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Stability analysis of time-varying delay systems in quadratic separation framework

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Summary. This paper deals with the stability analysis of linear time-delay systems. The time-delay is assumed to be a time-varying $\mathbb{C}^1$ function belonging to an interval and has a bounded derivative. Considering stable delay-free system, the quadratic separation framework is used to assess the maximal allowable value of the delay that preserves stability. To take into account the time-varying nature of the delay, the quadratic separation principle has been extended to cope with the general case of uncertain operators (instead of just uncertain matrices). Then, separation conditions lead to linear matrix inequality (LMI) which can be efficiently solved with Semi Definite Programming solvers in Matlab. The paper concludes with an illustrative academic example.

1 Introduction

Time-delay systems and their stability have been intensively studied since several decades. In the case of constant delay and unperturbed linear systems, efficient criteria based on roots location [12] allow to find the exact region of stability with respect to the value of the delay. For the case of uncertain linear systems, i.e. for proving robust stability, the problem has been partially solved, either by using Lyapunov functionals

[6] or robustness framework (small gain theory [6], IQCs [10] or quadratic separation [5]). All resulting stability conditions are based on convex optimization (Linear Matrix Inequality framework) and allow to conclude on stability region with respect to the delay and/or the uncertainty. These conditions are conservative, producing inner approximations of the stability regions. To reduce this inherent conservatism, new techniques have been proposed recently [4][5] by introducing redundant equations and new decision variables in the optimization problem at the expense of increasing the numerical burden.

Nevertheless, the upper cited results with reduced conservatism are up to now limited to constant delays. For time-varying delays, some results based on either Lyapunov-Krasovskii or IQC methodology have been successfully exploited [3][10][8] but, however, reveal to be very conservative in practice. The objective of the paper is to assess delay dependent stability of time-varying delay systems based on an extension of the quadratic separation principle [13]. Criteria are derived and expressed in terms of Linear Matrix Inequalities (LMIs) which may be solved efficiently with Semi-Definite Programming (SDP) solvers.

The derivation of results is based on redundant system modeling. Indeed, based on known interactions between delays, their variations and derivatives, redundant equations are introduced to construct a new modeling of the delay systems. To this end, an augmented state is considered which is composed of the original state vector and its derivatives. Then defining relationship between augmented states $\dot{x}$, $\ddot{x}$, the delay h and its derivative $\dot{h}$ as a set of integral quadratic constraints allows to produce some criteria with less conservatism.

After the introduction, the paper carries on with preliminaries on notations and definitions useful to present a new theorem on quadratic separation principle. In section 3, this latter prior result is exploited to derive a stability condition for time-varying delay systems. A numerical example that shows the effectiveness of the proposed criterion is provided in section 4.

2 Preliminaries

2.1 Notations

This section is devoted to notations and useful definitions considered throughout the paper. However, most of them stand for the mathematical formalism allowing the development of Theorem 1 (in section 2.2) and will not be required for the rest of the study. This latter fundamental theorem constitutes the main result for the design of stability criteria in following sections.

Let $L_2^m[0, +\infty] = L_2$ be the set of all measurable functions $f : \mathbb{R}^+ \to \mathbb{C}^m$ bounded with respect to the following norm $\|f\|_2 = \left(\int_0^\infty (f^*(t) f(t)) \right)^{1/2} dt < \infty$. Note that this norm corresponds to the inner product $\langle\ \rangle$ defined as $\langle f, g \rangle = \int_0^{+\infty} f^*(t) g(t) dt$. Hence, the norm is also defined as $\|f\|_2^2 = \langle f, f \rangle$. Introduce as well the truncation operator $\mathbb{P}_T$ such that:

$$\mathbb{P}_T(f) = f_T = \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases} \tag{1}$$

Definition 1. *The extended space L_{2e} consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{C}^m$ such that f_T belongs to L_2 for all $T \geq 0$.*

Definition 2. *A mapping $H : L_{2e} \to L_{2e}$ is said to be causal if $\mathbb{P}_T(Hf) = \mathbb{P}_T(Hf_T), \forall T \geq 0, \forall f \in L_{2e}$*

2.2 A new theorem in quadratic separation framework

Since the delay is time-varying, the previous results [4][5] on quadratic separation for time-delay systems analysis cannot be applied. Hence, in this paper the quadratic separation method is extended to handle not only the case of uncertain matrices but more generally uncertain operators. To this end, based on the inner product and the L_{2e} space a suitable theorem is proposed. This latter is used in the next section to derive stability conditions for time-varying delay systems.

Consider the interconnection defined by figure 1, where $\mathcal{E}(t)$ and $\mathcal{A}(t)$ are two, real

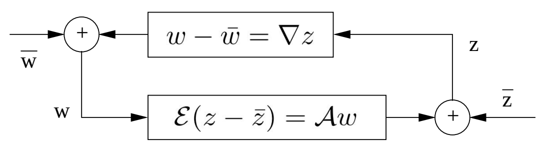

Fig. 1 Feedback system

valued, possibly non-square matrices depending on time and ∇ is a linear operator from L_{2e} to L_{2e}. For simplicity of notations, we assume in the present paper that $\mathcal{E}$ is full column rank. We are interested in looking for conditions that ensure the well-posedness of the interconnection as well as its stability. We first recall the definition of well-posedness:

Definition 3 (Well-posedness). *The system represented by the figure 1 is well-posed if internal signals are bounded and unique for bounded disturbances:*

$$\exists \bar{\gamma} > 0,\ \forall T \geq 0\, \forall (\bar{w}, \bar{z}) \in L_{2e} \left\| \begin{matrix} w_T \\ z_T \end{matrix} \right\|_2 \leq \bar{\gamma} \left\| \begin{matrix} \bar{w}_T \\ \bar{z}_T \end{matrix} \right\|_2 . \tag{2}$$

Following the ideas developed by [9] and then by [13], we propose the following theorem for testing the stability of such interconnection:

Theorem 1. *The interconnected system of Figure 1 is well posed and stable if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions*

$$\forall t,\ \left[\, \mathcal{E}(t)\ -\mathcal{A}(t) \,\right]^{\perp *} \Theta \left[\, \mathcal{E}(t)\ -\mathcal{A}(t) \,\right]^{\perp} > 0 \tag{3}$$

$$\forall t,\ \forall u \in L_{2e},\ \langle \begin{bmatrix} 1 \\ \bigtriangledown \end{bmatrix} u, \Theta \begin{bmatrix} 1 \\ \bigtriangledown \end{bmatrix} u \rangle \leq 0 \tag{4}$$

Proof Using (1) and the truncation operator, the overall feedback system is described by the equations

$$\begin{cases} w_T - \bar{w}_T = \mathbb{P}_T \nabla z \\ \mathcal{E}(z_T - \bar{z}_T) - \mathcal{A} w_T = 0 \end{cases}$$

Since the operator ∇ is supposed to be causal, we get

$$\begin{cases} w_T - \bar{w}_T = \mathbb{P}_T \nabla z_T \\ \mathcal{E}(z_T - \bar{z}_T) - \mathcal{A} w_T = 0 \end{cases} \tag{5}$$

The first inequality (3) implies the existence of some positive scalar ϵ such that

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} (\Theta - \epsilon 1) \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} \geq 0 \ . \tag{6}$$

Furthermore, from the second equation of the feedback system $\mathcal{E}(z-\bar{z})-\mathcal{A}w = 0$, there exists $y \in L_{2e}$ such that the vector $\begin{bmatrix} z-\bar{z} \\ w \end{bmatrix}$ can be written $\begin{bmatrix} z-\bar{z} \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} y$. Applying signal y_T to relation (6) yields to $\begin{pmatrix} z_T - \bar{z}_T \\ w_T \end{pmatrix}^* (\Theta - \epsilon 1) \begin{pmatrix} z_T - \bar{z}_T \\ w_T \end{pmatrix} \geq 0$ which implies also that there exists $\epsilon > 0$ such that

$$\langle \begin{pmatrix} z_T - \bar{z}_T \\ w_T \end{pmatrix}, (\Theta - \epsilon 1) \begin{pmatrix} z_T - \bar{z}_T \\ w_T \end{pmatrix} \rangle \geq 0 \tag{7}$$

The second inequality (4) implies also that there exists $\epsilon > 0$ such that

$$\langle \begin{bmatrix} z_T \\ w_T - \bar{w}_T \end{bmatrix}, (\Theta + \epsilon 1) \begin{bmatrix} z_T \\ w_T - \bar{w}_T \end{bmatrix} \rangle \leq 0 \tag{8}$$

Combining both inequalities (7) and (8), results in a quadratic constraint on the vector $X = \begin{pmatrix} w_T^* & z_T^* | \bar{w}_T^* & \bar{z}_T^* \end{pmatrix}^*$ such as

$$\langle X, \begin{bmatrix} 2\epsilon 1 & T_1 \\ T_1^* & T_2 \end{bmatrix} X \rangle \leq 0 \ .$$

with appropriate matrices T_1 and T_2 depending on Θ and ϵ. Take any $\tilde{\epsilon}$ such that $2\epsilon > \tilde{\epsilon} > 0$ and take a sufficiently large $\tilde{\gamma} > 0$ such that $\begin{bmatrix} \tilde{\epsilon} 1 & 0 \\ 0 & -\tilde{\gamma} 1 \end{bmatrix} \leq \begin{bmatrix} 2\epsilon 1 & T_1 \\ T_1^* & T_2 \end{bmatrix}$. It implies that $\langle X, \begin{bmatrix} \tilde{\epsilon} 1 & 0 \\ 0 & -\tilde{\gamma} 1 \end{bmatrix} X \rangle \leq 0$, which proves that :

$$\exists \tilde{\gamma} > 0, \forall (\bar{w}_T, \bar{z}_T) \left\| \begin{matrix} w_T \\ z_T \end{matrix} \right\|_2 \leq \bar{\gamma} \left\| \begin{matrix} \bar{w}_T \\ \bar{z}_T \end{matrix} \right\|_2 \ . \tag{9}$$

2.3 Problem statement

Consider the following time-varying delay system:

$$\begin{cases} \dot{x}(t) = A x(t) + A_d x(t - h(t)) & \forall t \geq 0, \\ x(t) = \phi(t) & \forall t \in [-h_m, 0] \end{cases} \tag{10}$$

where $x(t) \in \mathsf{R}^{\mathsf{n}}$ is the state vector, ϕ is the initial condition and A, $A_d \in \mathsf{R}^{\mathsf{n}\times\mathsf{n}}$ are constant matrices. The delay h is time-varying and the following constraints are assumed

$$h(t) \in [0, h_m] \quad \text{and} \quad |\dot{h}(t)| \leq d \leq 1, \tag{11}$$

where h_m and d are given scalar constants and may be infinite if delay independent condition and fast-varying delay condition are looked for. The key idea is to rewrite the time-varying delay system (10) as an interconnected system described by Figure 1. A first model taking $w = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}$ and $\nabla = \nabla_0 = \begin{bmatrix} \mathcal{I} & \\ & \mathcal{D} \end{bmatrix}$ is as follows

$$\begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} = \begin{bmatrix} \mathcal{I} & \\ & \mathcal{D} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} A & A_d \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}, \tag{13}$$

where $\mathcal{I}$ and $\mathcal{D}$ are two operators defined as

$$\mathcal{I} : x(t) \to \int_0^t x(u)du, \tag{14}$$

$$\mathcal{D} : x(t) \to x(t-h(t)). \tag{15}$$

In order to derive LMI conditions, one needs to have finite dimensional conditions for (4). These amounts, with conservatism, to expressions in terms of integral quadratic constraints (IQC) on both operators $\mathcal{D}$ and $\mathcal{I}$.

Lemma 1. *Integral quadratic constraints for the operators $\mathcal{I}$ and $\mathcal{D}$ are given by the following inequalities $\forall T > 0, \forall x \in L_{2e}, \forall P > 0, Q_0 > 0$:*

$$\langle \begin{bmatrix} 1 \\ \mathcal{I} \end{bmatrix} x, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \mathcal{I} \end{bmatrix} x \rangle < 0 \,, \langle \begin{bmatrix} 1 \\ \mathcal{D} \end{bmatrix} x, \begin{bmatrix} -Q_0 & 0 \\ 0 & Q_0(1-d) \end{bmatrix} \begin{bmatrix} 1 \\ \mathcal{D} \end{bmatrix} x \rangle < 0$$

Proof. Simple calculus shows that if $\forall x \in L_{2e}$, then

$$\begin{aligned} \langle \begin{bmatrix} 1 \\ \mathcal{I} \end{bmatrix} x, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \mathcal{I} \end{bmatrix} x \rangle &= -2 \int_0^{+\infty} x(t)^T P(\mathcal{I}x)dt = -2 \int_0^T \tfrac{d}{dt}(\mathcal{I}x)^T P(\mathcal{I}x)dt \\ &= -(\mathcal{I}x)^T P(\mathcal{I}x) < 0 \end{aligned}$$

$$\begin{aligned} &\langle \begin{bmatrix} 1 \\ \mathcal{D} \end{bmatrix} x, \begin{bmatrix} -Q_0 & 0 \\ 0 & Q_0 v \end{bmatrix} \begin{bmatrix} 1 \\ \mathcal{D} \end{bmatrix} x \rangle = -\int_0^{+\infty} x^T(u) Q_0 x(u) du + \int_0^{\infty} x_d^T(t) Q_0 x_d(t) v dt \\ &\leq -\int_0^{+T} x^T(t) Q_0 x(t) dt + \int_{-h(0)}^{T-h(T)} x^T(u) Q_0 x(u) du \leq -\int_{T-h(T)}^{T} x(u)^T Q_0 x(u) du < 0, \end{aligned}$$

where $x_d(t) = x(t-h(t))$ and $v = 1-d$.

Consequently, a conservative choice of separator that fulfils (4) for ∇_0 is of the type

$$\Theta = \left[\begin{array}{cc|cc} 0 & 0 & -P & 0 \\ 0 & -Q_0 & 0 & 0 \\ \hline -P & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_0(1-d) \end{array}\right], \tag{16}$$

with P and Q_0 two positive definite matrices. The stability of (10) can then be assessed (it provides only a sufficient condition) by the condition (3) with $\mathcal{E}$, $\mathcal{A}$ and Θ defined as in (13) and (16). This latter criterion does not contain any expression on the delay h and provides then with an independent of delay stability condition.

3 Stability Conditions

3.1 An classical result

If some bounds on the delay such that (11) are known, it may be more interesting to look for delay dependent and rate dependent stability condition. In the quadratic stability framework, the method consists in modeling the original system (10) as an interconnected system by adding some redundant equations to describe as much as possible all the relations between the extra signals. For example take:

$$\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_m) \\ w_1(t) \\ w_2(t) \end{bmatrix} = \nabla_1 \begin{bmatrix} \dot{x}(t) \\ x(t) \\ x(t) \\ \dot{x}(t) \\ \dot{x}(t) \end{bmatrix}, \quad \nabla_1 = \begin{bmatrix} \nabla_0 & & & \\ & \mathcal{D}_m & & \\ & & \nabla_a & \\ & & & \nabla_b \end{bmatrix}, \tag{17}$$

with $w_1 = x(t) - x(t-h(t))$, $w_2 = x(t) - x(t-h_m)$, ∇_0, $\mathcal{D}_m$, ∇_a, ∇_b are defined in (12), (19), (20), (21) respectively, and $\mathcal{E}z = \mathcal{A}w$, is expressed as

$$\begin{bmatrix} 1_{5n} \\ 0_{2n\times 5n} \end{bmatrix} \left(\begin{bmatrix} \dot{x}(t) \\ x(t) \\ x(t) \\ \dot{x}(t) \\ \dot{x}(t) \end{bmatrix} \right) = \mathcal{A} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h_m) \\ w_1(t) \\ w_2(t) \end{bmatrix} \text{ with } \mathcal{A} = \begin{bmatrix} A & A_d & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ A & A_d & 0 & 0 & 0 \\ A & A_d & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix} \tag{18}$$

The new interconnection operators are as follows:

$$\mathcal{D}_m : x(t) \to x(t-h_m), \tag{19}$$

$$\nabla_a \equiv (1-\mathcal{D})\mathcal{I} : x(t) \to \int_{t-h(t)}^{t} x(u)du, \tag{20}$$

$$\nabla_b \equiv (1-\mathcal{D}_m)\mathcal{I} : x(t) \to \int_{t-h_m}^{t} x(u)du. \tag{21}$$

Lemma 2. *An integral quadratic constraint for the operator $\mathcal{D}_m$ is given by the following inequality $\forall T > 0, \forall x \in L_{2e}, \forall Q_1 > 0$,*

$$\langle \begin{bmatrix} 1 \\ \mathcal{D}_m \end{bmatrix} x, \begin{bmatrix} -Q_1 & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} 1 \\ \mathcal{D}_m \end{bmatrix} x \rangle < 0$$

Proof. Omitted, see [5].

Then, concerning ∇_b, using previous results on time delay systems characterization with constant delay, we derive the following relations :

Lemma 3. *An integral quadratic constraint for the operator ∇_b is given by the following inequality $\forall T > 0, \forall x \in L_{2e}, \forall Q_3 > 0$,*

$$\langle \begin{bmatrix} 1 \\ \nabla_b \end{bmatrix} x, \begin{bmatrix} (c^2 - r^2)h_m^2 Q_3 & -ch_m Q_3 \\ -ch_m Q_3 & Q_3 \end{bmatrix} \begin{bmatrix} 1 \\ \nabla_b \end{bmatrix} x \rangle < 0. \tag{22}$$

where $(c; r) \in \mathcal{R}^2$ are two appropriate reals. For example, [5] propose the following values for c and r, $(c = 0.25; r = 0.75)$.

Proof. Omitted, see [5].

Finally, an IQC for ∇_a is constructed using ∇_b allows to link these two operators.

Lemma 4. *An integral quadratic constraint for the operator ∇_a is given by the following inequality $\forall T > 0, \forall x \in L_{2e}, \forall Q_2 > 0$,*

$$\langle \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \nabla_a & 0 \\ 0 & \nabla_b \end{array}\right] x, \Omega \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \nabla_a & 0 \\ 0 & \nabla_b \end{array}\right] x \rangle < 0 \text{ with } \Omega = \left[\begin{array}{cc|cc} -h_m^2 Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2Q_2 & -Q_2 \\ 0 & 0 & -Q_2 & Q_2 \end{array}\right] \tag{23}$$

Proof. Applying the Cauchy-Schwartz inequality on expression $\|(\nabla_b - \nabla_a)x\|_2^2 + \|(\nabla_a)x\|_2^2$ leads to

$$\begin{aligned} \int_0^\infty \| \int_{t-h(t)}^{t} x(u)du\|^2 + \| \int_{t-h_m}^{t-h(t)} x(u)du\|^2 dt &\leq \int_0^\infty h(t) \int_{t-h(t)}^{t} \|x(u)\|^2 du + (h_m - h(t)) \int_{t-h_m}^{t-h(t)} \|x(u)\|^2 dudt \\ &\leq \int_0^\infty h_m \int_{t-h(t)}^{t} \|x(u)\|^2 du + h_m \int_{t-h_m}^{t-h(t)} \|x(u)\|^2 dudt \\ &\leq \int_0^\infty h_m \int_{t-h_m}^{t} \|x(u)\|^2 du \leq h_m^2 \|x\|_2^2 \end{aligned}$$

Thus, the proposed inequality (23) stems from the following inequality:

$$\|(\nabla_b - \nabla_a)x\|_2^2 + \|(\nabla_a)x\|_2^2 \leq h_m^2 \|x\|_2^2 \tag{24}$$

Using all these inequalities, a choice of separator for ∇_1 that satisfies the relation (4) can then be written as :

$$\Theta = \left[\begin{array}{c|c} \Theta_{11} & \Theta_{12} \\ \hline * & \Theta_{22} \end{array}\right] \text{ with } \Theta_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -Q_0 & 0 & 0 & 0 \\ 0 & 0 & -Q_1 & 0 & 0 \\ 0 & 0 & 0 & -h_m^2 Q_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha Q_3 \end{bmatrix}, \tag{25}$$

$$\Theta_{12} = \begin{bmatrix} -P & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta Q_3 \end{bmatrix}, \quad \Theta_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & (1-d)Q_0 & 0 & 0 & 0 \\ 0 & 0 & Q_1 & 0 & 0 \\ 0 & 0 & 0 & 2Q_2 & -Q_2 \\ 0 & 0 & 0 & -Q_2 & Q_2+Q_3 \end{bmatrix} \tag{26}$$

where α and β are equal to $(c^2-r^2)h_m^2$ and $-ch_m$ respectively. Based on this IQC, the following delay dependent stability theorem is then proposed:

Theorem 2. *For given positive scalars d and h_m, if there exists positive definite matrices P, Q_i for $i = \{0,1,2,3\} \in \mathbb{R}^{n\times n}$, then the system (10) with a time varying delay constrained by (11) is asymptotically stable if the LMI (3) holds for $\dot{h}(t) = \{-d, d\}$ with Θ, $\mathcal{E}$ and $\mathcal{A}$ defined as (25) and (18).*

3.2 Main result

Consider the system (10) and its derivative $\ddot{x}(t) = A\dot{x}(t) + (1-\dot{h}(t))A_d\dot{x}(t-h(t))$. A new artificially augmented system can be derived:

$$E\dot{\varsigma}(t) = \bar{A}\varsigma(t) + \bar{A}_d(\dot{h}(t))\varsigma(t-h(t)) \tag{27}$$

with

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \\ 1 & 0 \end{bmatrix}, \bar{A}_d(\dot{h}) = \begin{bmatrix} (1-\dot{h})A_d & 0 \\ 0 & A_d \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \varsigma(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}. \tag{28}$$

The idea is to employ the same methodology as previously to the new time-varying delay system (27). Using the same uncertain transformation (18) with appropriate dimensions $w = \nabla_1 z$. with $w_1 = \varsigma(t) - \varsigma(t-h(t))$ and $w_2 = \varsigma(t) - \varsigma(t-h_m)$. Then, relations between signals must be specified: $\mathcal{E}z = \mathcal{A}(t)w$, such that

$$\begin{bmatrix} \varsigma(t) \\ \varsigma(t-h(t)) \\ \varsigma(t-h_m) \\ w_1(t) \\ w_2(t) \end{bmatrix} = \nabla_1 \begin{bmatrix} \dot{\varsigma}(t) \\ \varsigma(t) \\ \varsigma(t) \\ \dot{\varsigma}(t) \\ \dot{\varsigma}(t) \end{bmatrix}, \mathcal{E} = \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{A}(\dot{h}) = \begin{bmatrix} \bar{A} & \bar{A}_d(\dot{h}) & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}. \tag{29}$$

Adapting the dimensions of (25), this separator can be used to apply the theorem 1 to the interconnection (29). Thus, the delay dependent condition becomes:

$$[\mathcal{E} \quad -\mathcal{A}(\dot{h})]^{\perp^T} \Theta [\mathcal{E} \quad -\mathcal{A}(\dot{h})]^{\perp} > 0 \tag{30}$$

Furtermore, it can be shown (after few algebraic calculus) that (30) is linear (thus convex) in $\dot{h}$. So, it suffices to assess the condition on the bounds of $\dot{h}$ which are $-d$ and d. We state then the following theorem:

Theorem 3. *For given positive scalars d and h_m, if there exists positive definite matrices P, Q_i for $i = \{0,1,2,3\} \in \mathbb{R}^{2n\times 2n}$, then the system (10) with a time varying delay constrained by (11) is asymptotically stable if the LMI (3) holds for $\dot{h} = \{-d, d\}$ with $\mathcal{E}$ and $\mathcal{A}$ defined as (29). The separator is of the form (25) with appropriate dimensions.*

Table 1 The maximal allowable delays h_m for system (31)

d	0	0.1	0.2	0.5	0.8	1	$\forall d > 1$
Fridman et al (2002) [2]	4.472	3.604	3.033	2.008	1.364	0.999	0.999
Fridman et al (2006) [3]	1.632	1.632	1.632	1.632	1.632	1.632	1.632
Wu et al (2004) [15]	4.472	3.604	3.033	2.008	1.364	-	-
Kao et al (2005) [10]	4.472	3.604	3.033	2.008	1.364	0.999	-
He et al (2007) [7]	4.472	3.605	3.039	2.043	1.492	1.345	1.345
He et al (2007) [8]	4.472	3.605	3.039	2.043	1.492	1.345	1.345
Theorem 2	4.568	3.673	3.085	2.043	1.492	1.345	1.345
Theorem 3	4.568	3.740	3.263	2.536	2.183	2.034	-

4 Numerical example

considering the following numerical example

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \tag{31}$$

For various values of d, the maximal allowable delay, h_m, is computed performing a line search and solving the LMI at each step. To demonstrate the effectiveness of our criterion, results are compared to those obtained in [2], [3], [15], [7], [8] and [10]. All these papers, except the last one, use the Lyapunov theory in order to derive some stability analysis criteria for time delay systems. In [10], the stability problem is solved by a classical robust control approach: the IQC framework. The results are shown in Table 1.

Concerning the processing time, using solver SeDuMi [14] in Yalmip [11] environment, LMI conditions from Theorem 2 and Theorem 3 are solved in less than 1 second. These computations have been performed on a machine with an intel core 2 duo 3.40 GHz processor and 2GB RAM memory.

Numerical experiments show that Theorem 2 is close to [7], improvement is due to the choice ($c = 0.25; r = 0.75$) rather than a more classical choice ($c = 0; r = 1$) (see lemma 3 and [5]). Results for $d \geq 1$ are computed with Theorem 2 choosing $Q_0 = 0$ in (25) to make the criterion rate-independent. [3] gives a rate-independent criterion which may be interesting in certain cases when d is unknown. On the other hand, as no informations are taken into account about $\dot{h}(t)$, this is conservative for small delay variations. Note that, considering Theorem 2 without operators $\mathcal{D}_m$ and $(1 - \mathcal{D}_m) \circ \mathcal{I}$ provides the same results as [2].

Then, considering the augmented system (27) composed by the original system (10) and its derivative, Theorem 3 improves the maximal allowable delays. Indeed, using the same matrix of operator (17), conservatism is reduced thanks to the derivation of (10). As expected, this operation provides more information on the system and thus improves the stability analysis criteria.

5 Conclusion

In this paper, the problem of the delay dependent stability analysis of a time varying delay system has been studied by means of quadratic separation. Inspired from previous work on time delay systems with constant delay [4][5], stability criteria for time varying delay system are provided. Based on this first result, and using an augmented state (this methodology is also based on previous work [1]), new modelling of time delay systems are introduced which emphasizes the relation between $\dot{h}$ and signals $\dot{x}$ and $\ddot{x}$. The resulting criteria are then expressed in terms of a convex optimization problem with LMI constraints, allowing the use of efficient solvers. Finally, a numerical example shows that these methods reduced conservatism and improved the maximal allowable delay.

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