Construction of Lyapunov-Krasovskii functional for time-varying delay systems

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***Abstract*— This paper provides some new techniques to construct a Lyapunov-Krasovskii functional for time varying delay systems. The construction is based on a partitioning scheme of the time-varying delay leading to a new type of Lyapunov-Krasovskii functional. This functional is depending on an augmented state and also on an integral quadratic constraint added to reduce the conservatism of the proposed methodology. This approach is then extended to the robust case. Finally, some examples support our approach.**

I. INTRODUCTION

Delay systems and especially its asymptotic stability have been thoroughly studied since several decades [1], [2], [3] and references therein. The study of the delay phenomenon is motivated by its applied aspect. Indeed, many processes include dead-time phenomena in their dynamics such as biology, chemistry, economics, as well as population dynamics [2]. Moreover, processing time and propagation time in actuators and sensors generally induce such delays, especially if some devices are faraway from each other. That is the challenge of the stability of networked controlled systems [4] as well as networks control [5] [6].

In the case of constant delay, many different techniques lead to efficient algorithms (mainly based on LMIs) to test the stability of time delay system. It includes the robust approach (method based on the use of IQCs, separation approach or small gain like theorems [3]) and Lyapunov approach. In this last approach, we aim at finding a Lyapunov functional depending on the whole state of the system $x_t(\theta)$ which is not an easy task even for a linear time delay system with one delay. Indeed, for a linear time delay system, some general functional can be found [3] but is very difficult to handle. That is the reason why more simple and thus more conservative Lyapunov-Krasovskii functional (LKF) have been proposed. Generally, all these approach have to deal with two main difficulties (see [7] and [3]). The first one is the choice of the model transformation. The second problem lies on the bound of some cross terms which appear in the derivative of the Lyapunov functional. Mainly, two techniques have been proved to be efficient to reduce the conservatism. The first one adopt a discretizing scheme of the L.K. matrices [3]. At a price of an increasing number of variables to be optimized, the result tends to become a necessary and sufficient condition. Another interesting approach, developed in a Lyapunov and robust frameworks use an augmented state vector formulation to construct some new L.K.F. for the original system. Hence, in [8], a partitioning delay scheme is developed in order to construct a L.K.F which depends on a discretizing version of the whole state $x_t(\theta)$.

In the case of time varying delay, the results are much more scarce and the proposed methodologies are often conservative. In this paper, we aim at developing new type of LKF by fractionning the delay in order to take into account the whole state of the system. Even if this idea is not so new in the constant delay case it is much more complicated to elaborate if the delay is time varying. Indeed, due to the time varying nature of the delay, partitioning the delay and introducing an augmented state variable do not generally induce a good description of the original system. More particularly, it is not proved that using the state augmentation, we recover the original delayed state. That's the reason why, in the literature we can find the use of some slack variables to artificially construct linear relations between the augmented state formulation and the original state of the system. Here, we propose to cope with this problem by adding an integral quadratic constraint which take into account the relationship between the augmented delay state and the original delayed state.

Notations: For two symmetric matrices, A and B, $A > (\geq)\ B$ means that $A - B$ is (semi-) positive definite. A^T denotes the transpose of A. 1_n and $0_{m\times n}$ denote respectively the identity matrix of size n and null matrix of size $m \times n$. If the context allows it, the dimensions of these matrices are often omitted. For a given matrix $B \in R^{m\times n}$ such that $rank(B) = r$, we define $B^{\perp} \in R^{n\times(n-r)}$ the right orthogonal complement of B by $BB^{\perp} = 0$. $\|x(t)\|$ corresponds to the Euclidean norm of $x(t)$. We denote by L_2 the space of $\mathbb{R}^n$ valued functions of finite energy: $\|f\|^2_{L_2} = \int\limits_0^{\infty} |f(t)|^2 dt$. L_2^e the space of $\mathbb{R}^n$ valued functions of finite energy on finite interval. Defining an operator,a mapping from a normed space to another $\mathcal{D}: \ x \to \mathcal{D}[x]$, $\mathcal{D}^n[x]$ means that the operator $\mathcal{D}$ is applied n times to x. For instance, $\mathcal{D}^2[x]$ corresponds to $\mathcal{D}\left[\mathcal{D}[x]\right]$. A causal operator H from L_2^e to L_2^e is said to be bounded if $\|H\| = \sup_{f\in L_2} \frac{\|Hf\|}{\|f\|}$ is bounded. $x_t(.)$ is the function such that $\theta \to x_t(\theta) = x(t+\theta)$ and refers to the time delay system state. Finally, we denote by P_t the truncation operator $P_t[f](u) = f(u)$ if $u < t$ and 0 otherwise.

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II. A FIRST STEP TO A DISCRETIZATION SCHEME

Consider the following linear time delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_m, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathrm{R}^{\mathrm{n}}$ is the state vector, A, $A_d \in \mathrm{R}^{\mathrm{n}\times\mathrm{n}}$ are known constant matrices and ϕ is the initial condition. The delay, $h(t)$, is assumed to be a time-varying continuous function that satisfies

$$0 \leq h(t) \leq h_m \quad (2)$$

where $h_m > 0$ may be infinite if delay independent conditions are looked for. Furthermore, we also assume that a bound on the derivative of $\dot{h}(t)$ is provided :

$$|\dot{h}(t)| \leq d \leq 1 \quad (3)$$

with d a positive scalar.

Previous works on the stability analysis of time delay system with time-invariant delay have been proposed in [9] and in a quadratic separation [10] and Lyapunov-Krasovskii [3] frameworks, respectively. In these studies, the key idea to derive an efficient stability analysis criterion consists in delay fractioning. Indeed, it is shown that introducing redundant equations shifted in time by fractions of the delay reduce the conservatism of the stability condition.

The feature of this present contribution is to extend these previous results to the stability analysis of time-varying delay systems. Of course, the time-varying nature of the delay makes the task more complicated to deal with. Indeed, consider a time-invariant delay system,

$$\dot{x}(t) = Ax(t) + A_d x(t-h) \quad (4)$$

with h a positive constant scalar.
Applying the constant delay operator, $\mathcal{D}_{h/N} : \ x(t) \rightarrow x(t-h/N)$ to the state vector of (4) N times, the delayed state vector $x(t-h)$ of (4) is recovered. Then, a suitable choice of a Lyapunov-Krasovskii functional as explained in [9] leads to an efficient stability condition. This functional depends explicitly on a discretizing version of the whole state $[x(t), x(t-h/2), \ldots, x(t-(N-1)h/N), x(t-h)]'$. Considering a time-varying delay system (1), the fundamental difference is that applying time-varying delay operator, $\mathcal{D}_{h(t)/N} : \ x(t) \rightarrow x(t-h(t)/N)$ do not lead to a appropriate description of the state. For example, in the case of $N=2$, the fractioning scheme lead to the augmented state vector $[x(t), x(t-\frac{h(t)}{2}), x(t-\frac{h(t)}{2}-\frac{h(t-h(t)/2)}{2})]'$. The last component is deduced when the operator $\mathcal{D}_{h(t)/2}$ is applied two times to $x(t)$ ($\mathcal{D}^2_{h(t)/2}[x(t)]$) and is hardly suitable to describe the delayed instantaneous state $x(t-h)$. In order to describe properly the proposed methodology, the following is devoted to the case where the delay is partionned into two parts. Consider the following signals:

$$x_0(t) = x(t - \frac{h(t)}{2}), \quad (5)$$

$$x_1(t) = x(t - \frac{h(t)}{2} - \frac{h(t-h(t)/2)}{2}). \quad (6)$$

Given the signal $x_1(t)$, there is no apparent relationship with the delayed instantaneous state $x(t-h(t))$ and in order to clarify the relations between the two signals, we introduce an additional operator from L_2 to L_2,

$$\nabla : \ x(t) \rightarrow \int_{t-h(t)}^{t-\frac{h(t)}{2}-\frac{h(t-h(t)/2}{2}} x(u)du \quad (7)$$

that highlights the link between signals $x_1(t)$ and $x(t-h(t)$. Remarks that for a constant delay, ∇ is reduced to the null operator. Then, we can prove that the L_2 induced norm of the operator ∇ is bounded by $\frac{h_m d}{4}\sqrt{\left(\frac{1}{1-d}\right)}$. Indeed, the L_2-norm of the operator ∇ is defined by

$$\begin{aligned} \|\nabla(x)\|^2_{L_2} &= \int_0^\infty \left(\int_{t-h(t)}^{t-\frac{h(t)}{2}-\frac{h(t-h(t)/2}{2}} x(u)du \right)^2 dt \\ &= \int_0^\infty \left(\int_{t-h(t)}^{t-h(t)+\frac{1}{2}\delta(t)} x(u)du \right)^2 dt \end{aligned}$$

with $\delta(t) = h(t) - h(t - \frac{h(t)}{2})$ which can be bounded by

$$\delta(t) = \int_{t-\frac{h(t)}{2}}^t \dot{h}(u)du \leq \int_{t-\frac{h(t)}{2}}^t d \ du \leq \frac{h_m d}{2}$$

for all $t \in \mathbb{R}^+$. Then, the Cauchy-Schwarz inequality states that

$$\begin{aligned} \|\nabla(x)\|^2_{L_2} &\leq \int_0^\infty \frac{h_m d}{4} \int_{t-h(t)}^{t-h(t)+\frac{1}{2}\delta(t)} \|x(u)\|^2 du dt \\ &\leq \int_0^\infty \frac{h_m d}{4} \int_0^{h_m d/4} \|x(u+t-h(t))\|^2 du dt. \end{aligned}$$

Performing the substitution $s = u + t - h(t)$, we obtain

$$\begin{aligned} \|\nabla(x)\|^2_{L_2} &\leq \frac{h_m d}{4}\frac{1}{1-d} \int_0^{h_m d/4} \int_0^\infty \|x(s)\|^2 ds du \\ &\leq \left(\frac{h_m d}{4}\right)^2 \frac{1}{1-d}\|x\|^2_{L_2}. \end{aligned} \quad (8)$$

This last inequality concludes the proof.

Remark 1 *In order to use this inequality, we shall remark that d is supposed to be less than one. This technic is then not suitable for large variations of delay derivative, i.e. fast varying delay systems.*

Theorem 1 *Given scalars $h_m > 0$ and $0 \leq d \leq 1$, system (1) is asymptotically stable for any time-varying delay $h(t)$ satisfying (2) and (3) if there exists $n \times n$ positive definite matrices P, Q_0, Q_2, R_0, R_1 and a $2n \times 2n$ matrix $Q_1 > 0$ such that the following LMI holds:*

$$S^{\perp^T} \Gamma S^{\perp} < 0 \quad (9)$$

where

$$S = \begin{bmatrix} -1 & A & 0_n & 0_n & A_d \end{bmatrix} \quad \textit{and} \tag{10}$$

$$\Gamma = \begin{bmatrix} \mathbf{U} & \mathbf{P} & 0 & 0 & 0 \\ \mathbf{P} & \mathbf{V} & \frac{2}{h_m}\mathbf{R_1} & 0 & \frac{1}{h_m}\mathbf{R_0} \\ 0 & \frac{2}{h_m}\mathbf{R_1} & -\frac{2}{h_m}\mathbf{R_1} & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{Q_2} & \mathbf{Q_2} \\ 0 & \frac{1}{h_m}\mathbf{R_0} & 0 & \mathbf{Q_2} & \mathbf{W} \end{bmatrix} + \begin{bmatrix} 0 & 0_{n\times 2n} & 0_{n\times 2n} \\ 0_{2n\times n} & \mathbf{Q_1} & 0_{2n\times 2n} \\ 0_{2n\times n} & 0_{2n\times 2n} & 0_{2n\times 2n} \end{bmatrix} + \begin{bmatrix} 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times n} \\ 0_{2n\times 2n} & -(1-d/2)\mathbf{Q_1} & 0_{2n\times n} \\ 0_{n\times 2n} & 0_{n\times 2n} & 0 \end{bmatrix} \tag{11}$$

with

$$\begin{aligned} \mathbf{U} &= \left(\tfrac{h_m d}{4}\right)^2 \tfrac{1}{1-d}\mathbf{Q_2} + \tfrac{h_m}{2}\mathbf{R_1} + h_m\mathbf{R_0}, \\ \mathbf{V} &= \mathbf{Q_0} - \tfrac{2}{h_m}\mathbf{R_1} - \tfrac{1}{h_m}\mathbf{R_0}, \\ \mathbf{W} &= -(1-d)\mathbf{Q_0} - \mathbf{Q_2} - \tfrac{1}{h_m}\mathbf{R_0}. \end{aligned}$$

$S^\perp$ is an right orthogonal complement of S.

Proof: The proof is based on the Lyapunov-Krasovskii approach. The LKF considered is composed of the traditional terms used in the litterature (V_1, V_3 and V_5, see [11] [9] and references therein), some terms that take into account the delay partitionning (V_2 and V_4, see [8] for invariant delay case) and an integral quadratic constraint ($\Pi(t,x_t)$). Similarly as the input-output stability approach in time domain, adopted in [3], the IQC is constructed with the help of the L_2-norm of an operator. Let us define the following Lyapunov-Krasovskii functional candidate:

$$V(x) = V_1(x) + V_2(x) + V_3(x) + V_4(x) + V_5(x) \tag{12}$$

where

$$V_1(x) = x^T(t)\mathbf{P}x(t), \tag{13}$$

$$V_2(x) = \int_{t-\frac{h(t)}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{h(s)}{2}) \end{bmatrix}^T \mathbf{Q_1} \begin{bmatrix} x(s) \\ x(s-\frac{h(s)}{2}) \end{bmatrix} ds, \tag{14}$$

$$V_3(x) = \int_{t-h(t)}^{t} x^T(s)\mathbf{Q_0}x(s)ds, \tag{15}$$

$$V_4(x) = \int_{t-\frac{h_m}{2}}^{t}\int_{s}^{t} \dot{x}^T(u)\mathbf{R_1}\dot{x}(u)duds, \tag{16}$$

$$V_5(x) = \int_{t-h_m}^{t}\int_{s}^{t} \dot{x}^T(u)\mathbf{R_0}\dot{x}(u)duds, \tag{17}$$

and the IQC

$$\begin{aligned}\Pi(t,x_t) = \int_0^t & \left(\tfrac{h_m d}{4}\right)^2 \tfrac{1}{1-d}\dot{x}^T(s)\mathbf{Q_2}\dot{x}(s) \\ & -\nabla[\dot{x}(s)]^T\mathbf{Q_2}\nabla[\dot{x}(s)]ds. \end{aligned} \tag{18}$$

$\nabla[.]$ is an operator defined as (7). Since P, R_0, R_1, Q_i for $i=\{0,1,2\}$ are positive definite matrices, the functional $V_1(x)+V_2(x)+V_3(x)+V_4(x)+V_5(x)$ is also positive $\forall x \in \mathbb{R}^n$. Let us prove that $\Pi(t,x_t)$ is also a positive function. For this purpose, following a similar approach as for input-output stability analysis method in [12], let remark that $\int_0^t \nabla[\dot{x}(s)]^T\mathbf{Q_2}\nabla[\dot{x}(s)]ds = \|P_t[\nabla[Q_2^{1/2}\dot{x}(t)]]\|^2$ and as the operator ∇ is causal, this last expression can be expressed as $P_t[\nabla[Q_2^{1/2}\dot{x}(t)]]\|^2 = \|P_t[\nabla[P_t[Q_2^{1/2}\dot{x}(t)]]]\|^2$ using standard arguments [13] [12],

$$\|P_t[\nabla[P_t[Q_2^{1/2}\dot{x}(t)]]]\|^2 \le \|P_t\|^2\|\nabla\|^2\|P_t[Q_2^{1/2}\dot{x}(t)]\|^2.$$

with $\|P_t\|^2 < 1$, $\|\nabla\|^2 \le \left(\frac{h_m d}{4}\right)^2\frac{1}{1-d}$ and $\|P_t[Q_2^{1/2}\dot{x}(t)]\|^2 = \int_0^t \dot{x}^T(s)\mathbf{Q_2}\dot{x}(s)ds$. Regrouping all the terms proves that $\Pi(t,x_t)$ is positive definite. The derivative of the functional (12) along the trajectories of (1) leads to

$$\dot{V}(x) = \dot{V}_1(x) + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x) \tag{19}$$

where

$$\begin{aligned} \dot{V}_1(x) =& \dot{x}^T(t)\mathbf{P}x(t) + x^T(t)\mathbf{P}\dot{x}(t), \\ \dot{V}_2(x) =& \begin{bmatrix} x(t) \\ x_0(t) \end{bmatrix}^T \mathbf{Q_1} \begin{bmatrix} x(t) \\ x_0(t) \end{bmatrix} \\ & -(1-\frac{d}{2})\begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}^T \mathbf{Q_1} \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}, \\ \dot{V}_3(x) =& x^T(t)\mathbf{Q_0}x(t) - (1-d)x^T(t-h(t))\mathbf{Q_0}x(t-h(t)). \end{aligned} \tag{20}$$

with $x_0(t)$ and $x_1(t)$ defined as (5) and (6), respectively. Invoking the Jensen's inequality [3], terms $\dot{V}_4$ and $\dot{V}_5$ can be bounded by

$$\begin{aligned} \dot{V}_4(x) \le& \frac{h_m}{2}\dot{x}^T(t)\mathbf{R_1}\dot{x}(t) - \frac{2}{h(t)}w^T(t)\mathbf{R_1}w(t) \\ \le& \frac{h_m}{2}\dot{x}^T(t)\mathbf{R_1}\dot{x}(t) - \frac{2}{h_m}w^T(t)\mathbf{R_1}w(t) \\ \dot{V}_5(x) \le& h_m\dot{x}^T(t)\mathbf{R_0}\dot{x}(t) - \frac{1}{h(t)}v^T(t)\mathbf{R_0}v(t) \\ \le& h_m\dot{x}^T(t)\mathbf{R_0}\dot{x}(t) - \frac{1}{h_m}v^T(t)\mathbf{R_0}v(t) \end{aligned} \tag{21}$$

with $w(t) = x(t) - x(t-\frac{h(t)}{2})$ and $v(t) = x(t) - x(t-h(t))$. Invoking now the scaled small gain theorem presented in [3, p287] and considering the proposed IQC (18), the stability of (1) will be proved if the functional

$$\begin{aligned} W(t,x_t) = & \dot{V}(t,x_t) + \left(\tfrac{h_m d}{4}\right)^2 \tfrac{1}{1-d}\dot{x}^T(t)\mathbf{Q_2}\dot{x}(t) \\ & -\nabla(\dot{x})^T\mathbf{Q_2}\nabla(\dot{x}) \end{aligned} \tag{22}$$

is negative. In addition, this latter quantity can be expressed as $W(t,x_t) < \xi^T(t)\Gamma\xi(t)$ (gathering (20), (21) and (22)) with Γ defined as (11) and

$$\xi(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \\ x(t-\frac{h(t)}{2}) \\ x(t-\frac{h(t)}{2}-\frac{h(t-h(t)/2)}{2}) \\ x(t-h(t)) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ x(t) \\ x_0(t) \\ x_1(t) \\ x(t-h(t)) \end{bmatrix}. \tag{23}$$

Furthermore, using the extended variable $\xi(t)$ (23), system (1) can be rewritten as $S\xi = 0$ with S defined as (10). The original system (1) is asymptotically stable if for all ξ such that $S\xi = 0$, the inequality $\xi^T\Gamma\xi < 0$ holds. Using Finsler lemma [14], this is equivalent to $S^{\perp^T}\Gamma S^{\perp} < 0$, where $S^{\perp}$ is a right orthogonal complement of S, which concludes the proof. ■

III. MAIN RESULT

In the previous section a new condition for the time-varying delay systems analysis is obtained by means of extension of the state variables introducing a half delay. This methodology is now generalized by partitioning the interval $[t-h(t),\ t]$ into N parts.

Theorem 2 *Given scalars $h_m > 0$, $0 \le d \le 1$ and an integer $N > 0$, system (1) is asymptotically stable for any time-varying delay $h(t)$ satisfying (2) and (3) if there exists $n \times n$ positive definite matrices P, Q_0, Q_2, R_0, R_1 and a $Nn \times Nn$ matrix $Q_1 > 0$ such that the following LMI holds:*

$$S^{\perp^T}\Gamma S^{\perp} < 0 \tag{24}$$

$$\textit{where } S = \begin{bmatrix} -1 & A & 0_{n\times Nn} & A_d \end{bmatrix} \textit{ and} \tag{25}$$

$$\Gamma = \left[\begin{array}{ccc|c} \mathbf{U} & \mathbf{P} & 0 & 0_{n\times Nn} \\ \mathbf{P} & \mathbf{V} & \frac{N}{h_m}\mathbf{R_1} & 0 \quad \frac{1}{h_m}\mathbf{R_0} \\ 0 & \frac{N}{h_m}\mathbf{R_1} & -\frac{N}{h_m}\mathbf{R_1} & 0_{n\times Nn} \\ \hline 0_{Nn\times n} & \begin{matrix} 0 \\ \frac{1}{h_m}\mathbf{R_0} \end{matrix} & 0_{Nn\times n} & \mathbf{X} \end{array}\right] + \begin{bmatrix} 0 & 0_{n\times Nn} & 0_{n\times 2n} \\ 0_{Nn\times n} & \mathbf{Q_1} & 0_{Nn\times 2n} \\ 0_{2n\times n} & 0_{2n\times Nn} & 0_{2n\times 2n} \end{bmatrix} + \begin{bmatrix} 0_{2n\times 2n} & 0_{2n\times Nn} & 0_{2n\times n} \\ 0_{Nn\times 2n} & -(1-d/N)\mathbf{Q_1} & 0_{Nn\times n} \\ 0_{n\times 2n} & 0_{n\times Nn} & 0 \end{bmatrix} \tag{26}$$

with

$$\begin{aligned} \mathbf{U} &= \left(\tfrac{h_m d(N-1)}{2N}\right)^2 \tfrac{1}{1-d}\mathbf{Q_2} + \tfrac{h_m}{N}\mathbf{R_1} + h_m\mathbf{R_0}, \\ \mathbf{V} &= \mathbf{Q_0} - \tfrac{N}{h_m}\mathbf{R_1} - \tfrac{1}{h_m}\mathbf{R_0}, \\ \mathbf{X} &= \begin{bmatrix} 0_{n(N-2)\times n(N-2)} & 0_{n(N-2)\times n} & 0_{n(N-2)\times n} \\ 0_{n\times n(N-2)} & -\mathbf{Q_2} & \mathbf{Q_2} \\ 0_{n\times n(N-2)} & \mathbf{Q_2} & \mathbf{W} \end{bmatrix}, \\ \mathbf{W} &= -(1-d)\mathbf{Q_0} - \mathbf{Q_2} - \tfrac{1}{h_m}\mathbf{R_0}. \end{aligned}$$

$S^{\perp}$ is an right orthogonal complement of S.

Proof: Define the following Lyapunov-Krasovskii functional candidate:

$$V(x) = V_1(x) + V_3(x) + V_5(x) + V_7(x) + V_8(x) \tag{27}$$

where $V_1(x)$, $V_3(x)$, $V_5(x)$ are defined as (13), (15), (17) and

$$V_7(x) = \int_{t-\frac{h(t)}{N}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{h(s)}{N}) \\ x_1(t) \\ \vdots \\ x_{N-2}(t) \end{bmatrix}^T \mathbf{Q_1} \begin{bmatrix} x(s) \\ x(s-\frac{h(s)}{N}) \\ x_1(t) \\ \vdots \\ x_{N-2}(t) \end{bmatrix} ds, \tag{28}$$

$$V_8(x) = \int_{t-\frac{h_m}{N}}^{t} \int_{s}^{t} \dot{x}^T(u)\mathbf{R}\dot{x}(u)duds \tag{29}$$

as well as the IQC

$$\begin{aligned} \Pi_N(t,x_t) = \int_0^t \left(\tfrac{h_m d(N-1)}{2N}\right)^2 \tfrac{1}{1-d}\dot{x}^T(s)\mathbf{Q_2}\dot{x}(s) \\ - \nabla[\dot{x}(s)]^T\mathbf{Q_2}\nabla[\dot{x}(s)]ds \end{aligned} \tag{30}$$

with $x_i(t)$ and $\nabla[.]$ are defined as (31) and (35), respectively. As it has been stated in section I, the idea is to provide a LK functional that takes into account the state between t and $t-h(t)$. Thus, a discretization-like method is employed considering the state vector shifted by a fraction $\frac{h(t)}{N}$ of the delay. The discretized extended state is constructed with signals:

$$x_i(t) = \mathcal{D}^{(i+1)}_{h(t)/N}[x(t)]. \tag{31}$$

Note that these latter variables can be rewritten as $x_i(t) = x(t_i)$ where

$$\begin{aligned} t_i = \mathcal{D}^{(i+1)}_{h(t)/N}[t] &= t - a_0(i+1)h(t) + a_1(i+1)\delta(t) \\ &+ a_2(i+1)\delta(t+h_1(t)) + \ldots + a_i(i+1)\delta(t+h_{i-2}(t)) \end{aligned} \tag{32}$$

with

$$h_1(t) = -\frac{h(t)}{N},\ h_i(t) = \mathcal{D}^{(i-1)}_{h(t)/N}[-\frac{h(t)}{N}], \tag{33}$$

$$\delta(t) = h(t) - h(t-\frac{h(t)}{N}),\ a_j(i) = \begin{cases} \frac{i-j}{N}, & \text{if } i-j > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

Then, in order to emphasize the relationship between $x_{N-1}(t)$ and $x(t-h(t))$, we redefine the operator $\nabla[.]$ as

$$\nabla:\ x(t) \to \int_{t-h(t)}^{t_{N-1}} x(u)du. \tag{35}$$

Seeing that

$$\begin{aligned} t_{N-1} - (t-h(t)) &= a_1(N)\delta(t) + a_2(N)\delta(t+h_1(t)) \\ &\quad + \ldots + a_{(N-1)}(N)\delta(t+h_{N-3}(t)), \\ &\le [a_1(N) + \ldots + a_{(N-1)}(N)]\frac{h_m d}{N}, \\ &\le \frac{h_m d(N-1)}{2N} \end{aligned}$$

since $\delta(t)=\int_{t-\frac{h(t)}{N}}^{t}\dot{h}(s)ds \le \frac{h_m d}{N}$ and by the same way as (8), the following inequality is derived

$$\|\nabla[x]\|_{L_2}^2 \le \left(\frac{(N-1)h_m d}{2N}\right)^2 \left(\frac{1}{1-d}\right)\|x\|_{L_2}^2. \quad (36)$$

Using the same idea developed in the proof of Theorem 1, it can be easily proved that $V(x)$ (27) and the IQC (30) are positive functions for all $x \in \mathbb{R}^n$ and we have

$$\dot{V}(x)=\dot{V}_1(x)+\dot{V}_3(x)+\dot{V}_5(x)+\dot{V}_7(x)+\dot{V}_8(x) \quad (37)$$

where $\dot{V}_1(x)$, $\dot{V}_3(x)$, $\dot{V}_5(x)$ are defined as (20) (21) and

$$\begin{aligned}\dot{V}_7(x)=&\begin{bmatrix} x(t)\\ \vdots \\ x_{N-2}(t)\end{bmatrix}^T \mathbf{Q_1}\begin{bmatrix} x(t)\\ \vdots \\ x_{N-2}(t)\end{bmatrix}\\ &-(1-\frac{d}{2})\begin{bmatrix} x(t-\frac{h(s)}{N})\\ \vdots \\ x_{N-1}(t)\end{bmatrix}^T \mathbf{Q_1}\begin{bmatrix} x(t-\frac{h(s)}{N})\\ \vdots \\ x_{N-1}(t)\end{bmatrix},\\ \dot{V}_8(x)\le&\frac{h_m}{N}\dot{x}^T(t)\mathbf{R_1}\dot{x}(t)-\frac{N}{h_m}m^T(t)\mathbf{R_1}m(t)\end{aligned} \quad (38)$$

with $m(t)=x(t)-x(t-\frac{h(t)}{N})$. Invoking, as previously, the scaled small gain theorem presented in [3, p287] and considering the proposed IQC (30), the stability of (1) will be proved if the functional

$$\begin{aligned}W(t,x_t)= \;& \dot{V}(t,x_t)+\left(\frac{h_m d(N-1)}{2N}\right)^2\frac{1}{1-d}\dot{x}^T(t)\mathbf{Q_2}\dot{x}(t)\\ &-\nabla[\dot{x}]^T\mathbf{Q_2}\nabla[\dot{x}]\end{aligned} \quad (39)$$

is negative. In addition, this latter quantity can be expressed as $W(t,x_t)<\xi^T(t)\Gamma\xi(t)$ with Γ defined as (26) and

$$\xi(t)=\begin{bmatrix}\dot{x}(t)\\ x(t)\\ x(t-\frac{h(t)}{N})\\ \vdots\\ x_{N-1}(t)\\ x(t-h(t))\end{bmatrix}. \quad (40)$$

Then, using the extended variable $\xi(t)$ (40), system (1) can be rewritten as $S\xi=0$ with S defined as (25). As it has been stated in the proof of Theorem 1, the original system (1) is asymptotically stable if for all ξ such that $S\xi=0$, the inequality $\xi^T\Gamma\xi<0$ holds. Using Finsler lemma [14], this is equivalent to ${S^{\perp}}^T\Gamma S^{\perp}<0$, which concludes the proof. ∎

It is worthy to note that considering the LKF (12) with Q_1, Q_2 and R_1 set to 0, the classical results of the litterature [11] [15] [16] are recovered (related to the traditional LKF). Moreover, adding to this latter LKF the term $\int\limits_{t-h_m}^{t} x^T(s)\mathbf{Q_3}x(s)ds$ and performing the separation of the integral to $V_5(x)$ (17) i.e. estimating the derivative of $V_5(x)$ as $h_m\dot{x}^T(t)\mathbf{R_0}\dot{x}(t)-\int_{t-h(t)}^{t}\dot{x}^T(u)\mathbf{R_0}\dot{x}(u)du-\int_{t-h_m}^{t-h(t)}\dot{x}^T(u)\mathbf{R_0}\dot{x}(u)du$ rather than omitting the last term, lead to the results of [17]. Consequently, criteria provided in this paper are necessarily less pessimistic in the sense that results obtained are at least equivalent to the traditional stability conditions.

IV. ROBUSTNESS ISSUES

The proposed approach in Section III can be easily extended to the robust case. Indeed, while affine polytopic uncertain models are considered, the following system is defined:

$$\dot{x}(t)=A(\alpha)x(t)+A_d(\alpha)x(t-h(t)), \quad (41)$$

with $h(t)$ satisfying conditions (2) (3) and $\begin{bmatrix} A(\alpha) & A_d(\alpha)\end{bmatrix} = \sum_{i=1}^{\eta}\alpha_i\begin{bmatrix} A^{[i]} & A_d^{[i]}\end{bmatrix}$ where $\alpha = \begin{pmatrix}\alpha_1 & \dots & \alpha_\eta\end{pmatrix}$ belongs to the set $\Xi = \{\alpha_i \ge, \ \sum_{i=1}^{\eta}\alpha_i=1\}$. Note that the matrix S (25) is linear with respect to the model parameters $A^{[i]}$ and $A_d^{[i]}$. Thus, we denote the parameter dependent matrix

$$S(\alpha)=\sum_{i=1}^{\eta}\alpha_i S^{[i]}=\sum_{i=1}^{\eta}\alpha_i\begin{bmatrix}-1 & A^{[i]} & 0_{n\times Nn} & A_d^{[i]}\end{bmatrix}. \quad (42)$$

Theorem 3 *Given scalars $h_m>0$, $0\le d\le 1$ and an integer $N>0$, system (41) is asymptotically robustly stable for any time-varying delay $h(t)$ satisfying (2) and (3) if there exists $n\times n$ positive definite matrices $P^{[i]}$, $Q_0^{[i]}$, $Q_2^{[i]}$, $R_0^{[i]}$, $R_1^{[i]}$ and $Nn\times Nn$ matrices $Q_1^{[i]}>0$ and a $(N+3)n\times n$ matrix Y such that the following LMI hold for $i=\{1,2\dots\eta\}$:*

$$\Gamma^{[i]}+\mathbf{Y}S^{[i]}+{S^{[i]}}^T\mathbf{Y}^T<0$$

where $S^{[i]}$ are defined as in (42) and $\Gamma^{[i]}$ are structered as (26) with the according matrices $P^{[i]}$, $Q_0^{[i]}$, $Q_2^{[i]}$, $R_0^{[i]}$, $R_1^{[i]}$ and $Q_1^{[i]}$.

Due to space limitation the proof is omitted. Nevertheless, this latter is very similar to the one presented in [18] for linear systems.

V. EXAMPLES

Consider the following system,

$$\dot{x}(t)=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix}x(t)+\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}x(t-h(t)). \quad (43)$$

For this academic example, many results were obtained in the literature. For various d, the maximal allowable delay, h_m, is computed. To demonstrate the effectiveness of our criterion, results are compared against those obtained in [11], [1], [15], [17], [19] and [16]. All these papers, except the last one, use the Lyapunov theory in order to derive some stability analysis criteria for time delay systems. In [16], the stability problem is solved by a classical robust control approach: the IQC framework. Finally, [20] provides a stability criterion based on a new modelling of time delay systems considering an augmented state composed of the original state and its

TABLE I

THE MAXIMAL ALLOWABLE DELAYS h_m FOR SYSTEM (43)

d	0	0.1	0.2	0.5	0.8	Nb of var.
[11]	4.472	3.604	3.033	2.008	1.364	35
[1]	1.632	1.632	1.632	1.632	1.632	32
[15]	4.472	3.604	3.033	2.008	1.364	27
[16]	4.472	3.604	3.033	2.008	1.364	6
[17]	4.472	3.605	3.039	2.043	1.492	42
[19]	4.472	3.605	3.039	2.043	1.492	146
[20]	5,120	4,081	3,448	2,528	2,152	313
Theo 2 $N = 2$	5,717	4,286	3,366	2,008	1,364	22
Theo 2 $N = 4$	5,967	4,375	3,349	2,008	1,364	48
Theo 2 $N = 6$	6,120	4,396	3,321	2,008	1,364	90

TABLE II

THE MAXIMAL ALLOWABLE DELAYS h_m FOR SYSTEM (44)

d	0	0.05	0.1	0.2	0.3	0.5
[11]	∞	8.330	5.459	3.255	2.176	0.999
[1]	1.082	1.082	1.082	1.082	1.082	1.082
[15]	∞	8.330	5.459	3.255	2.176	0.999
[17]	∞	8.331	5.461	3.264	2.195	1.082
Theo 2	∞	10.311	6.095	3.295	2.176	0.999

derivative. Then, a suitable new type of LKF is derived which reduce the conservatism of the stability condition. The results are shown in Table I.

Then, considering the augmented state vector (40) by delay fractioning, Theorem 2 improves the maximal allowable delays for slow time-varying delays. Indeed, conservatism is reduced thanks to the discretization scheme. As expected, this operation provides more information on the system and thus improves the stability analysis criterion. Consider now the following system,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - h(t)). \quad (44)$$

The delay dependent stability analysis of system (44) has been studied and results are shown in table (II). System (44) is IOD stable (*independent of delay*) when the delay is constant. Once again, it is observed that Theorem 2 (with $N = 2$) improves the maximal bound on the delay which preserves the stability of (44) in the case of slow time-varying delays.

VI. CONCLUSION

In this paper, a new condition for the stability analysis of time-varying delay systems is proposed in the Lyapunov-Krasovskii framework. This latter criterion is formulated in terms of LMI which can be solved efficiently. Inherent conservatism of the Lyapunov-Krasovskii approach is reduced with the use of the delay fractioning methodology. Then, additional terms for the Lyapunov functional are required in order to describe as well as possible the system making the links between the different considered signals. Finally, a numerical example shows that this method reduced conservatism and improved the maximal allowable delay for slow time-varying delays.

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