Optimization over trace polynomials

Victor Magron (LAAS CNRS)
Joint work with Igor Klep and Jurij Volcic

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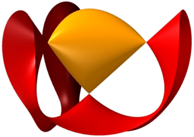
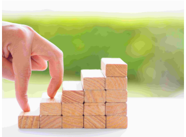

What is a trace polynomial?

An element of $\mathbb{T} = \mathsf{T}\langle \underline{x} \rangle$

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Symmetric noncommutative variables $\underline{x} = (x_1, \ldots, x_n)$
& sums of product traces T

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sums of hermitian squares (SOHS): $f^\star f$ hermitian square
$S \subset \operatorname{Sym} \mathbb{T}$ $\quad X_j$ operators from finite von Neumann algebra
Constraints $\mathcal{D}_S = \{\underline{X} = (X_1, \ldots, X_n) : s(\underline{X}) \succcurlyeq 0, \quad \forall s \in S\}$

Optimization over $\mathbb{T}$: special cases

- **Eigenvalue** optimization 💡 no traces [Helton-McCallough 04, Navascuez-Pironio-Acin 08]

$$\lambda_{\min} = \inf \{ \langle a(\underline{X}) \mathbf{v}, \mathbf{v} \rangle : \underline{X} \in \mathcal{D}_S, \|\mathbf{v}\| = 1 \}$$

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- Finite-dimensional matrices [Klep-Spenko-Volcic 18]: $a \succ 0$ on $\mathcal{D}_S \Rightarrow a$ has weighted SOHS decomposition
- Univ case [Klep-Pascoe-Volcic]: $a \succcurlyeq 0 \Rightarrow a =$ SOHS/SOHS
- Multilinear case [Huber]

Motivation: quantum information

Entanglement in quantum mechanics
→ **upper bounds** for violation levels of Bell inequalities

[Fukuda-Nechita 14] limit output states with input having specific parameters → bounds for generalized tensor traces

[Pozas et al 19] "scalar extension" of NPA hierarchy → identify correlations not attainable in entanglement-swapping scenario

NC operators A_i, C_k satisfy causal constraints:

$$\mathrm{tr}(A_{i_1} \cdots A_{i_m} C_{k_1} \cdots C_{k_m}) - \mathrm{tr}(A_{i_1} \cdots A_{i_m})\, \mathrm{tr}(C_{k_1} \cdots C_{k_m}) = 0.$$

💡 Additional variables for each $\mathrm{tr}(w)$ but no convergence proof

Contribution

Theorem: T variant of Helton-McCullough Psatz

Let $S \subset \operatorname{Sym} \mathbb{T}$ and $a \in \mathbb{T}$. The Positivstellensatz-induced hierarchy of semidefinite programs produces a convergent increasing sequence with limit $\inf_{\mathcal{D}_S} a$.

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Theorem: T variant of Helton-McCullough Psatz

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Theorem: cyclic Positivstellensatz for $\mathbb{T}$

Let $\mathcal{M}^{\mathrm{cyc}}$ be an archimedean cyclic quadratic module & $a \in \mathrm{Sym}\,\mathbb{T}$. The following are equivalent

(i) $a \succcurlyeq 0$ on $\mathcal{D}_{\mathcal{M}^{\mathrm{cyc}}}$

(ii) $\forall \varepsilon > 0$, there exist SOS $s_1, s_2 \in \mathbb{R}[t]$, $q \in \mathcal{M}^{\mathrm{cyc}}$ such that

$$\mathrm{tr}(ay) + \varepsilon = \mathrm{tr}(s_1(a)y + s_2(a)(1-y)) + q$$

where y is an auxiliary symmetric free variable.

$\mathrm{tr}(ay) + \varepsilon$ is in the module generated by $\mathcal{M}^{\mathrm{cyc}}, y, 1-y$

Moment-sums of squares hierarchies

Non-cyclic Psatz for T

Cyclic Psatz for $\mathbb{T}$

SDP hierarchies

Moment-sums of squares hierarchies

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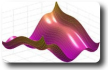
NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Theory

(Primal)		(Dual)
$\inf \int f \, d\mu$		$\sup \lambda$
with μ proba $\Rightarrow$	**INFINITE LP**	$\Leftarrow$ with $f - \lambda \geqslant 0$

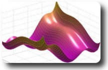
Moment-sums of squares hierarchies

NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Practice

(Primal **Relaxation**)

moments $\int \mathbf{x}^\alpha \, d\mu$



(Dual **Strengthening**)

$f - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f^*$

[Lasserre/Parrilo 01]

degree r & n vars $\Longrightarrow$ $\binom{n+2r}{n}$ **SDP** VARIABLES

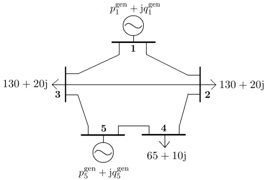
Numeric solvers $\Longrightarrow$ **Approx** Certificate

Success stories: Lasserre's hierarchy

MODELING POWER: **Cast** as ∞-*dimensional* LP over measures

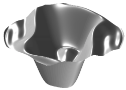
STATIC **Polynomial Optimization**

Optimal Powerflow $n \simeq \mathbf{10^3}$ [Josz et al 16]

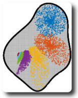


DYNAMICAL **Polynomial Optimization**

Regions of attraction [Henrion-Korda 14]



Reachable sets [Magron et al 19]

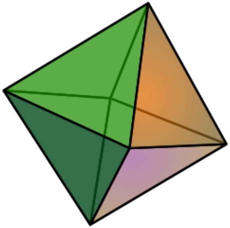


Conic Programming: LP

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d}$$

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij} z_j \geqslant d_i$"



Polyhedron

Conic Programming: SDP

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0$$

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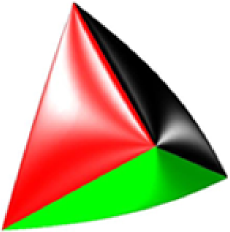

Spectrahedron

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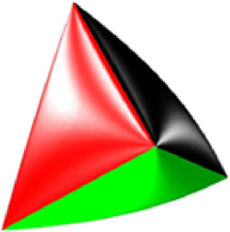

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SDP for Polynomial Optimization

Prove **polynomial inequalities** with SDP:

$$f = x_1^2 - 2x_1x_2 + x_2^2 \geqslant 0$$

Find $\mathbf{z}$ s.t. $f = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$x_1^2 - 2x_1x_2 + x_2^2 = z_1 x_1^2 + 2z_2 x_1x_2 + z_3 x_2^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$$

$$\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$$

SDP for Polynomial Optimization

Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

$$x_1^2 - 2x_1x_2 + x_2^2 = (x_1 \quad x_2) \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2)^2$$

Solving **SDP** $\implies$ Finding **Sums of Squares** certificates

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathcal{D}_S} f(\mathbf{x})$

Semialgebraic set $\mathcal{D}_S = \{\mathbf{x} \in \mathbb{R}^n : s_j(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{s_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{s_2}$$

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Sums of squares (SOS) σ_j

Quadratic module: $\mathcal{M}(S)_r = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \ \deg \sigma_j \, s_j \leqslant 2r \right\}$

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Hierarchy of SDP relaxations: $\lambda_r := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{M}(S)_r \right\}$

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Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

"No Free Lunch" Rule: $\binom{n+2r}{n}$ SDP variables

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Theorem: NC Putinar's Psatz [Helton-McCullough 02]

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$a - \lambda \mathbf{1} = \sum_i h_i^\star h_i + \sum_j \sum_i t_{ji}^\star s_j t_{ji}$ with h_i, t_{ji} of **bounded** degrees

Trace optimization

$$\mathrm{tr}_{\min} = \inf\{\mathrm{tr}(a(\underline{X})) : \underline{X} \in D_S\}$$

$$\begin{aligned} = \sup \quad & m \\ \text{s.t.} \quad & \mathrm{tr}(a(\underline{X}) - m) \geqslant 0\,, \quad \forall \underline{X} \in \mathcal{D}_S \end{aligned}$$

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Converging hierarchy with cyclic quadratic modules:

💡 replace "$\mathrm{tr}(a - m) \geqslant 0$ on $\mathcal{D}_S^{\mathrm{II}_1}$" by $a - m\mathbf{1} \in \mathcal{M}^{\mathrm{cyc}}(S)_r$

$\mathcal{M}^{\mathrm{cyc}}(S)_r$ = polynomials with same trace as some from $\mathcal{M}(S)_r$

Moment-sums of squares hierarchies

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Kadison-Dubois representation theorem

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Theorem: Kadison-Dubois [Marshall 08]

Given an Archimedean quadratic module $\mathcal{M} \subseteq \mathsf{T}$ & $a \in \mathsf{T}$:

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For $S \subseteq \mathsf{T}$, "augment" S with traces of hermitian squares:

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By induction: $\forall w \in \langle \underline{x} \rangle$, $m \pm \mathrm{tr}(w) \in \mathcal{M}(S(N))$ for some $m > 0$

$$w = x_j^{2k} \implies N^k + 1 + 2\,\mathrm{tr}(x_j^k) = \left(N^k - \mathrm{tr}(x_j^{2k})\right) + \mathrm{tr}\left((x_j^k + 1)^2\right)$$

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Theorem: Non-cyclic Psatz for T [Klep-M.-Volcic 20]

$a \geqslant 0$ on $\mathcal{D}_{S[N]}^{\mathrm{II}_1}$ $\quad \Leftrightarrow \quad$ $a + \varepsilon \in \mathcal{M}(S(N))$ for all $\varepsilon > 0$

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$\mathcal{M}^{\text{cyc}} \subseteq \operatorname{Sym} \mathbb{T}$ is a cyclic quadratic module if

$1 \in \mathcal{M}^{\text{cyc}}$, $\mathcal{M}^{\text{cyc}} + \mathcal{M}^{\text{cyc}} \subseteq \mathcal{M}^{\text{cyc}}$, $a^\star \mathcal{M}^{\text{cyc}} a \subseteq \mathcal{M}^{\text{cyc}}$ $\forall a \in \mathbb{T}$

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Proposition [Klep-M.-Volcic 20]

$\mathcal{M}^{\text{cyc}}$ is archimedean $\Leftrightarrow N - \sum_{i=1}^n x_j^2 \in \mathcal{M}^{\text{cyc}}$ for some $N > 0$

Positivity of elements in T

Theorem [Klep-M.-Volcic 20]

Let $\mathcal{M}^{\text{cyc}} \subseteq \operatorname{Sym} \mathbb{T}$ & $a \in$ T.

$a \geqslant 0$ on $\mathcal{D}^{\text{II}_1}_{\mathcal{M}^{\text{cyc}}} \Leftrightarrow a + \varepsilon \in \mathcal{M}^{\text{cyc}}$ for all $\varepsilon > 0$

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$x_1 \succcurlyeq 0$ on $\mathcal{D}^{\text{II}_1}_{\mathcal{M}^{\text{cyc}}}$ but $x_1 + \varepsilon \notin \mathcal{M}^{\text{cyc}}$

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Proposition [Klep-M.-Volcic 20]

Let $(\mathcal{F}, \tau)$ be a tracial pair and $X = X^* \in \mathcal{F}$. Tfae:

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(i) $X \succcurlyeq 0$

(ii) $\tau(XY) \geq 0$ for all positive semidefinite contractions $Y \in \mathcal{F}$

(iii) $\tau(Xp(X)^2) \geq 0$ for all $p \in \mathbb{R}[t]$

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(i) $a \succcurlyeq 0$ on $\mathcal{D}^{\mathrm{II}_1}_{\mathcal{M}^{\mathrm{cyc}}}$

(ii) $\forall \varepsilon > 0$, there exist SOS $s_1, s_2 \in \mathbb{R}[t]$, $q \in \mathcal{M}^{\mathrm{cyc}}$ such that

$$\mathrm{tr}(ay) + \varepsilon = \mathrm{tr}(s_1(a)y + s_2(a)(1-y)) + q$$

where y is an auxiliary symmetric free variable.

💡 $\mathrm{tr}(ay) + \varepsilon$ is in the module generated by $\mathcal{M}^{\mathrm{cyc}}, y, 1-y$

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Tracial words & moment matrices

$\mathbb{T}$-words = $\{\prod_i \operatorname{tr}(u_i) v \mid u_i, v \in \langle \underline{x} \rangle\}$ and T-words

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Tracial degree = up to cyclic equivalence

$\mathbf{W}_r^{\mathbb{T}}$ = vector of $\mathbb{T}$-words of with tracial degree $\leqslant r$

$n = 1$: $\mathbf{W}_2^{\mathbb{T}}$ contains $1, x_1, x_1^2, \operatorname{tr}(x_1), \operatorname{tr}(x_1^2), \operatorname{tr}(x_1)x_1$

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Reminder:

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Elements of $\mathcal{M}(S(N))_r$ are

$$a_1^2 s \quad a_2^2 \big(N^k - \mathrm{tr}(x_j^{2k})\big) \quad \mathrm{tr}(ff^\star)$$

for $s \in S$, $a_i \in \mathsf{T}$, $f \in \mathbb{T}$

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There is no duality gap and $a_r \to a_{\min}^{\mathrm{II}_1}$ as $r \to \infty$

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💡 Reduction from the general trace setting to the pure trace

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Conclusion and perspectives

CONVERGING HIERARCHIES to minimize pure trace polynomials

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Hierarchy for minimal eigenvalue problem?

APPLICATIONS IN QUANTUM INFORMATION?

Exploiting SPARSITY of cost and constraints?

Thank you for your attention!

https://homepages.laas.fr/vmagron

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