lionheart` and `the evil King John` are in the brotherhood relation in the model

### Models for FOL: Example

*Example* 6. Consider this new interpretation on the model about Yannick and his sisters in which

- *Richard* → `Yannick`
- *John* → `Marie`
- *Brother* → `the brother-sister relation`

Under this interpretation, $Brother(Richard, John)$ is true just in case `Yannick` and `Marie` are in the brother-sister relation in the model.

This interpretation is ok but the symbols are not very well-chosen!

### Universal quantification

Symbol: ∀ Syntax: ∀ $< variables >$ $< sentence >$ Semantics: ∀ $x$ $P$ is true in a model $m$ iff $P$ is true with $x$ being *each possible object* in the model

*Example* 7. *Quantified sentence*: $\forall x \quad Sister(x, Yannick) \Rightarrow ColorCar(FriendOf(x), blue)$

*Roughly speaking*, it is equivalent to:

$Sister(Marie, Yannick) \Rightarrow ColorCar(FriendOf(Marie), Blue) \wedge Sister(Claire, Yannick) \Rightarrow ColorCar(FriendOf(Claire), Blue$
$\wedge Sister(KingJohn, Yannick) \Rightarrow ColorCar(FriendOf(KingJohn), Blue) \wedge Sister(Blue, Yannick) \Rightarrow ColorCar(FriendOf(Blue), Bl$
$\wedge Sister(Yannick, Yannick) \Rightarrow ColorCar(FriendOf(Yannick), Blue) \wedge ...$

**A common mistake to avoid**

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$

*Example* 8.

$$\forall x \quad Sister(x, Yannick) \wedge ColorCar(FriendOf(x), Blue)$$

means "Everyone is the sister of Yannick and every friend of everyone has a blue car"

**Existential quantification**

Symbol: $\exists$

Syntax: $\exists < variables > \quad < sentence >$

Semantics: $\exists x \quad P$ is true in the model $m$ iff $P$ is true with $x$ beging some object in the model.

*Example* 9. Quantified sentence: $\exists x \quad Sister(x, Yannick) \wedge ColorCar(FriendOf(x), blue)$

*Roughly speaking,* it is equivalent to:

$Sister(Marie, Yannick) \wedge ColorCar(FriendOf(Marie), Blue) \vee Sister(Claire, Yannick) \wedge ColorCar(FriendOf(Claire), Blue)$
$\vee Sister(KingJohn, Yannick) \wedge ColorCar(FriendOf(KingJohn), Blue) \vee Sister(Blue, Yannick) \wedge ColorCar(FriendOf(Blue), Blue)$
$\vee Sister(Yannick, Yannick) \wedge ColorCar(FriendOf(Yannick), Blue) \vee ...$

**Another common mistake to avoid**

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$

*Example* 10.

$$\exists x \quad Sister(x, Yannick) \Rightarrow ColorCar(FriendOf(x), Blue)$$

This sentence is true if you find someone who is not my sister! It does not matter if this person has a blue car or not.

**Properties of quantifiers**

- $\forall x \forall y$ is the same as $\forall y \forall x$
- $\exists x \exists y$ is the same as $\exists y \exists x$
- $\exists x \forall y$ is *not* the same as $\forall y \exists x$
  - $\exists x \forall y \quad Loves(x, y)$
  - "There is a person who loves everyone in the world"
  - $\forall y \exists x \quad Loves(x, y)$
  - "Everyone in the world is loved by at least one person"

Each quantifier can be expressed using the other

- $\forall x \quad Likes(x, IceCream)$ is the same as $\neg \exists x \quad \neg Likes(x, IceCream)$
- $\exists x \quad Likes(x, Broccoli)$ is the same as $\neg \forall x \quad \neg Likes(x, Broccoli)$

### Fun with sentences

*Example* 11.
- Brothers are siblings

### Fun with sentences

*Example* 12.
- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

### Fun with sentences

*Example* 13.
- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- “Sibling” is symmetric

### Fun with sentences

*Example* 14.
- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- “Sibling” is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

### Fun with sentences

*Example* 15.
- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- “Sibling” is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- “One’s mother is one’s female parent”

### Fun with sentences

*Example* 16.
- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- “Sibling” is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- “One’s mother is one’s female parent”

$$\forall x, y \; Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$$

### Fun with sentences

*Example* 17.
- Brothers are siblings

$$\forall x, y \; Brother(x, y) \Rightarrow Sibling(x, y)$$

- “Sibling” is symmetric

$$\forall x, y \; Sibling(x, y) \Leftrightarrow Sibling(y, x)$$

- “One’s mother is one’s female parent”

$$\forall x, y \; Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y))$$

- “A first cousin is a child of a parent’s sibling”

### Fun with sentences

*Example* 18.
- Brothers are siblings
$$\forall x, y\ Brother(x,y) \Rightarrow Sibling(x,y)$$
- "Sibling" is symmetric
$$\forall x, y\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$$
- "One's mother is one's female parent"
$$\forall x, y\ Mother(x,y) \Leftrightarrow (Female(x) \wedge Parent(x,y))$$
- "A first cousin is a child of a parent's sibling"
$$\forall x, y\ FirstCousin(x,y) \Leftrightarrow \exists p, ps\ Parent(p,x) \wedge Sibling(ps,p) \wedge Parent(ps,y)$$

### Equality

$term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object.

*Example* 19.
- $1 = 2$ is satisfiable (if the symbols $1$ and $2$ refer to the same object in the interpretation)
- $2 = 2$ is valid

*Example* 20. Definition of $Sibling$ thanks to $Parent$:

$$\forall x, y\ Sibling(x,y) \Leftrightarrow (\neg(x = y) \wedge \exists m, f\ \neg(m = f) \wedge$$
$$Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y))$$

### Summary

- Knowledge representation language:
  - declarative, compositional, expressive, context-independent, unambiguous
- Model: set of objects, functions and their relation
- Knowledge-base in first-order logic
  - careful process
    1. analyzing the *domain* (objects, functions, relations),
    2. choosing a *vocabulary* (interpretation)
    3. encoding the *axioms* (what is known in $KB$) to support the desired *inferences*