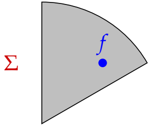
Two-player games between polynomial optimizers and semidefinite solvers

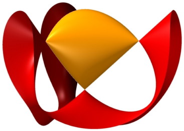
Victor Magron, CNRS–LAAS

Joint work with

Jean-Bernard Lasserre (CNRS–LAAS)
Mohab Safey El Din (Sorbonne Université)

Mosek Aps, Copenhagen, 21 January 2020



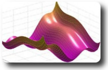


SDP for Polynomial Optimization

NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Theory

(Primal) $\inf \int f \, d\mu$ with μ proba $\Rightarrow$



INFINITE LP

$\Leftarrow$ (Dual) $\sup \lambda$ with $p - \lambda \geqslant 0$

SDP for Polynomial Optimization

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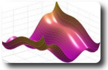
Practice

(Primal **Relaxation**)

moments $\int \mathbf{x}^\alpha \, d\mu$



(Dual **Strengthening**)

$f - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $f_d^\star \uparrow f^*$
[Lasserre/Parrilo 01]

degree d
n vars $\Longrightarrow \binom{n+d}{n}$ **SDP** VARIABLES

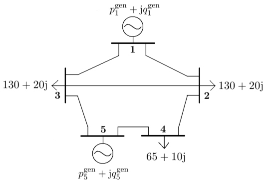
Numeric Solvers $\Longrightarrow$ **Approx** Certificate

Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as ∞-*dimensional* LP over measures

STATIC **Polynomial Optimization**

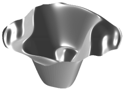
Optimal Powerflow $n \simeq \mathbf{10^3}$ [Josz et al 16]



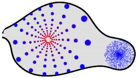
Roundoff Error $n \simeq \mathbf{10^2}$ [Magron et al 17]

DYNAMICAL **Polynomial Optimization**

Regions of attraction [Henrion et al 14]



Reachable sets [Magron et al 19]



APPROXIMATE OPTIMIZATION BOUNDS!

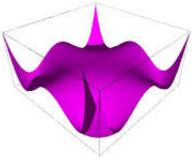
Two-player Games: Optimizers vs Solvers

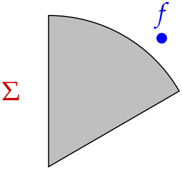
MOTZKIN POLYNOMIAL

sums of squares $= \Sigma$

$$f = \frac{1}{27} + x^2y^4 + x^4y^2 - x^2y^2$$

$f \geqslant 0$ but $f \notin \Sigma$





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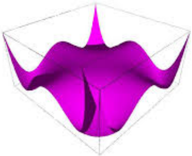

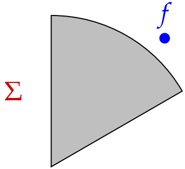

$$f^\star = \min_{(x,y)\in\mathbb{R}^2} f(x,y) = 0 \text{ for } |x^\star| = |y^\star| = \frac{\sqrt{3}}{3}$$

Lasserre's hierarchy:

- order 3 $\rightsquigarrow f_3^\star = -\infty$ unbounded SDP $\implies f \notin \Sigma$

Two-player Games: Optimizers vs Solvers

Motzkin polynomial

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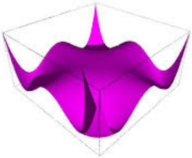

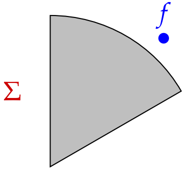

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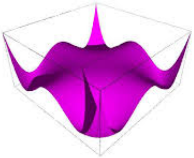

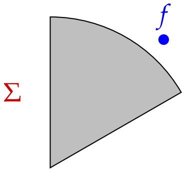

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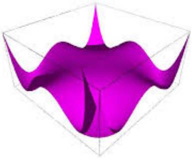
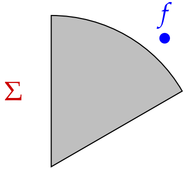
- order 4 $\rightsquigarrow f_4^\star = -\infty$
- order 5 $\rightsquigarrow f_5^\star \simeq -0.4$

Two-player Games: Optimizers vs Solvers

MOTZKIN POLYNOMIAL

sums of squares $= \Sigma$





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Lasserre's hierarchy:

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- order 4 $\rightsquigarrow f_4^\star = -\infty$
- order 5 $\rightsquigarrow f_5^\star \simeq -0.4$
- order 8 $\rightsquigarrow f_8^\star \simeq -10^{-8} \oplus$ extraction of $x^\star, y^\star$ **Paradox**?!

Two-player Games: Optimizers vs Solvers

Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$?

SDP for Polynomial Optimization

Optimization Game

Certification Game

Inaccurate SDP do Robust Optimization

$$f^\star = \inf \sum_\alpha f_\alpha \, \mathbf{x}^\alpha$$

Moment matrix $\mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

Accurate SDP Relaxations

(Primal **Relaxation**)	(Dual **Strengthening**)
$\inf_{\mathbf{y}} \sum_\alpha f_\alpha \, y_\alpha$	$\sup \ \lambda$
s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$	$f - \lambda = \sigma$
$y_0 = 1$	$\sigma \in \Sigma_d$

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$$\sup \ \lambda$$
$$f_\alpha - \lambda 1_{\alpha=0} = \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle$$
$$\mathbf{Q} \succcurlyeq 0$$

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Inaccurate SDP Relaxations

(Primal **Relaxation**)

(Dual **Strengthening**)

$$\sup \ \lambda$$

$$| \, f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \, | \leqslant \varepsilon$$

$$\mathbf{Q} \succcurlyeq -\eta \mathbf{I}$$

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Moment matrix $\mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

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Inaccurate SDP Relaxations

(Primal **Relaxation**)

$$\inf_{\mathbf{y}} \sum_\alpha f_\alpha \, y_\alpha + \eta \langle \mathbf{M}_d(\mathbf{y}), \mathbf{I} \rangle + \varepsilon \|\mathbf{y}\|_1$$

$$\text{s.t. } \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$$

$$y_0 = 1$$

(Dual **Strengthening**)

$$\sup \ \lambda$$

$$| \, f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \, | \leqslant \varepsilon$$

$$\mathbf{Q} \succcurlyeq -\eta \mathbf{I}$$

Priority to Trace Equalities: $\varepsilon = 0$

$$\tilde{f} = f + \eta \sum_{\beta} \mathbf{x}^{2\beta}$$

Inaccurate SDP Relaxations

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$$\begin{aligned} \inf_{\mathbf{y}} \quad & \sum_{\alpha} f_\alpha \, y_\alpha + \eta \langle \mathbf{M}_d(\mathbf{y}), \mathbf{I} \rangle \\ \text{s.t.} \quad & \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0 \\ & y_0 = 1 \end{aligned}$$

(Dual **Strengthening**)

$$\begin{aligned} \sup \quad & \lambda \\ & f_\alpha - \lambda 1_{\alpha = 0} - \langle \mathbf{B}_\alpha, \mathbf{Q} - \eta \mathbf{I} \rangle = 0 \\ & \mathbf{Q} \succcurlyeq 0 \end{aligned}$$

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(Primal **Relaxation**)	(Dual **Strengthening**)
$\inf_{\mathbf{y}} \sum_{\alpha} \tilde{f}_{\alpha} y_{\alpha}$	$\sup \lambda$
s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$	$\tilde{f} - \lambda = \sigma$
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Priority to Trace Equalities: $\varepsilon = 0$

$$\mathbf{B}_\infty(f, \eta) := \{ f + \theta \sum_\beta \mathbf{x}^{2\beta} : \mid \theta \mid \leqslant \eta \}$$

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Theorem [Lasserre-Magron 19]

Inaccurate SDP relaxations of the **robust** problem

$$\max_{\tilde{f} \in \mathbf{B}_\infty(f,\eta)} \min_{\mathbf{x}} \tilde{f}(\mathbf{x})$$

Priority to Trace Equalities: $\varepsilon = 0$

Theorem [Lasserre 06]

For fixed n, any $f \geqslant 0$ can be approximated arbitrarily closely by SOS polynomials.

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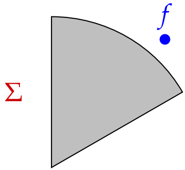
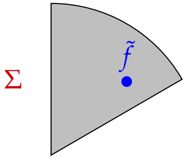


$$\tilde{f} = f + \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$$



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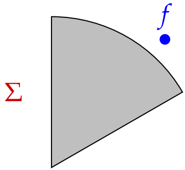

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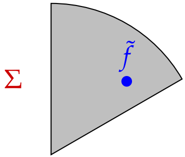
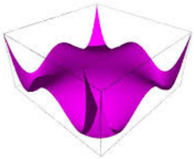


At fixed η, when $d \nearrow$, $\tilde{f} \in \Sigma$!



$$f + 10^{-7} \sum_{|\beta| \leqslant 4} \mathbf{x}^{2\beta} \in \Sigma$$

Paradox Explanation

Priority to SDP Inequalities: $\eta = 0$

Inaccurate SDP Relaxations

(Primal **Relaxation**)	(Dual **Strengthening**)
$\inf_{\mathbf{y}} \sum_{\alpha} f_\alpha y_\alpha + \varepsilon \|\mathbf{y}\|_1$	$\sup \ \lambda$
s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$	$\mid f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \mid \leqslant \varepsilon$
$y_0 = 1$	$\mathbf{Q} \succcurlyeq 0$

Priority to SDP Inequalities: $\eta = 0$

$$\mathbf{B}_\infty(f, \varepsilon) := \{\tilde{f} : \|\tilde{f} - f\|_\infty \leqslant \varepsilon\}$$

Inaccurate SDP Relaxations

(Primal **Relaxation**)	(Dual **Strengthening**)
$\inf_{\mathbf{y}} \sum_\alpha f_\alpha y_\alpha + \varepsilon \|\mathbf{y}\|_1$	$\sup_{\lambda, \tilde{f}} \lambda$
s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$	$\mid \tilde{f}_\alpha - f_\alpha \mid \leqslant \varepsilon$
$y_0 = 1$	$\tilde{f} - \lambda \in \Sigma_d$

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Theorem (Lasserre-Magron)

Inaccurate SDP relaxations of the **robust** problem

$$\max_{\tilde{f} \in \mathbf{B}_\infty(f, \varepsilon)} \min_{\mathbf{x}} \tilde{f}(\mathbf{x})$$

A Two-player Game Interpretation

max − min ROBUST OPTIMIZATION

Player 1 (solver) picks $\tilde{f} \in \mathbf{B}_\infty(f)$ $\rightsquigarrow$ **SDP leads**

Player 2 (optimizer) picks an SOS $\rightsquigarrow$ **User follows**

A Two-player Game Interpretation

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Convex SDP relaxations $\implies$ $\boxed{\max - \min = \min - \max}$

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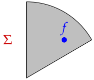
SDP for Polynomial Optimization

Optimization Game

Certification Game

From Approximate to Exact Solutions

Win TWO-PLAYER GAME

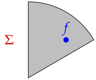


sum of squares of f?

≃ Output!

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



Hybrid Symbolic/Numeric Algorithms

sum of squares of $f - \varepsilon$?

Error Compensation

$\simeq \quad \rightarrow \quad =$

$\simeq$ Output!

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i {f_i}^2$?

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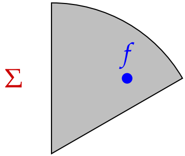
Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 = ???$$

Round & Project Algorithm [Peyrl-Parrilo 08]



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

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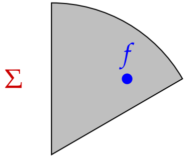

$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Find $\tilde{\mathbf{Q}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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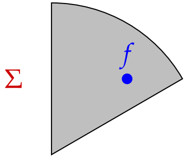

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Exact $Q \implies f_{\alpha+\beta} = \sum_{\alpha'+\beta'=\alpha+\beta} Q_{\alpha',\beta'}$

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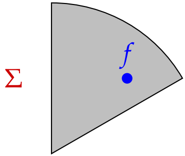

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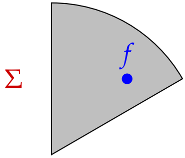
$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

Exact $Q \implies f_{\alpha+\beta} = \sum_{\alpha'+\beta'=\alpha+\beta} Q_{\alpha',\beta'}$

1. **Rounding step** $\hat{Q} \leftarrow \texttt{round}\left(\tilde{Q}, \hat{\delta}\right)$

Round & Project Algorithm [Peyrl-Parrilo 08]



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Find $\tilde{\mathbf{Q}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq {\mathbf{v}_D}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

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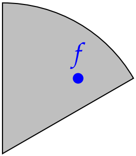
2. **Projection step**

   $Q_{\alpha,\beta} \leftarrow \hat{Q}_{\alpha,\beta} - \frac{1}{\eta(\alpha+\beta)}\left(\sum_{\alpha'+\beta'=\alpha+\beta} \hat{Q}_{\alpha',\beta'} - f_{\alpha+\beta}\right)$

Round & Project Algorithm [Peyrl-Parrilo 08]

Σ



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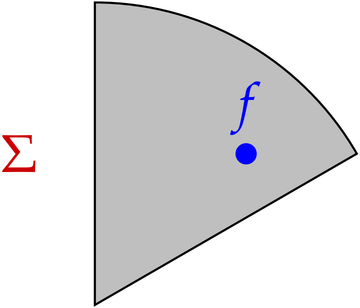
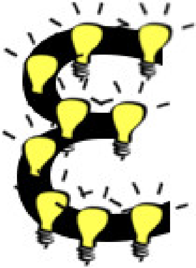
Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = \mathbf{v}_D{}^T(X)\, \mathbf{Q}\, \mathbf{v}_D(X)$ and $\mathbf{Q} \succcurlyeq 0$

Our Alternative Approach





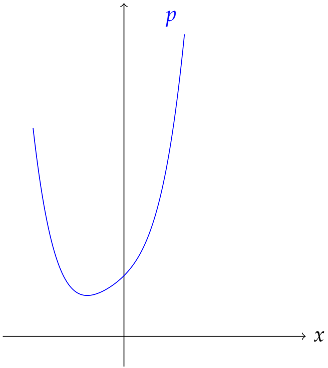
Perturbation idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

RealCertify with $n = 1$ [Chevillard et. al 11]

$f \in \mathbb{Q}[X]$, $\deg f = d = 2k$, $f > 0$



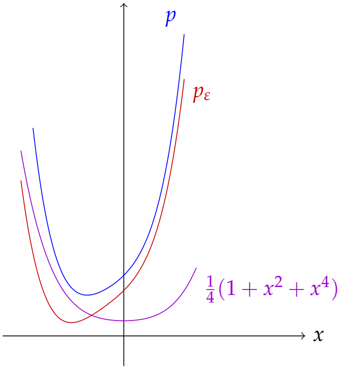
$f = 1 + X + X^2 + X^3 + X^4$

`RealCertify` with $n = 1$ [Chevillard et. al 11]

$f \in \mathbb{Q}[X]$, $\deg f = d = 2k$, $f > 0$

PERTURB: find $\varepsilon \in \mathbb{Q}$ s.t.

$$f_\varepsilon := f - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$f = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$f > \frac{1}{4}(1 + X^2 + X^4)$$

RealCertify with $n = 1$ [Chevillard et. al 11]

$f \in \mathbb{Q}[X]$, $\deg f = d = 2k$, $f > 0$

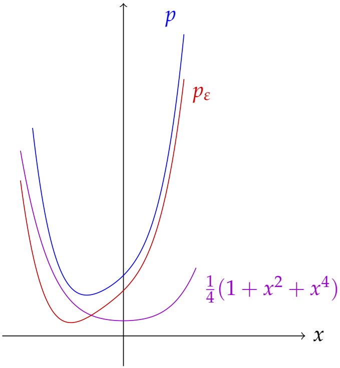
PERTURB: find $\varepsilon \in \mathbb{Q}$ s.t.

$$f_\varepsilon := f - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$

SDP Approximation:

$$f - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

ABSORB: small enough u_i
$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS

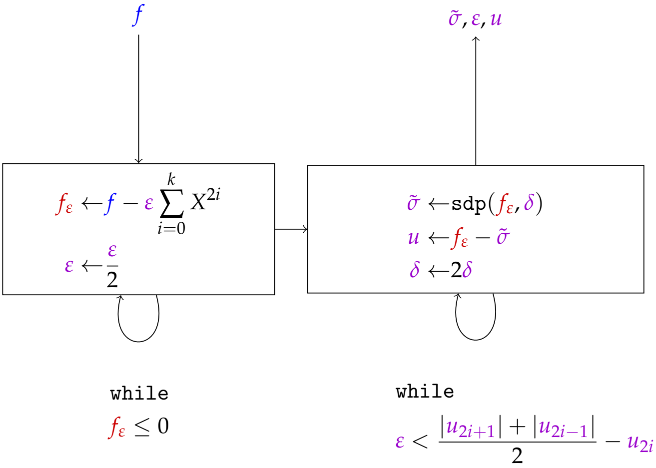


$$f = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$f > \frac{1}{4}(1 + X^2 + X^4)$$

`RealCertify` with $n = 1$: SDP Approximation



RealCertify with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

RealCertify with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

`RealCertify` with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
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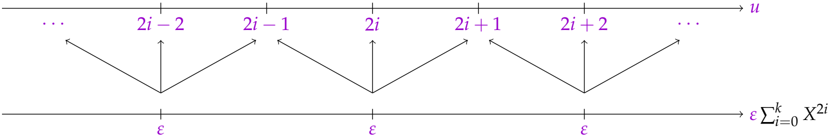

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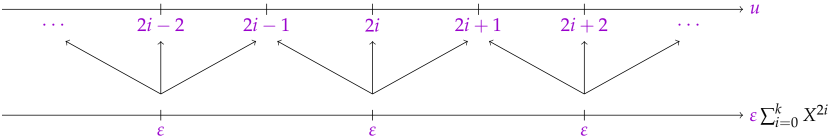
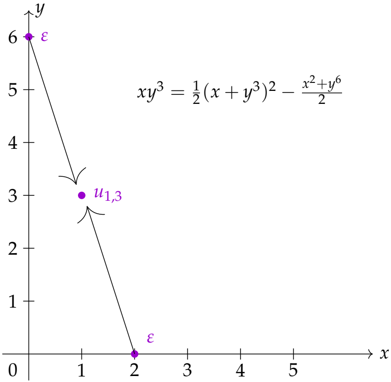


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

RealCertify with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



RealCertify with $n \geqslant 1$: Absorbtion

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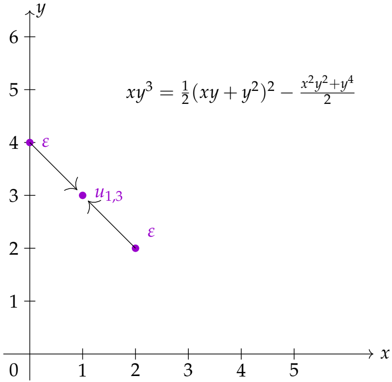

`RealCertify` with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

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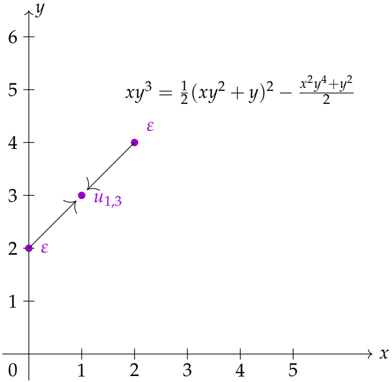


`RealCertify` with $n \geqslant 1$: Absorbtion

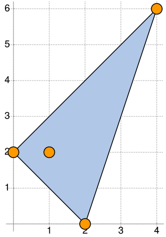
$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?

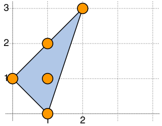
$f = 4x^4y^6 + x^2 - xy^2 + y^2$

spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\text{spt}(f))$



Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$ [Reznick 78]



RealCertify: Benchmarks

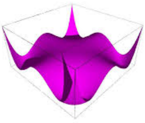
- RAGLib (critical points) [Safey El Din]
- SamplePoints (CAD) [Moreno Maza-Alvandi et al.]

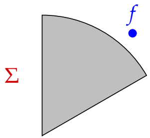
Id	n	d	RealCertify τ_1 (bits)	RealCertify t_1 (s)	RoundProject τ_2 (bits)	RoundProject t_2 (s)	RAGLib t_3 (s)	CAD t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	598.	–
f_{1}	10	4	344 347	2.45	8 374 082	4.59	–	–

Two-player Games: Perspectives

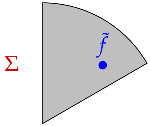
Optimization Game

Solvers **OUTPUT** inaccurate certificates ⇒ extract solutions





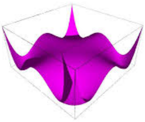
$$\tilde{f} = f + \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$$



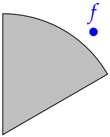
Two-player Games: Perspectives

Optimization Game

Solvers **OUTPUT** inaccurate certificates $\Rightarrow$ extract solutions



Σ



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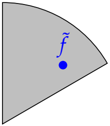


Certification Game

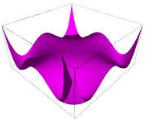
Optimizers **INPUT** inaccurate $\tilde{f} = f - \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$

$\Longrightarrow$ exact certificates

Two-player Games: Perspectives

OPTIMIZATION GAME

Solvers **OUTPUT** inaccurate certificates $\Rightarrow$ extract solutions



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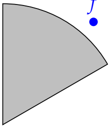

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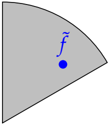


CERTIFICATION GAME

Optimizers **INPUT** inaccurate $\tilde{f} = f - \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$

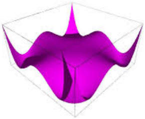
$\Longrightarrow$ exact certificates

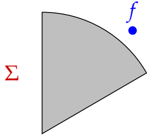
- Better arbitrary-precision SDP solvers
- Extension to other relaxations, sums of hermitian squares

Two-player Games: Perspectives

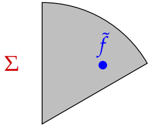
Optimization Game

Solvers **OUTPUT** inaccurate certificates $\Rightarrow$ extract solutions





$$\tilde{f} = f + \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$$



Certification Game

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Crucial need for polynomial systems certification

Available PhD/Postdoc Positions

End

Thank you for your attention!

gricad-gitlab:RealCertify

https://homepages.laas.fr/vmagron

- Lasserre & Magron. In SDP relaxations, inaccurate solvers do robust optimization, *SIOPT*. arxiv:1811.02879

- Magron & Safey El Din. On Exact Polya and Putinar's Representations, *ISSAC'18*. arxiv:1802.10339

- Magron & Safey El Din. RealCertify: a Maple package for certifying non-negativity, *ISSAC'18*. arxiv:1805.02201