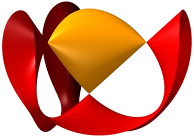
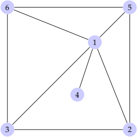
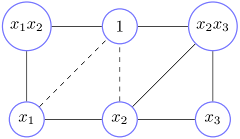
Sparse polynomial optimization: old and new

Victor Magron (LAAS CNRS)

Conférence PGMO
29 Novembre 2022







Dense polynomial optimization

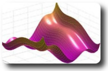
NP-hard NON CONVEX Problem $f_{\min} = \inf f(\mathbf{x})$

Theory

(Primal)

$$\inf \int f \, d\mu$$

with μ proba $\Rightarrow$



INFINITE-DIM

(Dual)

$$\sup \; b$$

$\Leftarrow$ with $f - b \geqslant 0$

Dense polynomial optimization

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Practice

(Primal **Relaxation**)

moments $\int \mathbf{x}^\alpha \, d\mu$

(Dual **Strengthening**)

$f - b =$ **sum of squares**

finite number $\Rightarrow$ **FINITE-DIM** $\Leftarrow$ **fixed** degree

[Lasserre '01] HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$

Based on representing positive polynomials [Putinar '93]

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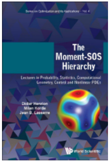

Attracted a lot of attention in optimization, applied mathematics, quantum computing, engineering, theoretical computer science

Sparse polynomial optimization

Structure exploitation with "SPARSE" cost f and constraints

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Correlative sparsity: few variable products in f

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$\rightsquigarrow f = x_1 x_2 + x_2 x_3 + \cdots + x_{99} x_{100}$

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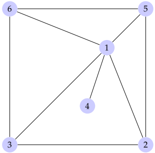

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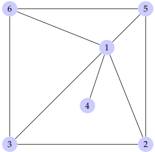

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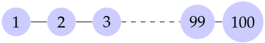
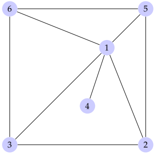


Term sparsity: few terms in f

$\rightsquigarrow f = x_1^{99} x_2 + x_1 x_2^{100}$



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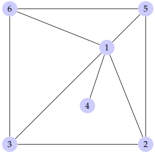

Ideal sparsity: constraints

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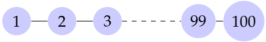

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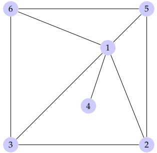

Ideal sparsity: constraints

$\rightsquigarrow x_1 x_2 = x_2 x_3 = 0$

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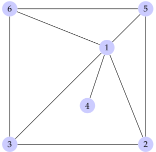

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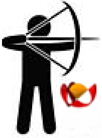
PERFORMANCE  VS  ACCURACY

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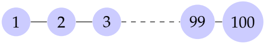

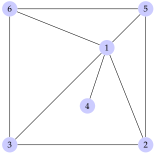

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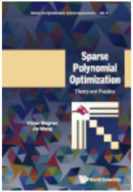
PERFORMANCE

VS

ACCURACY

Tons of applications: computer arithmetic, deep learning, entanglement, optimal power-flow, analysis of dynamical systems, matrix ranks



Moment-SOS hierarchies

Correlative sparsity

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Ideal sparsity

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Moment-SOS hierarchies: an example

NP hard General Problem: $f_{\min} := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

Semialgebraic set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1-x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1-x_2)}^{g_2}$$

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Sums of squares (SOS) σ_j

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Sums of squares (SOS) σ_j

Quadratic module: $\mathcal{M}(\mathbf{X})_d = \left\{ \sigma_0 + \sum_j \sigma_j g_j, \ \deg \sigma_j \, g_j \leqslant 2d \right\}$

Moment-SOS hierarchies

Hierarchy of SDP relaxations:

$$\lambda_d := \sup_{\lambda} \Big\{ \lambda : f - \lambda \in \mathcal{M}(\mathbf{X})_d \Big\}$$

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✗ **"No Free Lunch" Rule**: $\binom{n+2d}{n}$ SDP variables

Moment-SOS hierarchies

Correlative sparsity

Term sparsity

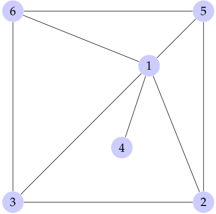
Ideal sparsity

Correlative sparsity

💡 Exploit few links between **variables** [Lasserre, Waki et al. ’06]

$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$

Chordal graph after adding edge $(3, 5)$



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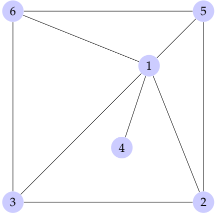


maximal cliques I_k

$I_1 = \{1, 4\}$
$I_2 = \{1, 2, 3, 5\}$
$I_3 = \{1, 3, 5, 6\}$

Dense SDP: 210 vars

Sparse SDP: 115 vars

Average size $\kappa \rightsquigarrow \kappa^{2d}$ vars

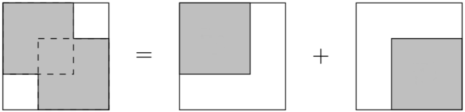
Correlative sparsity

Theorem [Griewank Toint '84]

Chordal graph G with maximal cliques I_1, I_2

$Q_G \succcurlyeq 0$ with nonzero entries at edges of G

$\Longrightarrow \; Q_G = {P_1}^T Q_1 P_1 + {P_2}^T Q_2 P_2$ with $Q_k \succcurlyeq 0$ indexed by I_k



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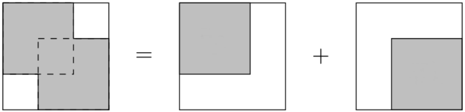

Sparse $f = f_1 + f_2$ where f_k involves **only** variables in I_k

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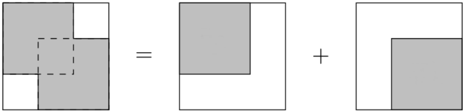

Sparse $f = f_1 + f_2$ where f_k involves **only** variables in I_k

Theorem: Sparse Putinar's representation [Lasserre '06]

$f > 0$ on $\{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$
chordal graph G with cliques I_k $\implies$
ball constraints for each $\mathbf{x}(I_k)$

$$f = \sigma_{01} + \sigma_{02} + \sum_j \sigma_j g_j$$

SOS σ_{0k} "sees" vars in I_k
σ_j "sees" vars from g_j

Extension to noncommutative optimization

Self-adjoint noncommutative (NC) variables $x = (x_1, \ldots, x_n)$

Theorem [Helton & McCullough '02]

$f \succcurlyeq 0 \Leftrightarrow f$ SOS (all positive polynomials are sums of squares)

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BAD NEWS: there is **no** sparse analog! [Klep Magron Povh '21]

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Theorem: Sparse NC Positivstellensatz [Klep Magron Povh '21]

$f = \sum_k f_k$, f_k depends on $x(I_k)$
$f > 0$ on $\{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$
chordal graph with cliques I_k
ball constraints for each $\mathbf{x}(I_k)$

$\Longrightarrow$

$$f = \sum_{k,i} (s_{ki}^\star s_{ki} + \sum_{j \in J_k} {t_{ji}}^\star g_j t_{ji})$$

s_{ki} "sees" vars in I_k
t_{ji} "sees" vars from g_j

Application to violation of Bell inequalities

$\mathbf{I}_{3322}$ Bell inequality (entanglement in quantum information)

$$f = x_1(x_4 + x_5 + x_6) + x_2(x_4 + x_5 - x_6) + x_3(x_4 - x_5) - x_1 - 2x_4 - x_5$$

Maximal violation levels $\rightarrow$ **upper bounds** on $\lambda_{\max}$ of f on $\{x : x_i^2 = x_i, x_i x_j = x_j x_i$ if $i \in \{1, 2, 3\}, j \in \{4, 5, 6\}\}$

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💡 $I_k \to \{x_1, x_2, x_3, x_{k+3}\}$

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level	sparse	dense [Pál & Vértesi '18]
2	0.2550008	0.2509397

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2	0.2550008	0.2509397
3	0.2511592	0.2508756

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Maximal violation levels $\to$ **upper bounds** on λ_{max} of f on
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💡 $I_k \to \{x_1, x_2, x_3, x_{k+3}\}$

level	sparse	dense [Pál & Vértesi '18]
2	0.2550008	0.2509397
3	0.2511592	0.2508756
3'		0.250875**4** (1 day)

Application to violation of Bell inequalities

$\mathbf{I}_{3322}$ Bell inequality (entanglement in quantum information)

$$f = x_1(x_4 + x_5 + x_6) + x_2(x_4 + x_5 - x_6) + x_3(x_4 - x_5) - x_1 - 2x_4 - x_5$$

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4	0.2508917	
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6	0.250875397718	(1 hour)

PERFORMANCE  VS  ACCURACY

Moment-SOS hierarchies

Correlative sparsity

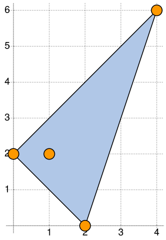
Term sparsity

Ideal sparsity

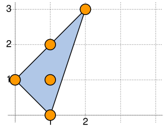
Term sparsity: unconstrained

$f = 4x_1^4x_2^6 + x_1^2 - x_1x_2^2 + x_2^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton polytope $\mathscr{B} = \text{conv}\,(\text{spt}(f))$



Squares in SOS decomposition $\subseteq \frac{\mathscr{B}}{2} \cap \mathbb{N}^n$
[Reznick '78]



$$f = \begin{pmatrix} x_1 & x_2 & x_1x_2 & x_1x_2^2 & x_1^2x_2^3 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1x_2^2 \\ x_1^2x_2^3 \end{pmatrix}$$

Term sparsity: unconstrained

[Postdoc Wang '19-21] ANR Tremplin-ERC

$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 9x_2^2x_3 - 45x_2x_3^2 + 142x_2^2x_3^2$$

[Reznick '78] $\rightarrow$ Newton polytope method

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COPS

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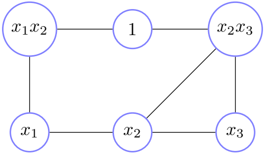
Term sparsity pattern graph G



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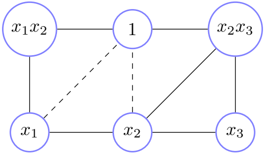
+ chordal extension G'



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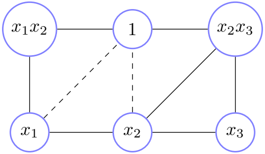

Replace Q by $Q_{G'}$ with nonzero entries at edges of G'

$\rightsquigarrow$ 6 + 9 = 15 "unknown" entries in $Q_{G'}$

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At step d of the hierarchy, tsp graph G has

Nodes V = monomials of degree $\leqslant d$

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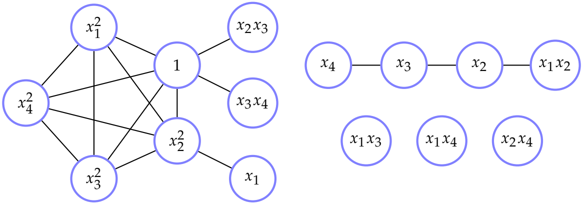
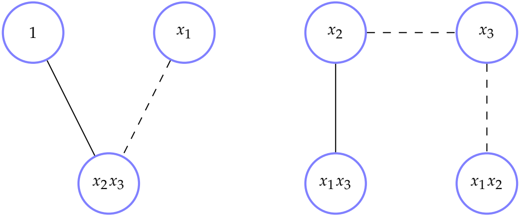
An example with $d = 2$
$f = x_1^4 + x_1 x_2^2 + x_2 x_3 + x_3^2 x_4^2$
$g_1 = 1 - x_1^2 - x_2^2 - x_3^2 \quad g_2 = 1 - x_3 x_4$



Term sparsity: support extension

$\alpha' + \beta' = \alpha + \beta$ and $(\alpha, \beta) \in E \Rightarrow (\alpha', \beta') \in E$



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By iteratively performing support extension & chordal extension

$$G^{(1)} = G' \subseteq \cdots \subseteq G^{(\ell)} \subseteq G^{(\ell+1)} \subseteq \cdots$$

Two-level hierarchy of lower bounds for $f_{\min}$, indexed by sparse order ℓ and relaxation order d

Term sparsity

CONVERGENCE GUARANTEES

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- handles Combo with correlative sparsity

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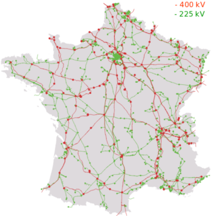
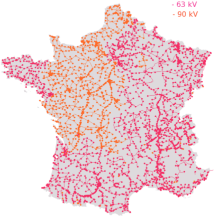
- choice of the CHORDAL EXTENSION: min / max

Application to AC optimal power-flow

Minimize active power injections of an alternating current transmission network under physical + operational constraints





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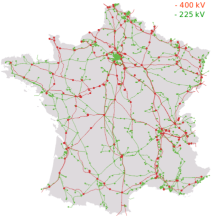

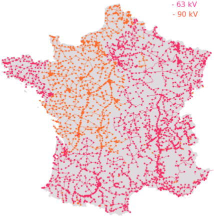

Artificial version of the control problem for electricity transmission network

Application to AC optimal power-flow

Network = Graph with buses N, *from* edges E, *to* edges E^R

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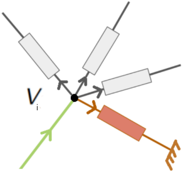

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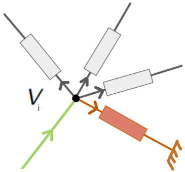

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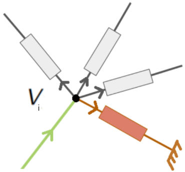

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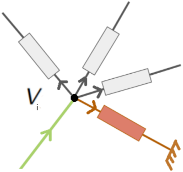

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$\rightsquigarrow$ leads to power-flow equations

Application to AC optimal power-flow

mb: maximal block size

gap: the optimality gap w.r.t. local optimal solution

n	m	CS ($d = 2$)			CS-TSSOS ($d = 2, \ell = 1$)		
		mb	time	gap	mb	time	gap
1112	4613	231	3114	0.85%	39	46.6	0.86%
		496	–	–	31	410	0.25%
4356	18257	378	–	–	27	934	0.51%
6698	29283	1326	–	–	76	1886	0.47%

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Theorem [Korda-Laurent-M-Steenkamp '22]

Ideal-sparse hierarchies provide better bounds than the dense ones

ACCURACY

Application to matrix ranks

Given a symmetric nonnegative matrix A, find the smallest r s.t.

$$A = \sum_{\ell=1}^{r} a_\ell {a_\ell}^T \qquad \text{for } a_\ell \geqslant 0$$

r is called the completely positive rank

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✓ Relax/convexify with a linear program over measures

$$r \geqslant \inf_{\mu} \{ \int_{K_A} 1 d\mu : \int_{K_A} x_i x_j d\mu = A_{ij} \; (i,j \in V) , \quad \text{supp}(\mu) \subseteq K_A \}$$

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$$K_A = \{ \mathbf{x} : \sqrt{A_{ii}} x_i - x_i \geqslant 0 \,, \quad A_{ij} - x_i x_j \geqslant 0 \ (i,j) \in E_A \,, \\ x_i x_j = 0 \ (i,j) \in \overline{E}_A \,, \quad A - \mathbf{x}\mathbf{x}^T \succcurlyeq 0 \}$$

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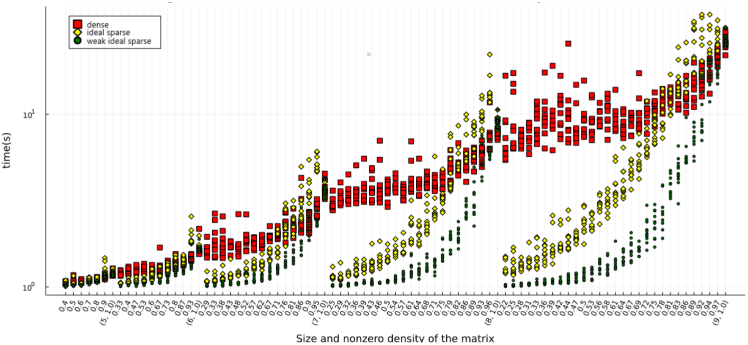
Random instances, order 2

Application to matrix ranks

Random instances, order 2



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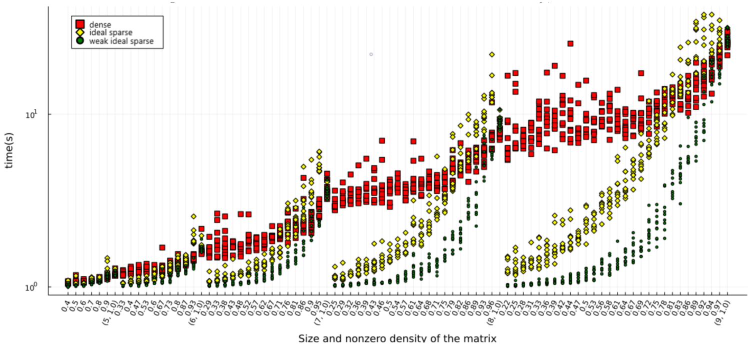

PERFORMANCE  **AND**  ACCURACY

Conclusion and perspectives

SPARSITY EXPLOITING CONVERGING HIERARCHIES to minimize large-scale polynomials

FAST IMPLEMENTATION IN JULIA: TSSOS

💡 Combine correlative & term sparsity $\rightsquigarrow$ solves problems with thousand variables

Take-away

Why should you do polynomial optimization?

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Why should you do polynomial optimization?

💡 powerful & accurate MODELING tool for many applications

Take-away

Why should you do polynomial optimization?

💡 powerful & accurate MODELING tool for many applications

💡 EFFICIENCY guaranteed on structured applications: deep learning, quantum information, energy networks

Thank you for your attention!

https://homepages.laas.fr/vmagron

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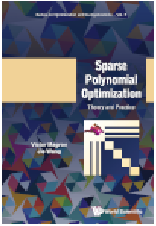

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