Semidefinite Approximations of Projections and Polynomial Images of Semialgebraic Sets

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The Problem

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$
- A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$,
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$
- $\deg f = d := \max\{\deg f_1, \ldots, \deg f_m\}$
- $\mathbf{F} := f(\mathbf{S}) \subseteq \mathbf{B}$, with $\mathbf{B} \subset \mathbb{R}^m$ a box or a ball
- Tractable approximations of $\mathbf{F}$?

The Problem

- Includes important special cases:

1. $m = 1$: **polynomial optimization**

$$\mathbf{F} \subseteq [\inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}), \sup_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})]$$

2. Approximate **projections** of $\mathbf{S}$ when $f(\mathbf{x}) := (x_1, \ldots, x_m)$

3. **Pareto curve** approximations

   For f_1,f_2 two conflicting criteria: $(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x})\, f_2(\mathbf{x}))^\top \right\}$

The Problem

3 **Pareto curve**: set of *weakly Edgeworth-Pareto optimal points*

$$(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) \, f_2(\mathbf{x}))^\top \right\}$$

Definition

A point $\bar{\mathbf{x}} \in \mathbf{S}$ is called a *weakly Edgeworth-Pareto (EP) optimal point* of Problem $\mathbf{P}$, when there is no $\mathbf{x} \in \mathbf{S}$ such that $f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}}), \; j = 1, 2.$

The Problem

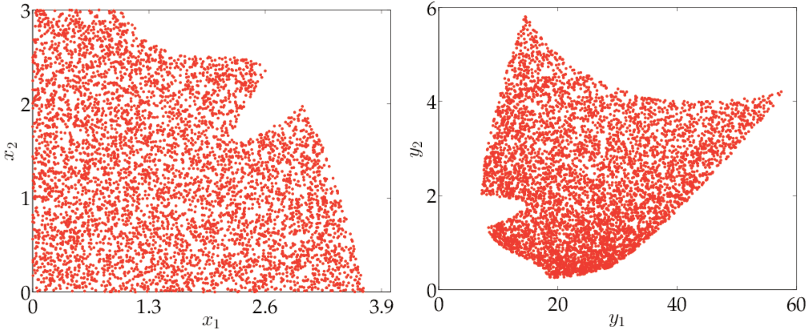
$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5$,

$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85$,

$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\}$.

$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2$,

$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4$.



Previous work

1. Exact description of projections with computer algebra

   - Real quantifier elimination (QE) [Tarski 51, Collins 74, Bochnak-Coste-Roy 98]
   - CAD: computational complexity $(sd)^{2^{O(n)}}$ for a finite set of s polynomials
   - Variant QE under radicality, equidimensionality [Hong-Safey 12]

Previous work

2 Scalarization methods for computing Pareto curve

- Numerical discretization schemes: modified Polak method [Pol 76]
- Iterative Eichfelder-Polak algorithm [Eich 09]
- Normal-boundary intersection method to find uniform spread of points [Das Dennis 98]

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications

- **No discretization** is required

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- Two different methods:
  1. Existential QE: $\mathbf{F} \subseteq \mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\}$
  2. Image measure supports: $\mathbf{F} \subseteq \mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$

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- Strong convergence guarantees

- Compute q_k or w_k with **Semidefinite programming** (SDP)

Polynomial Optimization

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

- $p^* := \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})$: NP hard

- Sums of squares $\Sigma[\mathbf{x}]$
  e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$

- $\mathcal{Q}(\mathbf{S}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^{l} \sigma_j(\mathbf{x}) g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$

- **REMEMBER**: $f \in \mathcal{Q}(\mathbf{S}) \implies \forall \mathbf{x} \in \mathbf{S}, f(\mathbf{x}) \geqslant 0$

Problem reformulation

- Borel σ-algebra $\mathcal{B}$ (generated by the open sets of $\mathbb{R}^n$)
- $\mathcal{M}_+(\mathbf{S})$: set of probability measures supported on $\mathbf{S}$.
  If $\mu \in \mathcal{M}_+(\mathbf{S})$ then
  1. $\mu : \mathcal{B} \to [0,1], \mu(\emptyset) = 0$
  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any countable $(B_i) \subset \mathcal{B}$
  3. $\int_{\mathbf{S}} \mu(d\mathbf{x}) = 1$
- $\text{supp}(\mu)$ is the smallest set $\mathbf{S}$ such that $\mu(\mathbb{R}^n \backslash \mathbf{S}) = 0$

Problem reformulation

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \int_{\mathbf{S}} f \, d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{S}$ if there exists a finite measure μ supported on $\mathbf{S}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{S}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n .$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{S})$: space of probability measures supported on $\mathbf{S}$
- $\mathcal{Q}(\mathbf{S})$: quadratic module

Polynomial Optimization Problems (POP)

$$
\begin{array}{lll}
\text{(Primal)} & & \text{(Dual)} \\
\inf \int_{\mathbf{S}} f \, d\mu & = & \sup \lambda \\
\text{s.t.} \quad \mu \in \mathcal{M}_+(\mathbf{S}) & & \text{s.t.} \quad \lambda \in \mathbb{R} \,, \\
 & & \qquad f - \lambda \in \mathcal{Q}(\mathbf{S})
\end{array}
$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{S})$
- Truncated quadratic module $\mathcal{Q}_k(\mathbf{S}) := \mathcal{Q}(\mathbf{S}) \cap \mathbb{R}_{2k}[\mathbf{x}]$

Polynomial Optimization Problems (POP)

$$
\begin{array}{llll}
 & \text{(Moment)} & & \text{(SOS)} \\
\inf & \sum_\alpha f_\alpha \, \mathbf{z}_\alpha & = \quad \sup & \lambda \\
\text{s.t.} & \mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) \succcurlyeq 0 \,, \quad 0 \leqslant j \leqslant l, & \text{s.t.} & \lambda \in \mathbb{R} \;, \\
 & \mathbf{z}_1 = 1 & & f - \lambda \in \mathcal{Q}_k(\mathbf{S})
\end{array}
$$

Lasserre's Hierarchy of SDP relaxations

$$\ell_{\mathbf{z}}(q) : q \in \mathbb{R}[\mathbf{x}] \mapsto \sum_{\alpha} q_\alpha \mathbf{z}_\alpha$$

- Moment matrix
  $\mathbf{M}(\mathbf{z})_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := \ell_{\mathbf{z}}(\mathbf{x}^\alpha \, \mathbf{x}^\beta) = \mathbf{z}_{\alpha+\beta}$
- Localizing matrix $\mathbf{M}(g_j \, \mathbf{z})$ associated with g_j
  $\mathbf{M}(g_j \, \mathbf{z})_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := \ell_{\mathbf{z}}(g_j \, \mathbf{x}^\alpha \, \mathbf{x}^\beta) = \sum_\gamma g_{j,\gamma} \, \mathbf{z}_{\alpha+\beta+\gamma}$

Lasserre's Hierarchy of SDP relaxations

- $\mathbf{M}_k(\mathbf{z})$ contains $\binom{n+2k}{n}$ variables, has size $\binom{n+k}{n}$
- Truncated matrix of order $k = 2$ with variables x_1, x_2:

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} \quad 1 & | & x_1 & x_2 & | & x_1^2 & x_1x_2 & x_2^2 \end{array} \\
\mathbf{M}_2(\mathbf{z}) = \begin{array}{c} 1 \\ - \\ x_1 \\ x_2 \\ - \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} &
\left(\begin{array}{cccccccc}
1 & | & z_{1,0} & z_{0,1} & | & z_{2,0} & z_{1,1} & z_{0,2} \\
- & - & - & - & - & - & - & \\
z_{1,0} & | & z_{2,0} & z_{1,1} & | & z_{3,0} & z_{2,1} & z_{1,2} \\
z_{0,1} & | & z_{1,1} & z_{0,2} & | & z_{2,1} & z_{1,2} & z_{0,3} \\
- & - & - & - & - & - & - & - \\
z_{2,0} & | & z_{3,0} & z_{2,1} & | & z_{4,0} & z_{3,1} & z_{2,2} \\
z_{1,1} & | & z_{2,1} & z_{1,2} & | & z_{3,1} & z_{2,2} & z_{1,3} \\
z_{0,2} & | & z_{1,2} & z_{0,3} & | & z_{2,2} & z_{1,3} & z_{0,4}
\end{array}\right)
\end{array}
$$

Lasserre's Hierarchy of SDP relaxations

- Consider $g_1(\mathbf{x}) := 2 - x_1^2 - x_2^2$. Then $v_1 = \lceil \deg g_1 / 2 \rceil = 1$.

$$\mathbf{M}_1(g_1\,\mathbf{z}) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x_1 & x_2 \end{array} \\ \begin{pmatrix} 2 - z_{2,0} - z_{0,2} & 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{0,1} - z_{2,1} - z_{0,3} \\ 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{2,0} - z_{4,0} - z_{2,2} & 2z_{1,1} - z_{3,1} - z_{1,3} \\ 2z_{0,1} - z_{2,1} - z_{0,3} & 2z_{1,1} - z_{3,1} - z_{1,3} & 2z_{0,2} - z_{2,2} - z_{0,4} \end{pmatrix} \end{array}$$

$$\begin{aligned} \mathbf{M}_1(g_1\,\mathbf{z})(3,3) = \ell(g_1(\mathbf{x}) \cdot x_2 \cdot x_2) &= \ell(2x_2^2 - x_1^2 x_2^2 - x_2^4) \\ &= 2z_{0,2} - z_{2,2} - z_{0,4} \end{aligned}$$

Lasserre's Hierarchy of SDP relaxations

- Truncation with moments of order at most $2k$
- $v_j := \lceil \deg g_j / 2 \rceil$
- Hierarchy of semidefinite relaxations:

$$\left\{ \begin{array}{rcl} \inf_{\mathbf{z}} \ell_{\mathbf{z}}(f) & = & \sum_\alpha \int_{\mathbf{S}} f_\alpha \, \mathbf{x}^\alpha \, \mu(d\mathbf{x}) = \sum_\alpha f_\alpha \, \mathbf{z}_\alpha \\ \mathbf{M}_k(\mathbf{z}) & \succcurlyeq & 0 \,, \\ \mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) & \succcurlyeq & 0 \,, \quad 1 \leqslant j \leqslant l, \\ \mathbf{z}_1 & = & 1 \,. \end{array} \right.$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$
(SDP) \begin{cases} \mathcal{P}: & \inf_{\mathbf{z}} & \sum_\alpha c_\alpha \mathbf{z}_\alpha \\ & \text{s.t.} & \sum_\alpha F_\alpha \mathbf{z}_\alpha - F_0 \succcurlyeq 0 \\ & & \\ \mathcal{D}: & \sup_{\mathbf{Y}} & \text{Trace}\,(F_0\, \mathbf{Y}) \\ & \text{s.t.} & \text{Trace}\,(F_\alpha\, \mathbf{Y}) = c_\alpha \;, \quad \mathbf{Y} \succcurlyeq 0 \;. \end{cases}
$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } f(\mathbf{x}) = \mathbf{y}\} \ ,$$

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } \|\mathbf{y} - f(\mathbf{x})\|_2^2 = 0\} \ ,$$

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } h_f(\mathbf{x}, \mathbf{y}) \geqslant 0\} \ ,$$

with

$$h_f(\mathbf{x}, \mathbf{y}) := -\|\mathbf{y} - f(\mathbf{x})\|_2^2 \ .$$

Approximation of sets defined with "∃"

Existential QE: approximate $\mathbf{F}$ as closely as desired [Lasserre 14]

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \quad ,$$

for some polynomials $q_k \in \mathbb{R}_{2k}[\mathbf{y}]$.

A hierarchy of outer approximations of F

- Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k-truncated quadratic module
- **REMEMBER**:
  $q - h_f \in \mathcal{Q}_k(\mathbf{K}) \Longrightarrow \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geqslant 0$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

- Hierarchy of Semidefinite programs:

$$\inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K})) \right\} .$$

A hierarchy of outer approximations of F

Assuming the existence of solution q_k, the sublevel sets

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \supseteq \mathbf{F} \ ,$$

provide a sequence of certified outer approximations of $\mathbf{F}$.

A hierarchy of outer approximations of F

Assuming the existence of solution q_k, the sublevel sets

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provide a sequence of certified outer approximations of $\mathbf{F}$.

It comes from the following:

- q_k feasible solution, $q_k - h_f \in \mathcal{Q}_k(\mathbf{K})$
- $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q_k(\mathbf{y}) \geqslant h_f(\mathbf{x}, \mathbf{y}) \iff \forall \mathbf{y}, q_k(\mathbf{y}) \geqslant h(\mathbf{y}) \ .$

Strong convergence property

Theorem

Assuming that $\mathring{\mathbf{S}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{K})$ is Archimedean,

1. The sequence of optimal solutions (q_k) converges to h w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0 \ , (q_k \to_{L_1} h)$$

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$$\lim_{k\to\infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0 \ , (q_k \to_{L_1} h)$$

2. $$\lim_{k\to\infty} \mathrm{vol}(\mathbf{F}_k^1 \backslash \mathbf{F}) = 0 \ .$$

Strong convergence property

Proof of existence

1. Existence of optimal q_k by **Slater's condition**

Strong convergence property

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1. Existence of optimal q_k by **Slater's condition**
   - Dual SDP:

$$
\begin{aligned}
\rho_k^* := \sup_{\mathbf{z}} \quad & \ell_{\mathbf{z}}(h_f) \\
\text{s.t.} \quad & \mathbf{M}_k(\mathbf{z}) \succcurlyeq 0, \\
& \mathbf{M}_{k-v_j}(g_j\, \mathbf{z}) \succcurlyeq 0, \quad j = 1, \ldots, l, \\
& \ell_{\mathbf{z}}(\mathbf{y}^{\beta}) = z_{\beta}^{\mathbf{B}}, \quad \forall \beta \in \mathbb{N}_{2k}^{m}.
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   - Strictly feasible $\mathbf{z}$: moments of Lebesgue measure $\lambda_{\mathbf{K}}$

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\end{aligned}
$$

   - Strictly feasible $\mathbf{z}$: moments of Lebesgue measure $\lambda_{\mathbf{K}}$
   - $q = 0$ feasible for Primal SDP:

$$
\rho_k := \inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K}) \right\} .
$$

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   - By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \to_{L_1} h$

Strong convergence property

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   - By Monotone Convergence Theorem, $f_k \to_{L_1} h$.
   - By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \to_{L_1} h$
   - Apply Putinar's Positivstellensatz to $p_k - h_f + \epsilon / \operatorname{vol}(\mathbf{B})$:

$$p_k - h_f + \epsilon / \operatorname{vol}(\mathbf{B}) = \sum_{j=0}^{l} \sigma_j g_j$$

Strong convergence property

Proof of volume convergence

2. Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geqslant -1/r\}$

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   - $\text{vol}\, \mathbf{F}(r) \to \text{vol}\, \mathbf{F}$

   - $\lim_{k \to \infty} \text{vol}\, \mathbf{F}_k^1 \leqslant \text{vol}\, \mathbf{F}(r)$

Strong convergence property

Proof of volume convergence

2. Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geqslant -1/r\}$

- $\operatorname{vol} \mathbf{F}(r) \to \operatorname{vol} \mathbf{F}$
- $\lim_{k\to\infty} \operatorname{vol} \mathbf{F}_k^1 \leqslant \operatorname{vol} \mathbf{F}(r)$
- $\operatorname{vol} \mathbf{F} \leqslant \lim_{k\to\infty} \operatorname{vol} \mathbf{F}_k^1 \leqslant \operatorname{vol} \mathbf{F}(r)$

Infinite dimensional LP formulation

- **Pushforward** $f_{\#} : \mathcal{M}(\mathbf{S}) \to \mathcal{M}(\mathbf{B})$:

$$f_{\#}\mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{S} : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{B}), \forall \mu_0 \in \mathcal{M}(\mathbf{S})$$

- $f_{\#}\mu_0$ is the **image measure** of μ_0 under f

Infinite dimensional LP formulation

$$
\begin{aligned}
p^* := \sup_{\mu_0,\mu_1,\hat{\mu}_1} \quad & \int_{\mathbf{B}} \mu_1 \\
\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}} \,, \\
& \mu_1 = f_{\#}\mu_0 \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}) \,.
\end{aligned}
$$

Lebesgue measure on $\mathbf{B}$ is $\lambda_{\mathbf{B}}(d\mathbf{y}) := \mathbf{1}_{\mathbf{B}}(\mathbf{y})\, d\mathbf{y}$

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$$
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\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}} \,, \\
& \mu_1 = f_{\#} \mu_0 \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}) \,.
\end{aligned}
$$

Lemma

Let μ_1^* be an optimal solution of the above LP.
Then $\mu_1^* = \lambda_{\mathbf{F}}$ and $p^* = \mathrm{vol}\, \mathbf{F}$.

LP Primal-dual conic formulation

The LP can be cast as follows:

$$\begin{aligned} p^* = \sup_x \quad & \langle x, c \rangle_1 \\ \text{s.t.} \quad & \mathcal{A}\, x = b\,, \\ & x \in E_1^+\,, \end{aligned}$$

LP Primal-dual conic formulation

The LP can be cast as follows:

$$\begin{aligned} p^* = \sup_{x} \quad & \langle x, c \rangle_1 \\ \text{s.t.} \quad & \mathcal{A}\, x = b\,, \\ & x \in E_1^+\,, \end{aligned}$$

with

- $E_1 := \mathcal{M}(\mathbf{S}) \times \mathcal{M}(\mathbf{B})^2 \quad F_1 := \mathcal{C}(\mathbf{S}) \times \mathcal{C}(\mathbf{B})^2$
- $x := (\mu_0, \mu_1, \hat{\mu}_1) \quad c := (0, 1, 0) \in F_1 \quad b := (0, \lambda_{\mathbf{B}})$
- the linear operator $\mathcal{A} : E_1 \to E_2$ given by

$$\mathcal{A}\,(\mu_0, \mu_1, \hat{\mu}_1) := \begin{bmatrix} -f_{\#}\mu_0 + \mu_1 \\ \mu_1 + \hat{\mu}_1 \end{bmatrix}.$$

LP Primal-dual conic formulation

Primal LP

$$
\begin{aligned}
p^* = \sup_{x} \quad & \langle x, c \rangle_1 \\
\text{s.t.} \quad & \mathcal{A}\, x = b\,, \\
& x \in E_1^+\,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^* = \inf_{y} \quad & \langle b, y \rangle_2 \\
\text{s.t.} \quad & \mathcal{A}'\, y - c \in \mathcal{C}_+(\mathbf{B})^2\,.
\end{aligned}
$$

with

- $y := (v, w) \in \mathcal{M}(\mathbf{B})^2$
- $\mathcal{A}'\,(v, w) := \begin{bmatrix} -v \circ f \\ v + w \\ w \end{bmatrix}.$

LP Primal-dual conic formulation

Primal LP

$$
\begin{aligned}
p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \quad & \int \mu_1 \\
\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}}\,, \\
& \mu_1 = f_{\#}\mu_0\,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S})\,, \\
& \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B})\,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^* := \inf_{v, w} \quad & \int w(\mathbf{y})\, \lambda_{\mathbf{B}}(d\mathbf{y}) \\
\text{s.t.} \quad & v(f(\mathbf{x})) \geqslant 0, \quad \forall \mathbf{x} \in \mathbf{S}\,, \\
& w(\mathbf{y}) \geqslant 1 + v(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbf{B}\,, \\
& w(\mathbf{y}) \geqslant 0, \quad \forall \mathbf{y} \in \mathbf{B}\,, \\
& v, w \in \mathcal{C}(\mathbf{B})\,.
\end{aligned}
$$

Zero duality gap

Lemma

$$p^* = d^*$$

Strong convergence property

Strengthening of the dual LP:

$$
\begin{aligned}
d_k^* := \inf_{v,w} \quad & \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}} \\
\text{s.t.} \quad & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\
& w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), \\
& w \in \mathcal{Q}_k(\mathbf{B}), \\
& v, w \in \mathbb{R}_{2k}[\mathbf{y}].
\end{aligned}
$$

Strong convergence property

Theorem

Assuming that $\mathring{\mathbf{F}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

1. The sequence (w_k) converges to $\mathbf{1}_{\mathbf{F}}$ w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 \ .$$

Strong convergence property

Theorem

Assuming that $\mathring{\mathbf{F}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

1. The sequence (w_k) converges to $\mathbf{1_F}$ w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1_F}| d\mathbf{y} = 0 \ .$$

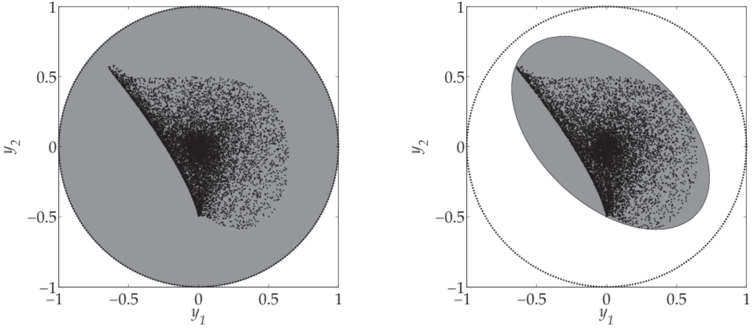
2. Let $\mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$. Then,

$$\lim_{k \to \infty} \mathrm{vol}(\mathbf{F}_k^2 \backslash \mathbf{F}) = 0 \ .$$

Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1 x_2, x_2 - x_1^3)/2$$



$\mathbf{F}_1^1$ $\mathbf{F}_1^2$

Polynomial image of the unit ball

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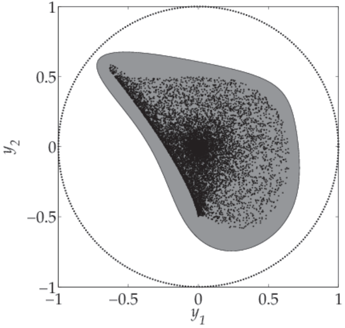
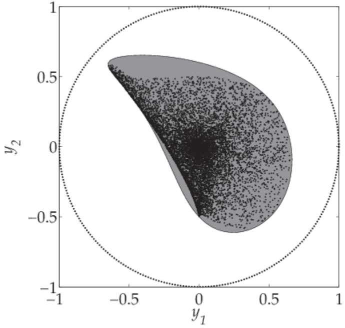


$\mathbf{F}_2^1$



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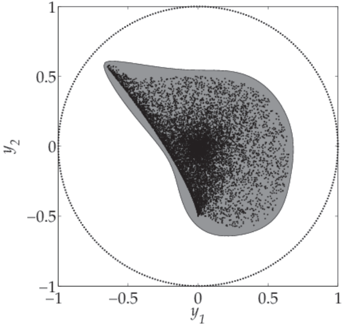
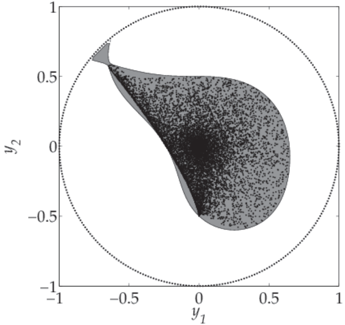


$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

Polynomial image of the unit ball

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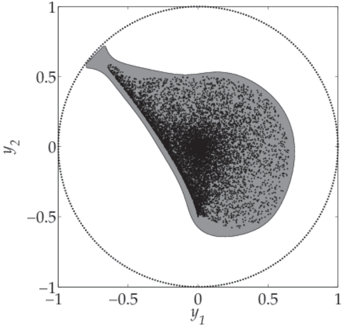
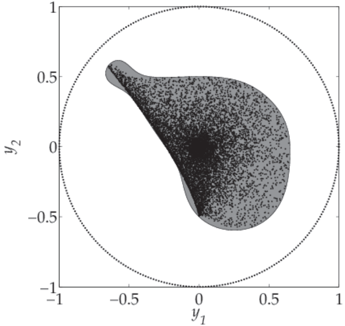
$$f(\mathbf{x}) := (x_1 + x_1 x_2, x_2 - x_1^3)/2$$



$\mathbf{F}_4^1$



$\mathbf{F}_4^2$

Semialgebraic set projections

Simpler formulation:

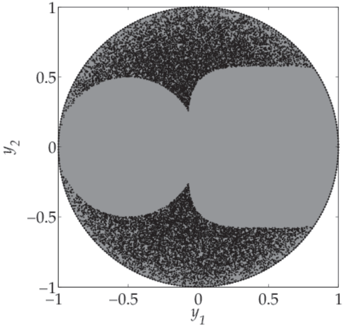
$$\begin{aligned} d_k^* := \inf_{v,w} \quad & \sum_{\beta \in \mathbb{N}^m_{2k}} w_\beta z^{\mathbf{B}}_\beta \\ \text{s.t.} \quad & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\ & w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), \\ & w \in \mathcal{Q}_k(\mathbf{B}), \\ & v, w \in \mathbb{R}_{2k}[\mathbf{y}]. \end{aligned}$$

$$\begin{aligned} \inf_{w} \quad & \sum_{\beta \in \mathbb{N}^m_{2k}} w_\beta z^{\mathbf{B}}_\beta \\ \text{s.t.} \quad & w - 1 \in \mathcal{Q}_k(\mathbf{S}), \\ & w \in \mathcal{Q}_k(\mathbf{B}), \\ & w \in \mathbb{R}_{2k}[x_1, \ldots, x_m]. \end{aligned}$$

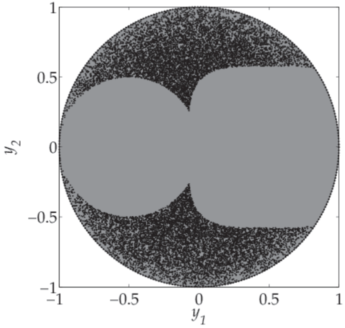
Semialgebraic set projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on $\mathbb{R}^2$ of the semialgebraic set

$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leqslant 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geqslant 0,$$
$$1/9 - (x_1 - 1/2)^4 - x_2^4 \geqslant 0\}$$



$\mathbf{F}_2^1$



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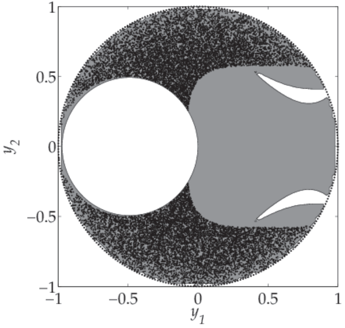
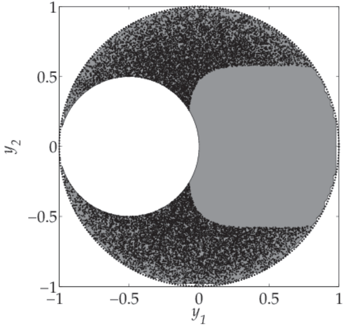


$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

Semialgebraic set projections

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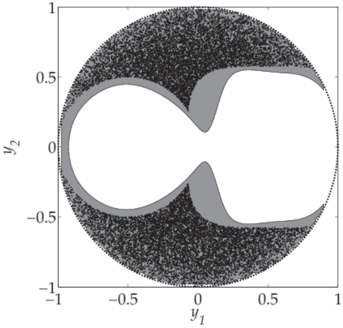
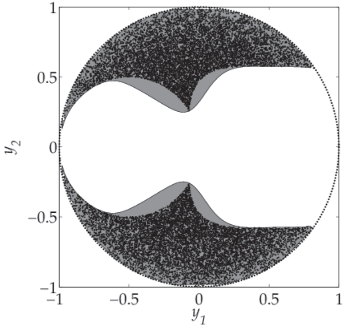


$\mathbf{F}_4^1$

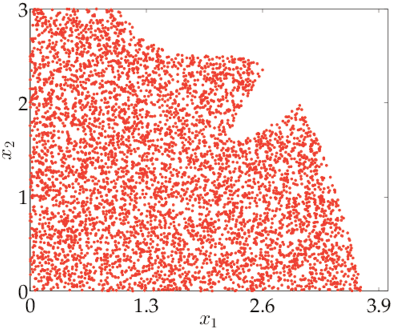


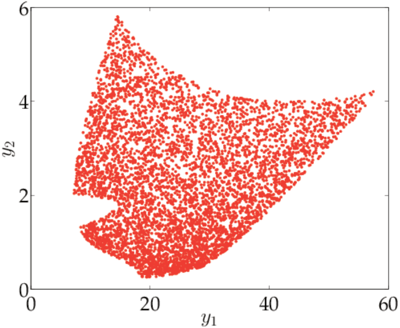
$\mathbf{F}_4^2$

Bicriteria Optimization Problems

$$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 \ ,$$
$$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 \ ,$$
$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\} \ .$$

$$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 \ ,$$
$$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 \ .$$



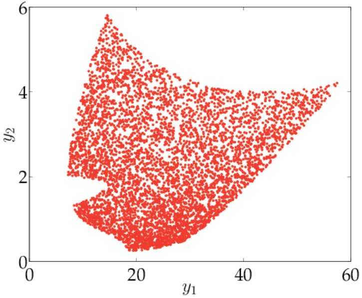


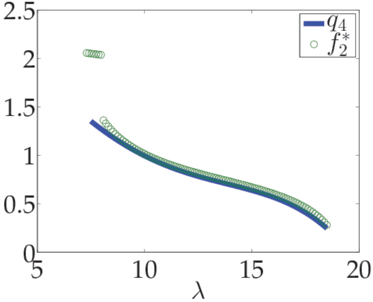
Previous Contributions

- Numerical schemes that **avoid computing finitely many points**.
- Pareto curve approximation with polynomials, **convergence guarantees** in L_1-norm

V. Magron, D. Henrion, J.B. Lasserre. Approximating Pareto Curves using Semidefinite Relaxations. *Operations Research Letters*. arxiv:1404.4772, April 2014.

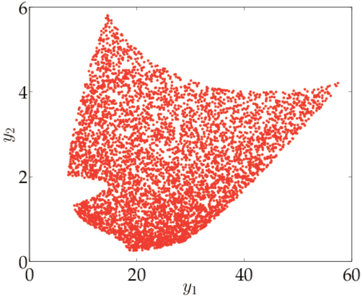
Previous Contributions

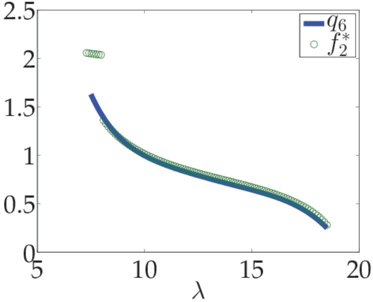




Degree 4

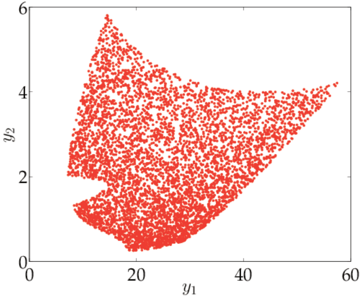
Previous Contributions

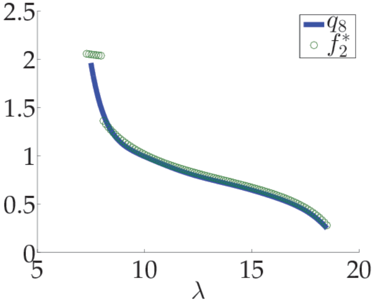




Degree 6

Previous Contributions

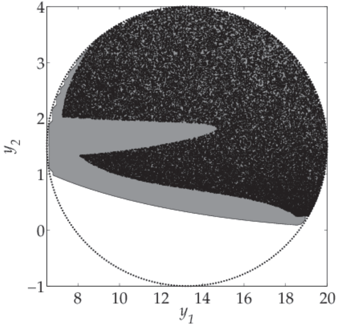




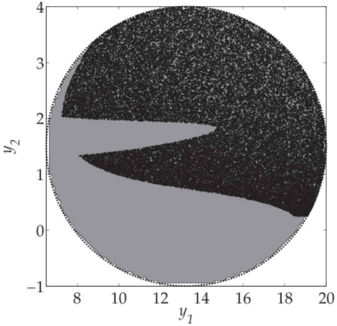
Degree 8

Approximating Pareto curves

Back on our previous nonconvex example:



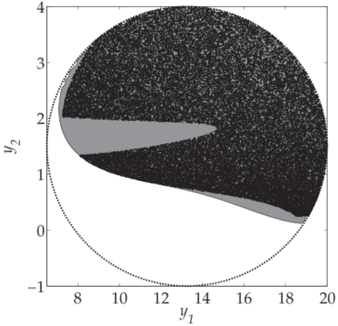
$\mathbf{F}_1^1$



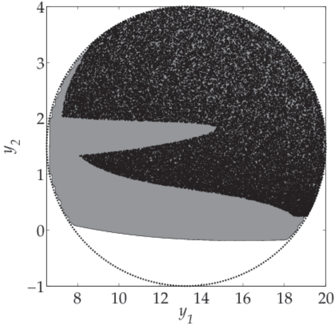
$\mathbf{F}_1^2$

Approximating Pareto curves

Back on our previous nonconvex example:



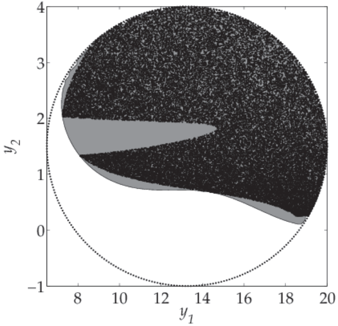
$\mathbf{F}_2^1$



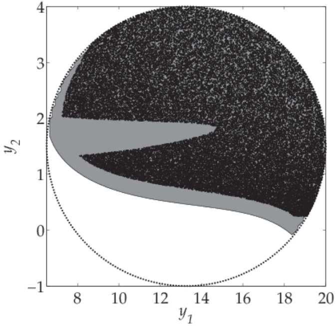
$\mathbf{F}_2^2$

Approximating Pareto curves

Back on our previous nonconvex example:



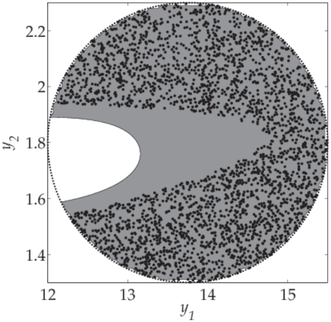
$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

Approximating Pareto curves

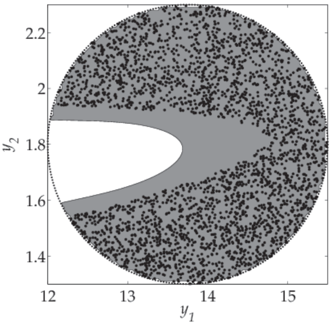
"Zoom" on the region which is hard to approximate:



$\mathbf{F}_4^1$

Approximating Pareto curves

"Zoom" on the region which is hard to approximate:

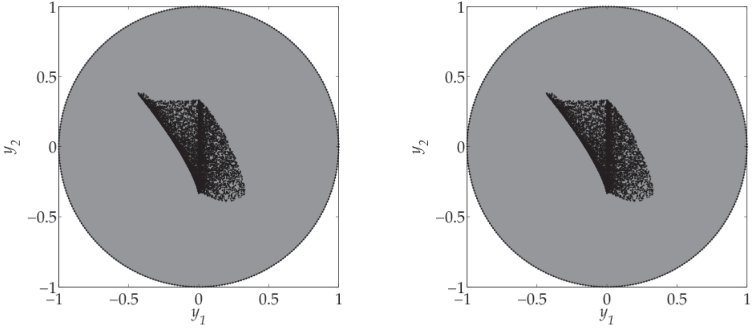


$\mathbf{F}_5^1$

Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1 x_2, x_1^2), x_2 - x_1^3)/3$$



$\mathbf{F}_1^1$ $\mathbf{F}_1^2$

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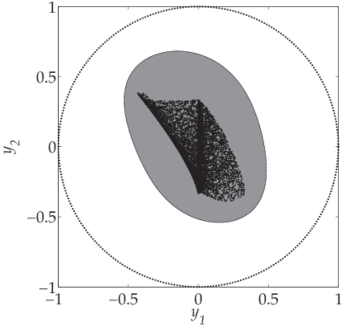
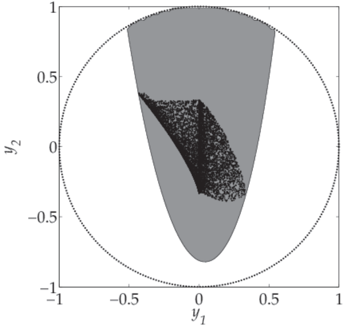


$\mathbf{F}_2^1$



$\mathbf{F}_2^2$

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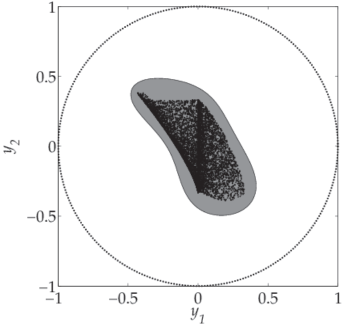
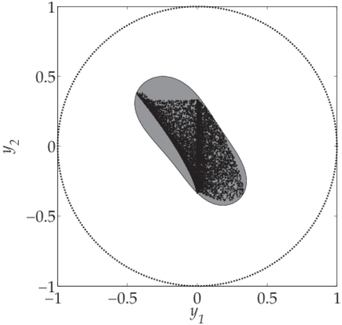


$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

Semialgebraic image of semialgebraic sets

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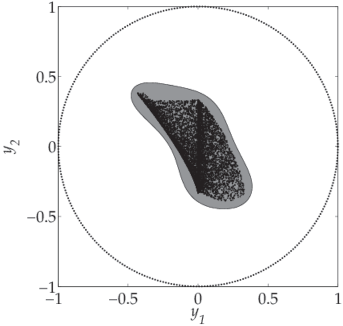
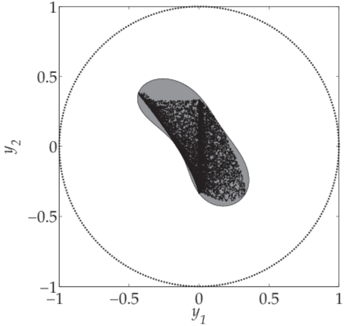


$\mathbf{F}_4^1$



$\mathbf{F}_4^2$

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Conclusion

- **Unifying** framework: projections, Pareto curves
- Computational complexity
  1. Method 1: $\binom{n+m+2k}{2k}$ SDP variables
  2. Method 2: $\binom{n+2kd}{2kd}$ SDP variables
- **Structure sparsity** can be exploited

Conclusion

Further research:

- Alternative positivity certificates LP/SDP
  1. Less computationally demanding than SDP
  2. More efficient than LP (as generic convergence cannot occur)

End

V. Magron, D. Henrion, J.B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. oo:2014.10.4606, October 2014.

Thank you for your attention!

cas.ee.ic.ac.uk/people/vmagron