# Design of Lyapunov based controllers as TCP AQM

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***Abstract*—The paper considers the design of robust AQM for the congestion problem of a router. Using a state space representation of a linearized fluid model of TCP, we provide an AQM which stabilises the queue length of the buffer for any delay less than a prescribed upper bound. Our results rely on the use of recently developed Lyapunov-Krasovskii functional and have been expressed in terms of Linear Matrix Inequalities which can be solved easily. Finally, an example extracted from the literature and simulations support our study.**

## I. INTRODUCTION

Over a past few years, problems have arisen with regard to Quality of Service (QoS) issues in Internet traffic congestion control [1], [2], [3]. AQM mechanism, which supports the end-to-end congestion control mechanism of Transmission Control Protocol (TCP), has been actively studied by many researchers. AQM mechanism controls the queue length of a router by actively dropping packets. Various mechanisms have been proposed in the literature such as Random Early Detection (RED) [4], Random Early Marking (REM) [5], BLUE [6], Adaptive Virtual Queue (AVQ) [7] and many others [8]. Their performances have been evaluated [9], [8] and empirical studies have shown the effectiveness of these mechanisms [10]. During the last few years, significant research have been devoted to the use of control theory to develop more efficient AQM. Using dynamical model developed by [11], some P (*Proportional*), PI (*Proportional Integral*) [12], PID (*Proportional Integral Derivative*) [13] have been designed as well as robust control framework issued [14]. Nevertheless, most of these papers do not take into account the delay and ensure the stability in closed loop for all delays which could be conservative.

The study of congestion problem with time delay systems framework is not new and has been succesfully exploited. In [15], [16], using Lyapunov-Krasovskii theory, the global stability analysis of the non linear model of TCP is performed. In [17], a delay dependent state feedback controller is provided by compensation of the delay with a memory feedback control. This latter methodology is interesting in theory but hardly suitable in practice. In [18], [19], robust AQM are derived using time delay system approach. The first one builds a state feedback controller based on Lyapunov-Krasovskii theory. The second one designs an output feedback in robust control framework, especially $H_\infty$ control.

Based on recently developed Lyapunov functional for delayed systems, we construct two new AQM stabilizing the TCP model. The first one is called IOD-AQM (*Independent Of Delay*) and it deals with the control of TCP for all delays in the loop. Because of the space limitation, this latter is not presented and invite the reader to [20]. The second one, DD-AQM (*Delay Dependent*) is devoted to the control of the TCP dynamics when an upperbound of the delay is known. In order to consider a more realistic case, extension to the robust case, where the delay is uncertain is considered using quadratic stabilization framework.

The paper is organized as follows. The second part discussed the uncertain mathematical model of a network supporting TCP. Section III is dedicated to the design of an AQM ensuring the robust stabilization of TCP as well as a certain level of performances. Section IV presents application of the exposed theory and simulation results using NS-2 [21]. Finally, Section V concludes the paper.

*Notations:* For two symmetric matrices, $A$ and $B$, $A > (\geq) B$ means that $A-B$ is (semi-) positive definite. $A^T$ denotes the transpose of $A$. $1_n$ and $0_{m\times n}$ denote respectively the identity matrix of size $n$ and null matrix of size $m \times n$. If the context allows it, the dimensions of these matrices are often omitted. For a given matrix $A \in R^{n\times n}$, $< A >$ stands for $A + A^T$.

## II. PROBLEM STATEMENT AND MODELISATION

### A. The linearized fluid-flow model of TCP

The fluid flow model of TCP considered here was introduced in [11], [12]. Based on this system, a new type of AQM will be constructed, which takes into account delays into the network.

Given the network parameters: number of TCP sessions, link capacity and propagation delay ($N$, $C$ and $T_p$ respectively), we define the set of operating points $(W_0, q_0, p_0)$ by $\dot{W} = 0$ and $\dot{q} = 0$:

$$\begin{cases} \dot{W} = 0 & \Rightarrow\ W_0^2 p_0 = 2 \\ \dot{q} = 0 & \Rightarrow\ W_0 = \frac{R_0 C}{N},\ R_0 = \frac{q_0}{C} + T_p \end{cases} \quad (1)$$

where $W(t)$ is the congestion window, $q(t)$ is the queue length at the congested router and $R(t)$ is the Round Trip Time (RTT) which represents the delay in TCP dynamics. $x_0$ denotes the value of the variable $x$ at the equilibrium point.

Assuming $N(t) \equiv N$ and $R(t) \equiv R_0$ as constants, the dynamic model of TCP can be approximated, at an equilibrium point, by the linear time delay system [11]:

$$\begin{cases} \delta\dot{W}(t) = -\dfrac{N}{R_0^2 C}\Big(\delta W(t) + \delta W(t-h(t))\Big) \\ \qquad -\frac{1}{R_0^2 C}\Big(\delta q(t) - \delta q(t-h(t))\Big) - \frac{R_0 C^2}{2N^2}\delta p(t-h(t)) \\ \delta\dot{q}(t) = \dfrac{N}{R_0}\delta W(t) - \dfrac{1}{R_0}\delta q(t) \end{cases} \tag{2}$$

where $\delta W \doteq W - W_0$, $\delta q \doteq q - q_0$ and $\delta p \doteq p - p_0$ are the state variables and input perturbations around the operating point. The model (2) is valid only if the variations of these new variables are kept enough small.

Note that the input of the model (2) corresponds to the drop probability of a packet, fixed by the AQM. This latter has for objective to regulate the queue size of the buffer. The purpose of this paper is to develop an AQM algorithm based on control theory, especially time delay system theory.

For synthesis problem (see section III), we consider an active queue management expressed as a state feedback

$$p(t) = p_0 + k_1\delta W(t) + k_2\delta q(t). \tag{3}$$

**Remark 1** *Using the second equation of (2), the perturbed congestion window can be expressed as* $\delta W = \frac{1}{N}\delta q + \frac{R_0}{N}\delta\dot{q}$. *As it has been shown [17], the state feedback appears to be a natural PD (Proportional Derivative) control law. In that case,* $\dot{q}$ *can be calculated as* $\dot{q} = -C + y(t)$ *where* $y(t) = \frac{N}{R_0}W(t)$ *is the aggregate flow at the link. This last value can be measured [17] [1].*

Note that the model considered here [11] results from a tradeoff between precision and simplicity. Indeed, this model is known to be unprecise but its simplicity allows to develop practical methods for control purpose. Remarks that there exist extensions to better models e.g. see [22] or [3] and references therein.

### B. Time delay system approach

In this paper, we choose to model the dynamics of the queue and the congestion window as a time delay system. Indeed, the delay is an intrinsic phenomenon in networks. Taking into account this characteristic, we expect to reflect as much as possible the TCP behavior and provide with more relevant analysis and synthesis methods.

The linearized TCP fluid model (2) can be rewritten as the following time delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h) + Bu(t-h) \\ x_0(\theta) = \phi(\theta), \text{ with } \theta \in [-h, 0] \end{cases} \tag{4}$$

with

$$A = \begin{bmatrix} -\frac{N}{R_0^2 C} & -\frac{1}{CR_0^2} \\ \frac{N}{R_0} & -\frac{1}{R_0} \end{bmatrix}, A_d = \begin{bmatrix} -\frac{N}{R_0^2 C} & \frac{1}{R_0^2 C} \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} -\frac{C^2 R_0}{2N^2} \\ 0 \end{bmatrix}. \tag{5}$$

where $x(t) = [\delta W^T(t) \quad \delta q^T(t)]^T$ is the state vector and $u(t) = \delta p(t)$ the input. $\phi(\theta)$ is the initial condition.

In the literature, there are mainly three methods to study time delay system stability: analysis of the characteristic roots, robust approach and Lyapunov theory. The latter will be considered because it is an effective and practical method which provides $LMI$ (Linear Matrix Inequalities [23]) criteria. To analyze and control our system (4), the Lyapunov-Krasovskii approach [24] is used which is an extension of the traditional Lyapunov theory.

In the literature, few articles using time delay systems approach to model TCP dynamic already appeared. In [18], a delay dependent robust stability condition was proposed and the design of a state feedback was derived. However, the criterion used is quite obsolete and thus conservative. Then, other papers design control laws based on predictor [25], [17]. The predictive approach is an interesting method theoretically but not in practice, moreover the delay has to be known exactly. [15], [16] and [19] use time delay system approach too and propose global stability analysis of the non linear model. However the synthesis of AQM is not considered.

In this paper, we aim at providing a method which allow to control system (4) with different objectives: giving conditions for the nominal or robust stabilization and improving dynamical performances, ensuring thus a better QoS.

### C. Polytopic uncertain model

The state space representation shows that the matrices $A$, $A_d$ and $B$ depend on network parameters. Especially, it depends on the RTT $R_0$, a significant parameter, which is quite difficult to estimate in practice. For a more rigorous study, it could be interesting to take into account some uncertainty on the delay $R_0$. Let then rewrite system (4) as following

$$\dot{x}(t) = A(R_0)x(t) + A_d(R_0)x(t-h) + B(R_0)u(t-h). \tag{6}$$

With the polytopic approach, the idea is to insure the stability for a set of systems. Let suppose that $R_0 \in [R_{0_{min}}, R_{0_{max}}]$, then the matrices $A$, $A_d$ and $B$ belong to a certain set

$$\Omega = \{[A, A_d, B] \mid R_0 \in [R_{0_{min}}, R_{0_{max}}]\}$$

and we aim at looking for an AQM (expressed in term of state feedback) which stabilizes system (6) for all matrices belonging to $\Omega$. However, the parameter $R_0$ doesn't appear linearly in the matrices $A$, $A_d$ and $B$. So that, the set $\Omega$ defined by the uncertainty is non convex.

A common idea in robust control theory is to look for a polytopic set $\mathcal{P}$ which includes the set $\Omega$. Using convexity property, it is much more easy to test the stability in closed loop for the overall polytop. If the stability of $\mathcal{P}$ is proved, then the stability of $\Omega$ is insured.

In order to create the polytop $\mathcal{P}$, we pose $\rho_1 = \frac{1}{R_0}$, $\rho_2 = \frac{1}{R_0^2}$ and $\rho_3 = R_0$. Since there are three uncertain parameters, the polytop will have $n_\omega = 8$ vertices. For a bounded value $R_0$, the new uncertain parameters $\rho_i$, $\forall i = \{1, 2, 3\}$ are bounded.

So, the matrices of the uncertain system (6) are defined as

$$\begin{aligned}
A &= \rho_1 \begin{bmatrix} 0 & 0 \\ N & -1 \end{bmatrix} + \rho_2 \begin{bmatrix} -\frac{N}{C} & -\frac{1}{C} \\ 0 & 0 \end{bmatrix} = \rho_1 A_0 + \rho_2 A_1, \\
A_d &= \rho_2 \begin{bmatrix} -\frac{N}{C} & \frac{1}{C} \\ 0 & 0 \end{bmatrix} = \rho_2 A_{d_0}, \\
B &= \rho_3 \begin{bmatrix} -\frac{C^2}{2N} \\ 0 \end{bmatrix} = \rho_3 B_0.
\end{aligned} \tag{7}$$

The set $\Omega$ is contained in $\mathcal{P}$,

$$\Omega \subset co\{\omega^{(i)},\ i = 1, 2, ..., 8\}. \tag{8}$$

where the $\omega^{(i)}$ are the vertices of $\mathcal{P}$.

## III. Stabilization using time delay system approach: *Delay dependent* AQM design

We have designed an uncertain model of TCP/AQM dynamics and this Section will be devoted to the construction of a robust AQM stabilizing a such model. We describe a delay dependent (*DD*) method which takes into account the size of the delay. Using an information on the delay, we expect a reduction of conservatism and an improvement of results.

The delay-dependent case starts from a system stable without delay and looks for the maximal delay that preserves stability. Generally, all methods involve a Lyapunov functional, and more or less tight techniques to bound some cross terms [26][24]. These choices of specific Lyapunov functionals and bounding techniques are the origin of conservatism. In the present paper, we choose a recent Lyapunov-Krasovskii functional (9) [27].

$$\begin{aligned}
V(x_t) &= x^T(t)Px(t) + \int_{t-\frac{h}{r}}^{t}\int_{\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta \\
&+ \int_{t-\frac{h}{r}}^{t} \begin{pmatrix} x(s) \\ x(s-\frac{1}{r}h) \\ \vdots \\ x(s-\frac{r-1}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(s) \\ x(s-\frac{1}{r}h) \\ \vdots \\ x(s-\frac{r-1}{r}h) \end{pmatrix} ds
\end{aligned} \tag{9}$$

where $P \in \mathsf{S}^{\mathsf{n}}$ is a positive definite matrix, $Q \in \mathsf{S}^{\mathsf{rn}}$ and $R \in \mathsf{S}^{\mathsf{n}}$ are two semi-positive definite matrices. $r \geq 1$ an integer corresponding to the discretization step. Using this functional, we propose the following.

**Theorem 1** *If there exist symmetric positive definite matrices $P$, $R \in \mathsf{R}^{\mathsf{n}\times\mathsf{n}}$, $Q \in \mathsf{R}^{\mathsf{rn}\times\mathsf{rn}}$, a matrix $X \in \mathsf{R}^{(\mathsf{r}+2)\mathsf{n}\times\mathsf{n}}$, a scalar $h_m > 0$, an integer $r \geq 1$ and a matrix $K \in \mathsf{R}^{\mathsf{m}\times\mathsf{n}}$ such that*

$$\mathbf{\Gamma} + \mathbf{X}S + S^T\mathbf{X}^T \prec 0 \tag{10}$$

*where*

$$\Gamma = \begin{bmatrix} \frac{h_m}{r}\mathbf{R} & \mathbf{P} & 0 & \dots & 0 \\ \mathbf{P} & -\frac{r}{h_m}\mathbf{R} & \frac{r}{h_m}\mathbf{R} & & \vdots \\ 0 & \frac{r}{h_m}\mathbf{R} & -\frac{r}{h_m}\mathbf{R} & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \mathbf{Q} & \vdots \\ 0 & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots \\ 0 & \dots \\ \vdots & \mathbf{Q} \end{bmatrix} \tag{11}$$

*and* $S = \begin{bmatrix} -1 & A & 0_{\mathsf{n}\times(\mathsf{r}-1)\mathsf{n}} & \widetilde{A_d} \end{bmatrix}$
*then, system (4) can be stabilized for all $h \leq h_m$ by the state feedback $u(t) = Kx(t)$. Applying this control law to (4), we get closed-loop system*

$$\begin{cases} \dot{x}(t) = Ax(t) + \widetilde{A_d}x(t-h) \\ x_0(\theta) = \phi(\theta),\ \textit{with } \theta \in [-h, 0] \end{cases} \tag{12}$$

*with* $\widetilde{A_d} = A_d + BK$.

*Proof:* It is always possible to rewrite (12) as $S\xi = 0$ where

$$\xi = \begin{bmatrix} \dot{x}(t) \\ x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \\ x(t-h) \end{bmatrix} \in \mathsf{R}^{(\mathsf{r}+2)\mathsf{n}} \tag{13}$$

and $S = \begin{bmatrix} -1 & A & 0_{\mathsf{n}\times(\mathsf{r}-1)\mathsf{n}} & \widetilde{A_d} \end{bmatrix}$

Using the extended variable $\xi(t)$ (13), the derivative of $V$ along the trajectories of system (12) leads to:

$$\begin{cases} \dot{V}(x_t) = \xi^T \begin{bmatrix} \frac{h}{r}\mathbf{R} & \mathbf{P} & 0 & \dots & 0 \\ \mathbf{P} & -\frac{r}{h}\mathbf{R} & \frac{r}{h}\mathbf{R} & & \vdots \\ 0 & \frac{r}{h}\mathbf{R} & -\frac{r}{h}\mathbf{R} & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \xi \\ +\xi^T \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \mathbf{Q} & \vdots \\ 0 & \dots & 0 \end{bmatrix} \xi - \xi^T \begin{bmatrix} 0 & \dots \\ 0 & \dots \\ \vdots & \mathbf{Q} \end{bmatrix} \xi \prec 0 \\ \text{such that} \begin{bmatrix} -1 & A & 0 & \cdots & 0 & \widetilde{A_d} \end{bmatrix} \xi = 0 \end{cases} \tag{14}$$

$$\Leftrightarrow \begin{cases} \dot{V}(x_t) = \xi^T \Gamma \xi \prec 0 \\ \text{such that} \begin{bmatrix} -1 & A & 0 & \cdots & 0 & \widetilde{A_d} \end{bmatrix} \xi = 0 \end{cases} \tag{15}$$

where $\Gamma \in \mathsf{S}^{(\mathsf{r}+2)\mathsf{n}}$ depends on $P$, $R$, $Q$ and the delay $h$.

Using projection lemma [28], there exists $X \in \mathsf{R}^{(\mathsf{r}+2)\mathsf{n}\times\mathsf{n}}$ such that (15) is equivalent to (10). ∎

**Remark 2** *In term of analysis, it is shown in [29] that for $r = 1$, this proposed function (9) is equivalent to the main classical results of the literature. Moreover, in the same paper it is proved that for $r > 1$ the results are less conservative.*

Nevertheless, applying a state feedback, we have $\widetilde{A_d} = A_d + BK$ with controller gain $K$ appearing as a decision variable. Then, the condition becomes a BMI. Indeed, the optimization problem of Theorem 1 could not be solved efficiently. In this case solutions could not be global and resolutions are not efficient. That's why in this paper, we propose a relaxation algorithm. The algorithm principle consists to alternate analysis and synthesis steps.

Let first define the *synthesis* $LMI$:

$$\mathbf{\Gamma} + < X \left[ \begin{array}{cccccc} -1 & A & 0 & \cdots & 0 & A_d + B\mathbf{K} \end{array} \right] > \prec 0 \quad (16)$$

where $K \in \mathsf{R}^{m\times n}$ and $X$ is the slack variable which has been fixed.

By the same way, we define the *analysis* $LMI$:

$$\mathbf{\Gamma} + < \mathbf{X} \left[ \begin{array}{cccccc} -1 & A & 0 & \cdots & 0 & A_d + BK \end{array} \right] > \prec 0 \quad (17)$$

where $K$ is fixed. Then, we propose the following algorithm.

**Algorithm**:

- Slack variable initialization, $X = X_0$

1) We solve the *synthesis* optimization

$$\begin{cases} h^s_{max_i} = \max_{\mathbf{P},\mathbf{Q},\mathbf{R},\mathbf{K_i}} \{h_m\} \\ \text{s.t. } LMI\ (16) \end{cases}$$

A matrix gain called $K_i$ is derived.

2) We solve the *analysis* optimization with $K = K_i$.

$$\begin{cases} h^a_{max_i} = \max_{\mathbf{P},\mathbf{Q},\mathbf{R},\mathbf{X_i}} \{h_m\} \\ \text{s.t. } LMI\ (17) \end{cases}$$

The new slack variable is derived $X_i$.

- We test if $h^a_{max_i} = h^s_{max_i}$.
- if true, there is no improvement on the maximal size of the allowable delay: end of the algorithm
- if false, the process is reiterated to step (1) with a new slack variable and upperbound of the delay.

**Remark 3** *At any step, one always has $h^a_{max_i} \geq h^s_{max_i}$. Consequently, throughout the progression of the algorithm the upperbound $h_m$ can not regress.*

Notes that the main problem, which is common in relaxation methods, remains the initialization of slack variables (or $K$).

For a performance purpose, in time delay system case, it is quite difficult to insure a certain level of performances. Since, time delay systems have an infinity of poles, one way to improve performances is the $\alpha$-stability, i.e. such that $\mathcal{R}_e$(poles)$\leq -\alpha$. Thus, let us consider a new system:

$$z(t) = e^{\alpha t} x(t), \quad (18)$$

with $\alpha$ a positive scalar and $x(t)$ the state vector of our initial system (4). If the stability of the new system (18) is proved then the initial system (4) is $\alpha$-stable.

## IV. Application to TCP/AQM dynamics and validation through NS-2

In this section, we are going first to consider the nominal system in order to expose the control principle. Then, we will extend our methods to the robust case. Finally, we will try also to improve dynamic performances by placing poles of the closed loop in a certain region.

### A. Numerical example

Considering a widely used numerical illustration extracted from [12], where the desired queue size is $q_0 = 175$ packets with $T_p = 0.2$ second and $C = 3750$ packets/s (corresponds to a 15 Mb/s link with an average packet size of 500 bytes). Then, for a load of $N = 60$ TCP sessions, we have $W_0 = 15$ packets, $p_0 = 0.008$, $R_0 = 0.246$ seconds. The following open loop system is obtained.

$$\begin{aligned} \begin{bmatrix} \delta\dot{W}(t) \\ \delta\dot{q}(t) \end{bmatrix} &= \begin{bmatrix} -0.2644 & -0.0044 \\ 243.9024 & -4.0650 \end{bmatrix} \begin{bmatrix} \delta W(t) \\ \delta q(t) \end{bmatrix} \\ &+ \begin{bmatrix} -0.2644 & 0.0044 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta W(t-h(t)) \\ \delta q(t-h(t)) \end{bmatrix} \end{aligned} \quad (19)$$

In a general way, for various network parameters, it appears that the open loop system (4) is *IOD* stable [20]. However, in order to avoid congestion and to regulate queue size at a desired level in spite of uncertainty on delay, an AQM has to be implanted.

### B. DD Synthesis

Using the relaxation algorithm previously exposed, we get the following results table I for the robust delay dependent case where a common Lyapunov-Krasovskii functional is found for each vertice of the polytop (8). Considering quadratic stability framework [23], condition (10) has to be verified on each vertice with the same matrices $P$, $Q$, $R$, $X$ and $K$. It is shown that an IOD stabilizing state feedback $K$ can always be found (for nominal case) [20]. This latter gain can constitute a starting point for the algorithm. Although this method may not provides a global optimum it affords a systematic technique for the algorithm initialization.

| r | $[\mathbf{R_{0_{min}}}, \mathbf{R_{0_{max}}}]$ | Gain K | $\mathbf{h_m}$ |
|---|---|---|---|
| 1 | $[0.1, 0.45]$ | $10^{-3}[-0.589 \;\; 0.0244]$ | 0.56 |
| 1 | $[0.1, 0.5]$ | $10^{-3}[-0.321 \;\; 0.0204]$ | 0.48 |
| 2 | $[0.1, 0.45]$ | $10^{-3}[-0.575 \;\; 0.0240]$ | 0.62 |
| 2 | $[0.1, 0.5]$ | $10^{-3}[-0.272 \;\; 0.0193]$ | 0.52 |

Table I
*DD* state feedback gains to stabilize a polytop

**Remark 4**
- *If $R_{0_{max}} > h_m$, then system (6) is stable only for $R_0 \in [R_{0_{min}}, h_m]$ since $R_0$ is the RTT and corresponds also to the delay.*
- *As expected, we obtain better results for $r = 2$, since $h_{max}$ is larger.*

Our results can be compared with results from [18] where a robust delay dependent stabilisation is designed. In [18], the system in closed loop is shown to be robustly stable for $R_0 \in [0, 0.216]$ while the proposed criterion of Theorem 1 robustly stabilizes the system for $R_0 \in [0.1, 0.5]$.

### C. Performances

We choose in this part to place the poles of the closed loop system in a convex region using the $\alpha$-stability principle. In order to improve the system performances (4), we apply our algorithm replacing matrices $A$, $A_d$ and $B$ by $(\alpha \mathsf{1}+\mathsf{A})$, $e^{\alpha h}A_d$ and $e^{\alpha h}B$ respectively. However, that substitution introduces some terms including the delay $h$, we can not thus insure the criterion validity $\forall h \leq h_{max}$.

The idea is to use the polytopic approach once again in order to insure the performances on a set of delays. Let give $\delta = e^{\alpha h}$, and for $h \in [h_{min}, h_{max}]$ we have $\delta \in [\delta_{min}, \delta_{max}]$. The method consists to solve the algorithm on two vertices (there will be one additional $LMI$ at each analysis and synthesis steps). Considering example (19), the algorithm provides results of table II.

| $\alpha$ | $[\mathbf{h_{min}}, \mathbf{h_{max}}]$ | **Gain K** |
|---|---|---|
| 1 | $[0.1, 0.9]$ | $10^{-3}[0.355 \quad 0.0084]$ |
| 2 | $[0.1, 0.36]$ | $10^{-3}[2.34 \quad 0.0054]$ |
| 3 | $[0.1, 0.22]$ | $10^{-3}[4.23 \quad 0.0041]$ |

Table II
*DD* STATE FEEDBACK GAINS FOR $\alpha$-STABILITY

Note that depending on the network topology, it is not necessary to have a lower bound $\delta_{min}$ too small (since a minimal propagation delay is unavoidable).

**Remark 5** *For $\alpha = 3$, the maximal allowable delay $h_{max}$ is lower than the nominal delay $R_0$ (delay at the equilibrium point). Consequently, the corresponding gain $K$ is not valid.*

It is also possible to improve performances for a set of systems, such that the polytop $\mathcal{P}$ defined in Section II-C. As previously, the $\alpha$-stability condition must be tested on each vertice of the polytop. The objective is to obtain the desired dynamic on a whole set. Considering the same example than previously, results (with $r = 1$) of table III are obtained.

| $\alpha$ | $[\mathbf{R0_{min}}, \mathbf{R0_{max}}]$ | **Gain K** | $\mathbf{h_m}$ |
|---|---|---|---|
| 1 | $[0.2, 0.3]$ | $10^{-3}[1.144 \quad 0.0069]$ | 0.70 |
| 2 | $[0.2, 0.3]$ | $10^{-3}[3.003 \quad 0.0045]$ | 0.25 |

Table III
*DD* STATE FEEDBACK GAINS FOR ROBUST $\alpha$-STABILITY

One remarks that for $\alpha = 2$ the maximal allowable delay is only $0.25$ (since $h_{max} = 0.25 < R_{o_{max}}$).

So, we are able to provide a constant matrix gain which allows to stabilize the queue length at a target value despite of some uncertainty on the delay RTT and with a certain level of performance.

### D. Simulations

We aim at proving the effectiveness of our method using NS-2 [21], a network simulator widely used in the communication community. Taking values from the previous numerical example, we apply our AQM based on a state feedback. The target queue length $q_0$ is 175 packets while buffer size is $800$. The average packet length is $500$ Kbytes. The default transport protocol is TCP-New Reno without ECN marking.

For the convenience of comparison, we adopt the same values and network configuration than [12] which design a PI controller (*Proportional-Integral*). This PI is configured as follow, the coefficients $a$ and $b$ are fixed at $1.822e-5$ and $1.816e-5$ respectively, the sampling frequency is $160$Hz.

In the figure 1, we apply the gain $K$ from the table III which ensures *DD* robust stability and performances with $\alpha = 1$. We compare our result with PI AQM provided by [12] (see figure 1). It appears that our proposed AQM regulates faster than the PI-AQM which presents larger oscillations and a more important peak phenomenon.

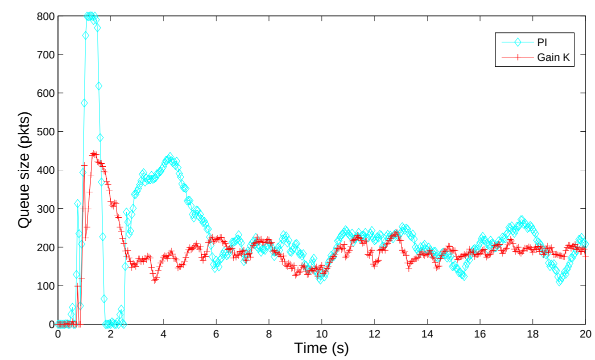


Figure 1. *Time evolution of the queue length: comparison between PI and state feedback Gain K ($\alpha = 1$).*

The table IV shows different characteristics concerning the response of the closed loop system with the PI-AQM and the state feedback AQM. We observe that the $K$ controller keeps the queue size closed to the desired length whereas the PI allows a larger distribution around the equilibrium point.

| | Settling time | Mean | Standard deviation |
|---|---|---|---|
| PI | $\approx 6$ s | 186.68 pkts | 78.86 pkts |
| Gain K | $\approx 3$ s | 175.54 pkts | 44.95 pkts |

Table IV
RESPONSES OF CLOSED LOOP SYSTEM WITH PI AND K CONTROLLERS

We also test the gain $K$ from the table II which ensures *DD* stability and improves performances with $\alpha = 2$ (see figure 2). However, we observe that the response is quite similar. This can be explained by the fact that our results are conservative and we don't know exactly where poles are located. Moreover, our criteria have been developed for linear system whereas TCP/AQM behaviour is non linear. Simulation of perturbed system is reported in figure 3. In this case, the propagation delay has been increased by 20 ms and the queue size is still stable without degradation of performance.

For more important pertubations (on the delay $R_0$), the system is still stable but the steady state changes since we converge to a new equilibrium point.

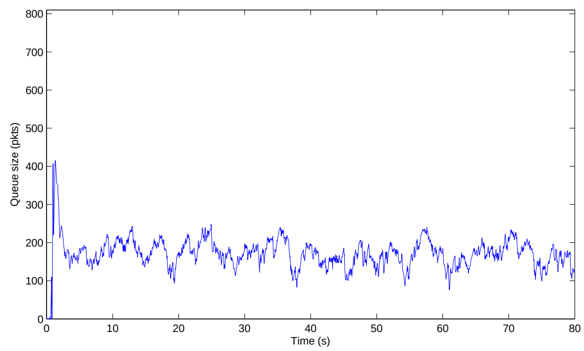


Figure 2. *Time evolution of the queue length for gain $K$ calculated from DD nominal stabilization and performances ($\alpha = 2$).*

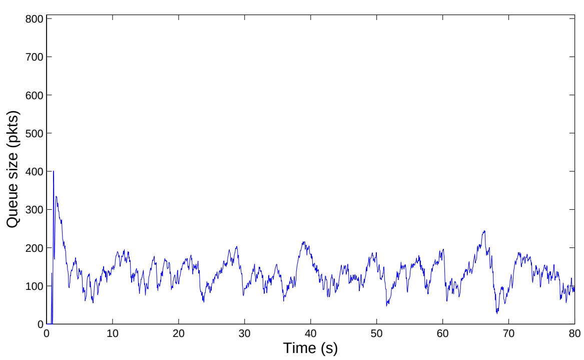


Figure 3. *Time evolution of the queue length for gain $K$ calculated from DD robust stabilization and performances ($\alpha = 1$) with a perturbation on the delay.*

## V. Conclusion and future works

In this preliminary work, we have proposed the construction of a robust AQM for the congestion problem in communications networks. The developed AQM have been established by using Lyapunov Krasovskii theory and semi definite programming to solve the Linear Matrix Inequalities. Note that the proposed methods have been extended to the robust case where the delay in the loop is uncertain. Finally, this methodology has been validated using NS simulator.

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