

# Semidefinite Approximations of Reachable Sets for Polynomial Systems

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Joint work with

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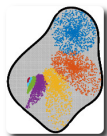
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Khalil Ghorbal (IRISA)

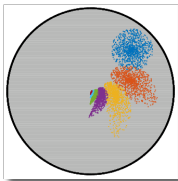
TU Berlin, Institut für Mathematik

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# The RS Problem in Discrete Time

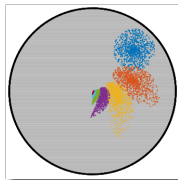
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Initial conditions  $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geq 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$

# The RS Problem in Discrete Time

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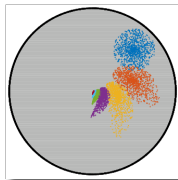


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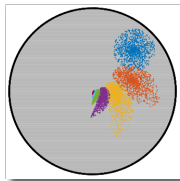
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Reachable Set (RS) of admissible trajectories

$\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t \in \mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$

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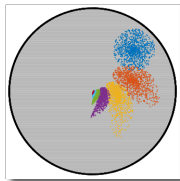
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$\mathbf{X}^\infty = \bigcup_{t \in \mathbb{N}} f^t(\mathbf{X}_0) \subseteq \mathbf{X} \subset \mathbb{R}^n$  (box or ball)

Tractable approximations of RS  $\mathbf{X}^\infty$  ?

# The RS Problem in Continuous Time

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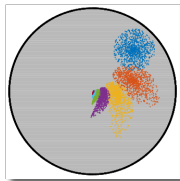


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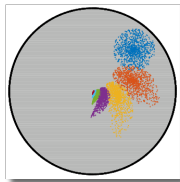
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$\mathbf{X}^T := \{(\mathbf{x}(t)) : \dot{\mathbf{x}} = f(\mathbf{x}), \forall t \in [0, T], \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{X}_0\}$

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$\mathbf{X}^\infty := \lim_{t \rightarrow \infty} \mathbf{X}^T \subseteq \mathbf{X} \subset \mathbb{R}^n$  (box or ball)

Tractable approximations of RS  $\mathbf{X}^\infty$  ?



# Motivations

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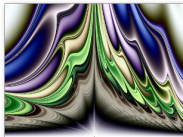
- Occurs in several contexts :

- 1 program analysis: fixpoint computation

```
toyprogram (x1, x2)  
  requires (0.25 ≤ x1 ≤ 0.75 && 0.25 ≤ x2 ≤ 0.75)  
  ;  
  while (x12 + x22 ≤ 1) {  
    x1 = x1 + 2x1x2;  
    x2 = 0.5(x2 - 2x13);  
  }
```

- 2 hybrid systems, biology: Neuron Model, Growth Model

- 3 control: integrator, Hénon map



## Related work: RS

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- 1 Contractive methods based on LP relaxations and polyhedra projection [Bertsekas 72]
- 2 Extension to nonlinear systems [Harwood et al. 16]
- 3 Bernstein/Krivine-Handelman representations [Ben Sassi et al. 15, Ben Sassi et al. 12]

⊕ LP relaxations  $\implies$  scalability

⊖ Convex approximations of nonconvex sets  $\implies$  coarse

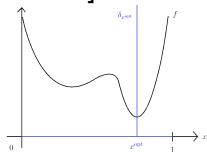
⊖ No convergence guarantees (very often)

# Related work: Lasserre hierarchy

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💡 **Cast** a polynomial optimization problem as an *infinite-dimensional* LP over measures [Lasserre 2001]

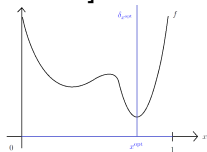
$$f^* := \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f(\mathbf{x}) d\mu$$



# Related work: Lasserre hierarchy

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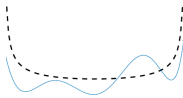
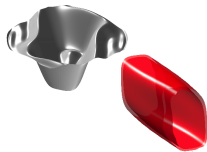


↪ Regions of attraction [Henrion-Korda 14]

↪ Maximum invariants [Korda et al. 13]

↪ Invariant 1D densities [Henrion 2012]

↪ Maximal positively invariant sets [Oustry-Tacchi-Henrion 2019]



## Related work: Lasserre hierarchy

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- 5 SDP approximation of polynomial images of semialgebraic sets [M.-Henrion-Lasserre 15]

- $\mathbf{X}_1 := f(\mathbf{X}_0) \subseteq \mathbf{X}$ , with  $\mathbf{X} \subset \mathbb{R}^n$  a box or a ball  
 $\implies$  Discrete-time system with a single iteration

- 💡 Approximation of image measure supports  
 $\implies$  certified SDP over approximations of  $\mathbf{X}_1$

- $\mathbf{X}_t := f^t(\mathbf{X}_0)$

⊖  $\deg f^t = d \times t \implies$  very expensive computation

⊖ Would only approximate  $\mathbf{X}_t$  and not  $\mathbf{X}^\infty$

# Contributions

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- General framework to approximate  $X^\infty$ 
  - ⊕ **No discretization** is required

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- General framework to approximate  $\mathbf{X}^\infty$ 
  - ⊕ **No discretization** is required
- Infinite-dimensional LP formulation
  - 💡 support of measures solving Liouville's Equation
- Finite-dimensional SDP relaxations
- $\mathbf{X}^\infty \subseteq \mathbf{X}_r^\infty = \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geq 1\}$ 
  - ⊕ Strong convergence guarantees  
 $\lim_{r \rightarrow \infty} \text{vol}(\mathbf{X}_r^\infty \setminus \mathbf{X}^\infty) = 0$
  - ⊕ Compute  $w_r$  by solving one **semidefinite program**



The RS Problem in Discrete Time

The RS Problem in Continuous Time

Motivations

**Infinite LP Formulation for Polynomial Optimization**

Infinite LP Formulation for RS

Application Examples

Conclusion and Perspectives

# What is Semidefinite Programming?

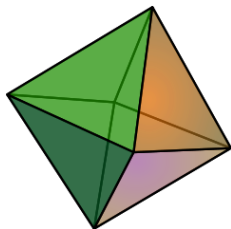
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- Linear Programming (LP):

$$\begin{array}{ll}\min_{\mathbf{z}} & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} & \mathbf{A} \mathbf{z} \geq \mathbf{d} .\end{array}$$

- Linear cost  $\mathbf{c}$

- Linear inequalities “ $\sum_i A_{ij} z_j \geq d_i$ ”



**Polyhedron**

# What is Semidefinite Programming?

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- Semidefinite Programming (SDP):

$$\begin{array}{ll}\min_{\mathbf{z}} & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 .\end{array}$$

- Linear cost  $\mathbf{c}$
- Symmetric matrices  $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities “ $\mathbf{F} \succcurlyeq 0$ ”  
( $\mathbf{F}$  has nonnegative eigenvalues)



**Spectrahedron**

# What is Semidefinite Programming?

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## ■ Semidefinite Programming (SDP):

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**Spectrahedron**

# SDP for Polynomial Optimization

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- Prove **polynomial inequalities** with SDP:

$$f(a, b) := a^2 - 2ab + b^2 \geq 0 .$$

- Find  $\mathbf{z}$  s.t.  $f(a, b) = \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} .$

- Find  $\mathbf{z}$  s.t.  $a^2 - 2ab + b^2 = z_1 a^2 + 2z_2 ab + z_3 b^2 \quad (\mathbf{A} \mathbf{z} = \mathbf{d})$

- $\underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\mathbf{F}_1} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

# SDP for Polynomial Optimization

---

- Choose a cost  $\mathbf{c}$  e.g.  $(1, 0, 1)$  and solve:

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0, \quad \mathbf{A} \mathbf{z} = \mathbf{d}. \end{aligned}$$

- Solution  $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$  (eigenvalues 0 and 2)
- $a^2 - 2ab + b^2 = (a \ b) \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2.$
- Solving **SDP**  $\implies$  Finding **SUMS OF SQUARES** certificates

# SDP for Polynomial Optimization

---

**NP hard General Problem:**  $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Semialgebraic set  $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_l(\mathbf{x}) \geq 0\}$

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 $\mathbf{X} = [0, 1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geq 0, \quad x_2(1 - x_2) \geq 0\}$



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$$\underbrace{x_1 x_2}_f = -\frac{1}{8} + \frac{1}{2} \overbrace{\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \frac{1}{2} \overbrace{x_1(1 - x_1)}^{\sigma_1 \quad g_1} + \frac{1}{2} \overbrace{x_2(1 - x_2)}^{\sigma_2 \quad g_2}$$

# SDP for Polynomial Optimization

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- Sums of squares (SOS)  $\sigma_i$

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- Sums of squares (SOS)  $\sigma_i$

- Bounded degree:

$$\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^l \sigma_j g_j, \text{ with } \deg \sigma_j g_j \leq 2r \right\}$$

# SDP for Polynomial Optimization

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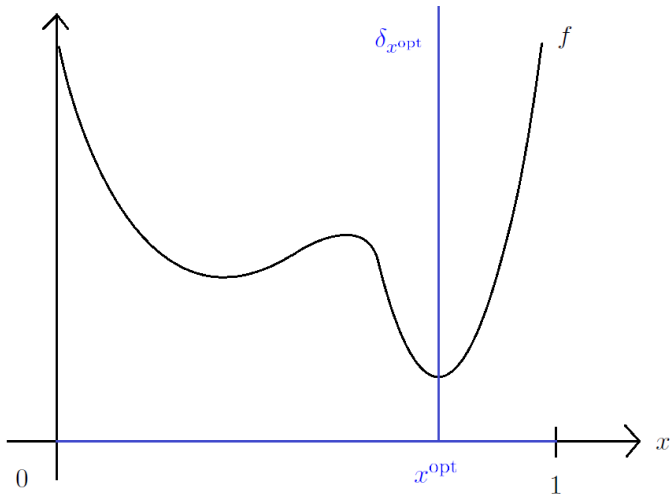
- **Hierarchy of SDP relaxations:**

$$m_r := \sup_m \left\{ m : f - m \in \mathcal{Q}_r(\mathbf{X}) \right\}$$

- Convergence guarantees  $m_r \uparrow f^*$  [Lasserre 01]
- Can be computed with SDP solvers (CSDP, SDPA)
- **“No Free Lunch” Rule:**  $\binom{n+2r}{n}$  SDP variables

# Primal-dual Moment-SOS [Lasserre 01]

$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f d\mu$$



# Primal-dual Moment-SOS [Lasserre 01]

---

- Let  $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$  be the monomial basis

## Definition

A sequence  $\mathbf{z}$  has a representing measure on  $\mathbf{X}$  if there exists a finite measure  $\mu$  supported on  $\mathbf{X}$  such that

$$\mathbf{z}_\alpha = \int_{\mathbf{X}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n.$$

# Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{X})$ : space of **probability measures** supported on  $\mathbf{X}$
- $\mathcal{Q}(\mathbf{X})$ : combining **sums of squares** and polynomials  $g_j$  from  $\mathbf{X}$

## Polynomial Optimization Problems (POP)

$$\begin{array}{ll} \text{(Primal)} & \text{(Dual)} \\ \inf \int_{\mathbf{X}} f d\mu & = \sup m \\ \text{s.t. } \mu \in \mathcal{M}_+(\mathbf{X}) & \text{s.t. } m \in \mathbb{R}, \\ & f - m \in \mathcal{Q}(\mathbf{X}) \end{array}$$

# Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences  $\mathbf{z}$  of measures in  $\mathcal{M}_+(\mathbf{X})$
- Truncated quadratic module  $\mathcal{Q}_r(\mathbf{X})$

## Lasserre's Hierarchy for Polynomial Optimization

| (Moment)                                                                         | (SOS)                                 |
|----------------------------------------------------------------------------------|---------------------------------------|
| $\inf \sum_{\alpha} f_{\alpha} \mathbf{z}_{\alpha}$                              | $= \sup m$                            |
| s.t. $\mathbf{M}_{r-r_j}(g_j \mathbf{z}) \succcurlyeq 0, \quad 0 \leq j \leq l,$ | s.t. $m \in \mathbb{R},$              |
| $\mathbf{z}_0 = 1$                                                               | $f - m \in \mathcal{Q}_r(\mathbf{X})$ |



The RS Problem in Discrete Time

The RS Problem in Continuous Time

Motivations

Infinite LP Formulation for Polynomial Optimization

**Infinite LP Formulation for RS**

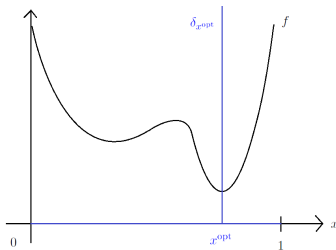
Application Examples

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# Characterizing the RS

## CHARACTERIZE A VALUE

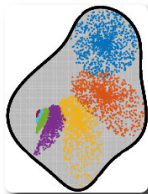
$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f d\mu$$



Dirac measure  $\mu^* = \delta_{x^{\text{opt}}}$

## CHARACTERIZE A SET

?

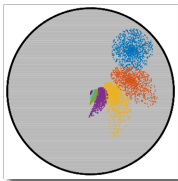


Lebesgue measure  $\mu^* = \lambda_{\mathbf{X}^\infty}$

# Occupation Measures and Liouville's Equation

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t) \quad \mathbf{x}_0 \in \mathbf{X}_0$$

$$\mathbf{x}_1 = f(\mathbf{x}_0) \dots \mathbf{x}_t = f(\mathbf{x}_{t-1})$$



■ Let  $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$

■ **Pushforward**  $f_{\#} : \mathcal{M}_+(\mathbf{X}_0) \rightarrow \mathcal{M}_+(\mathbf{X})$

$$\mu_1(\mathbf{A}) = f_{\#} \mu_0(\mathbf{A}) := \mu_0(f^{-1}(\mathbf{A}))$$

■  $f_{\#} \mu_0$  is the **image measure** of  $\mu_0$  under  $f$

# Occupation Measures and Liouville's Equation

---

- Let  $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$  and

$$\mu_1 = f_{\#} \mu_0$$

...

$$\mu_t = f_{\#} \mu_{t-1}$$

$$\nu_t = \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} f_{\#}^i \mu_0$$

- The measures  $\mu_t, \nu_t, \mu_0$  satisfy **Liouville's Equation**:

$$\mu_t + \nu_t = f_{\#} \nu_t + \mu_0$$

# Occupation Measures and Liouville's Equation

---

- Lebesgue measure  $\lambda_{\mathbf{x}_t}$  on  $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$  s.t.  $\lambda_{\mathbf{x}_t} = f_{\#}^t \mu_{0,t}$   
 $\implies \lambda_{\mathbf{x}_t}$  satisfies **Liouville's Equation**.

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- Lebesgue measure  $\lambda_{\mathbf{x}^T}$  on  $\mathbf{X}^T := \bigcup_{t=0}^T \mathbf{X}_t$   
 $\implies \lambda_{\mathbf{x}^T}$  satisfies **Liouville's Equation** by superposition

# Occupation Measures and Liouville's Equation

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■ Lebesgue measure  $\lambda_{\mathbf{x}_t}$  on  $\mathbf{X}_t = f^t(\mathbf{X}_0)$

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 $\implies \lambda_{\mathbf{x}^T}$  satisfies **Liouville's Equation** by superposition

$$\lambda_{\mathbf{x}^T} + \nu^T = f_{\#} \nu^T + \mu_0^T$$

average **occupation measure**  $\nu^T$ : measures time spent in  $\mathbf{X}^T$

# Volume Assumption

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## Discrete Time

Define  $\mathbf{Y}^0 := \mathbf{X}^0$  and  $\mathbf{Y}^t := \mathbf{X}_t \setminus \mathbf{X}^{t-1}$ .

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T t \operatorname{vol} \mathbf{Y}^t < \infty.$$



# Volume Assumption

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## Discrete Time

Define  $\mathbf{Y}^0 := \mathbf{X}^0$  and  $\mathbf{Y}^t := \mathbf{X}_t \setminus \mathbf{X}^{t-1}$ .

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T t \operatorname{vol} \mathbf{Y}^t < \infty.$$

## Lemma

Under **Volume Assumption**,  $\lambda_{\mathbf{X}^\infty}$  satisfies **Liouville's Equation**

# Volume Assumption

## Discrete Time

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## Lemma

Under **Volume Assumption**,  $\lambda_{\mathbf{X}^\infty}$  satisfies **Liouville's Equation**

## Proof

- $\lambda_{\mathbf{X}^T} = \sum_{t=0}^T \lambda_{\mathbf{Y}^t} \rightarrow \lambda_{\mathbf{X}^\infty}$  as  $T \rightarrow \infty$
- $\mu_t + \nu_t = f_{\#} \nu_t + \mu_{0,t} \implies \nu^T := \sum_{t=0}^T \nu_t$  has mass  $\leq \sum_{t=0}^T t \operatorname{vol} \mathbf{Y}^t$

# Volume Assumption

---

## Continuous Time

Define  $\tau(\mathbf{x})$  = minimal time to reach  $\mathbf{x}$ .

$$\frac{1}{\text{vol}(\mathbf{X})} \int_{\mathbf{X}^\infty} \tau(\mathbf{x}) d\mathbf{x} < \infty.$$

# Volume Assumption

---

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# Infinite Primal LP for Discrete RS

---

$$\begin{aligned} p^T &:= \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu \\ \text{s.t.} \quad &\int_{\mathbf{X}} \nu \leq T \text{vol } \mathbf{X} \\ &\mu + \nu = f_{\#} \nu + \mu_0 \\ &\mu \leq \lambda_{\mathbf{X}} \\ &\mu_0 \in \mathcal{M}_+(\mathbf{X}_0), \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X}) \end{aligned}$$

# Infinite Primal LP for Continuous RS

---

$$\begin{aligned} p^T &:= \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu \\ \text{s.t.} \quad &\int_{\mathbf{X}} \nu \leq T \text{vol } \mathbf{X} \\ &\mu + \text{div}(f\nu) = \mu_0 \\ &\mu \leq \lambda_{\mathbf{X}} \\ &\mu_0 \in \mathcal{M}_+(\mathbf{X}_0), \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X}) \end{aligned}$$

# Infinite Primal LP for Continuous RS

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$$\int_{\mathbf{X}} \nu(\mathbf{x}) \text{div } f\nu = - \int_{\mathbf{X}} \text{grad } \nu(\mathbf{x}) \cdot f(\mathbf{x}) d\nu$$

# Infinite Primal LP for Discrete/Continuous RS

---

## Lemma

Volume Assumption  $\implies$  optimal solution  $\mu^* = \lambda_{x^\infty}$



# Primal-dual LP in Discrete Time

---

Primal LP

$$\begin{aligned}
 p^T &:= \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu \\
 \text{s.t.} \quad &\int_{\mathbf{X}} \nu \leq T \text{vol } \mathbf{X} \\
 &\mu + \nu = f_{\#} \nu + \mu_0 \\
 &\mu \leq \lambda_{\mathbf{X}} \\
 &\mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \\
 &\mu, \nu \in \mathcal{M}_+(\mathbf{X})
 \end{aligned}$$

Dual LP

$$\begin{aligned}
 d^T &:= \inf_{u, v, w} \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\
 \text{s.t.} \quad &v \in \mathcal{C}_+(\mathbf{X}_0) \\
 &w - v - 1 \in \mathcal{C}_+(\mathbf{X}) \\
 &w \in \mathcal{C}_+(\mathbf{X}) \\
 &u + v \circ f - v \in \mathcal{C}_+(\mathbf{X}) \\
 &u \geq 0 \\
 &u \in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X})
 \end{aligned}$$

# Primal-dual LP in Continuous Time

---

## Primal LP

$$\begin{aligned} p^T &:= \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu \\ \text{s.t. } \int_{\mathbf{X}} \nu &\leq T \text{vol } \mathbf{X} \\ \mu + \text{div}(f\nu) &= \mu_0 \\ \mu &\leq \lambda_{\mathbf{X}} \\ \mu_0 &\in \mathcal{M}_+(\mathbf{X}_0) \\ \mu, \nu &\in \mathcal{M}_+(\mathbf{X}) \end{aligned}$$

## Dual LP

$$\begin{aligned} d^T &:= \inf_{u, v, w} \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\ \text{s.t. } v &\in \mathcal{C}_+(\mathbf{X}_0) \\ w - v - 1 &\in \mathcal{C}_+(\mathbf{X}) \\ w &\in \mathcal{C}_+(\mathbf{X}) \\ u + \text{grad } v \cdot f &\in \mathcal{C}_+(\mathbf{X}) \\ u &\geq 0 \\ u &\in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X}) \end{aligned}$$

# Zero Duality Gap

---

## Lemma

- 1  $p^T = d^T$  and  $\exists$  minimizing sequence  $(u_k, v_k, w_k)$  for dual LP.
- 2  $u_k = 0 \implies \text{Volume Assumption} \implies p^T = d^T = \text{vol } X^\infty$

# SDP Strengthening of the Dual LP

---

## Discrete Time

$$\begin{aligned} d_r^T &:= \inf_{u,v,w} \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\ \text{s.t. } & v \in \mathcal{Q}_r(\mathbf{X}_0) \\ & w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \\ & u + v \circ f - v \in \mathcal{Q}_{rd}(\mathbf{X}) \\ & w \in \mathcal{Q}_r(\mathbf{X}) \\ & u \geq 0 \end{aligned}$$

# SDP Strengthening of the Dual LP

---

## Continuous Time

$$\begin{aligned} d_r^T &:= \inf_{u,v,w} \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\ \text{s.t. } & v \in \mathcal{Q}_r(\mathbf{X}_0) \\ & w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \\ & u + \text{grad } v \cdot f \in \mathcal{Q}_{r+d}(\mathbf{X}) \\ & w \in \mathcal{Q}_r(\mathbf{X}) \\ & u \geq 0 \end{aligned}$$

# Strong Convergence Properties

---

## Theorem

Assume that  $X^0, X^\infty, X \setminus X^\infty$  have nonempty interior.

- 1 No duality gap between primal and dual SDP:  $p_r^T = d_r^T$ .

# Strong Convergence Properties

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$$\lim_{r \rightarrow \infty} \int_{\mathbf{x}} |w_r + u_r T - \mathbf{1}_{\mathbf{x}^\infty}| d\mathbf{x} = 0.$$

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- 3 Let  $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geq 0\} \supseteq \mathbf{X}^T$ .



# Strong Convergence Properties

## Theorem

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- 3 Let  $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geq 0\} \supseteq \mathbf{X}^T$ .
- 4  $u_r = 0 \Rightarrow \text{Volume Assumption} \Rightarrow \lim_{r \rightarrow \infty} \text{vol}(\mathbf{X}_r^\infty \setminus \mathbf{X}^\infty) = 0$ .

The RS Problem in Discrete Time

The RS Problem in Continuous Time

Motivations

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for RS

**Application Examples**

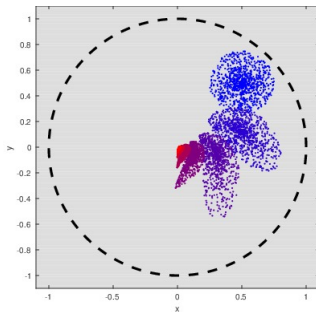
Conclusion and Perspectives

# Toy Example

Trajectories from  $\mathbf{x}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leq \frac{1}{4}\}$  under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2),$$

$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3),$$



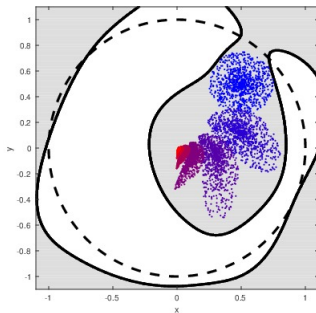
$\mathbf{x}_2^\infty$

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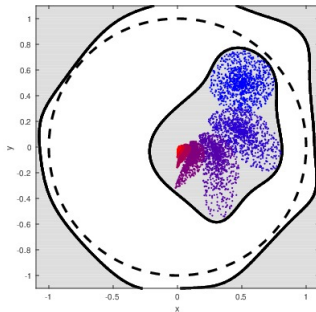
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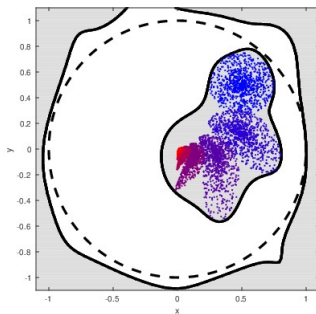
$\mathbf{x}_4^\infty$

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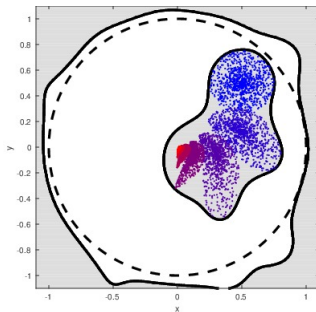
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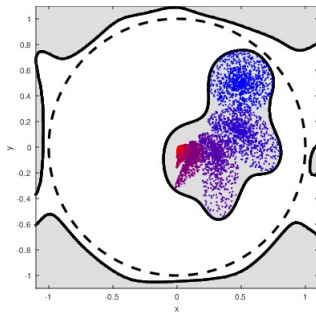
$\mathbf{x}_6^\infty$

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$\mathbf{x}_7^\infty$

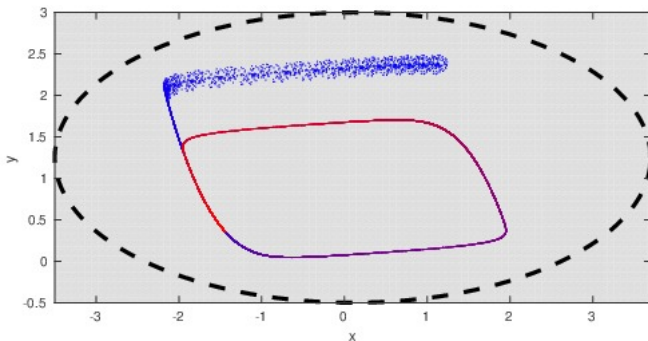


# FitzHugh-Nagumo Neuron Model

Trajectories from  $\mathbf{x}_0 := [1, 1.25] \times [2.25, 2.5]$  under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$



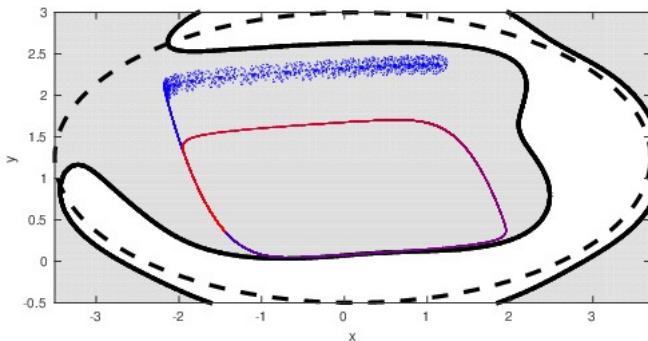
$$\mathbf{x}_2^\infty$$

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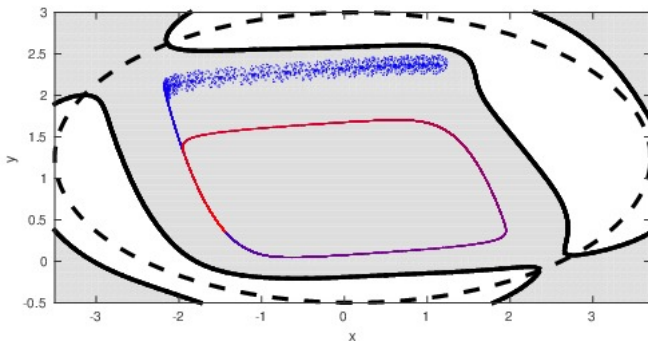
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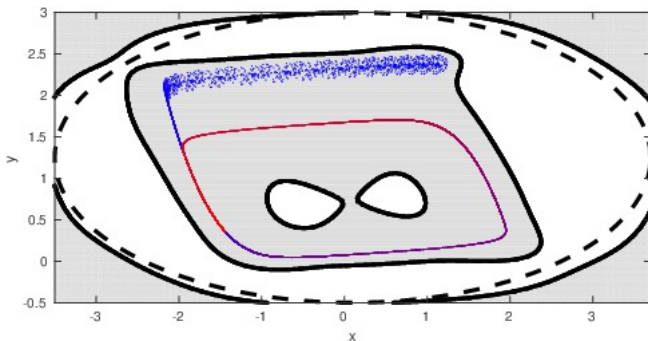
$\mathbf{x}_4^\infty$

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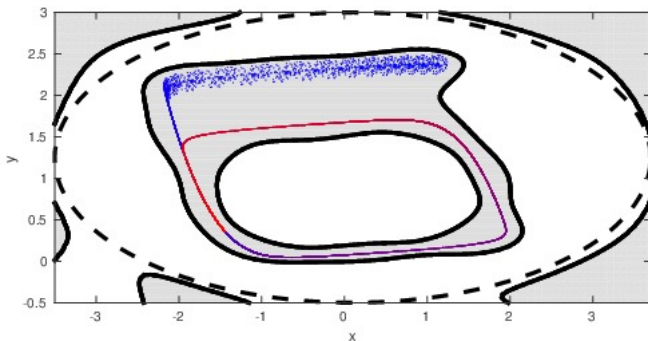
$\mathbf{x}_5^\infty$

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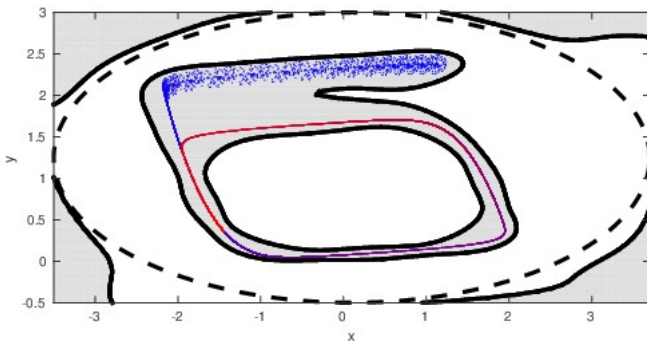
$\mathbf{x}_6^\infty$

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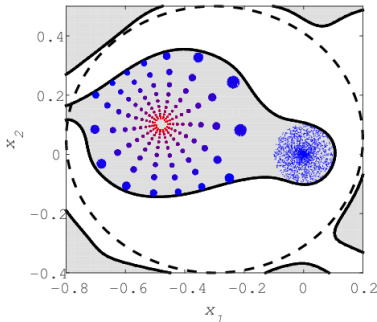


$\mathbf{X}_7^\infty$

Trajectories from  $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$  under

$$x_1^+ := x_1^2 - x_2^2 + c_1,$$

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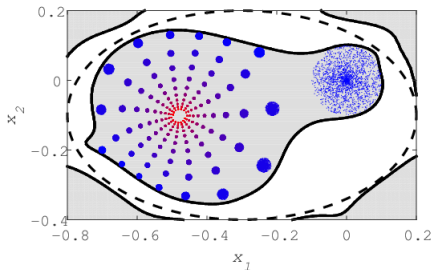


$\mathbf{X}_5^\infty$  with  $c_1 = -0.7$  and  $c_2 = 0.2$

Trajectories from  $\mathbf{x}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$  under

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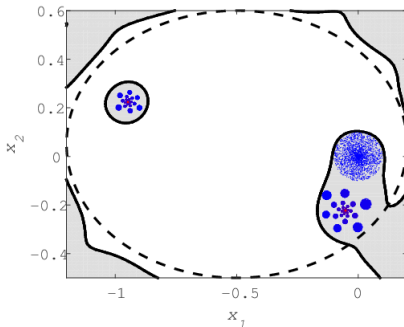
$\mathbf{x}_5^\infty$  with  $c_1 = -0.7$  and  $c_2 = -0.2$



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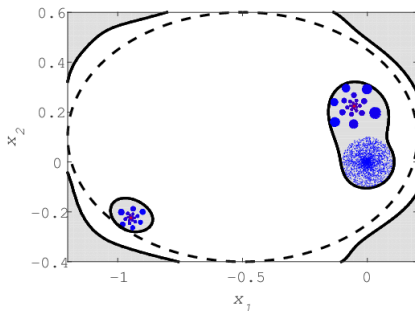


$\mathbf{x}_5^\infty$  with  $c_1 = -0.9$  and  $c_2 = 0.2$

Trajectories from  $\mathbf{x}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$  under

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$\mathbf{x}_5^\infty$  with  $c_1 = -0.9$  and  $c_2 = -0.2$

The RS Problem in Discrete Time

The RS Problem in Continuous Time

Motivations

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for RS

Application Examples

Conclusion and Perspectives

# Conclusion and Perspectives

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- ⊕ Certified Approximation of the **whole reachable set**  $X^\infty$
- ⊖ Computational complexity:  $\binom{n+2rd}{n}$  SDP variables
- ⊕ **Structure sparsity** may be exploited
- 💡 Exploiting Sparsity for Volume Computation [Tacchi et al. 19]

# Conclusion and Perspectives

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## Further research

- **Volume Assumption:**  $\lim_{T \rightarrow \infty} \sum_{t=0}^T t \operatorname{vol} \mathbf{Y}^t \leq \infty$   
always true?
- **Exact** certification:  $\mathbf{X}_r^T = \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geq 0\} \supsetneq \mathbf{X}^T$

# Perspectives: Exact Certificates

## APPROXIMATE SOLUTIONS

sum of squares of  $a^2 - 2ab + b^2$ ?



$(1.00001a - 0.99998b)^2!$



$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

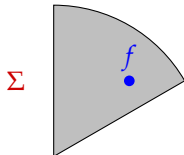
$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\boxed{\simeq \rightarrow = ?}$$

# Conclusion and Perspectives

## Win TWO-PLAYER GAME

- ↪ Univariate optimization [M.-Safey El Din-Schweighofer 18]
- ↪ Multivariate optimization [M.-Safey El Din 18]



sum of squares of  $f$ ?



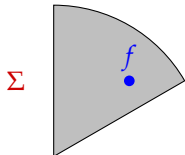
$\simeq$  Output!



# Conclusion and Perspectives

## Win TWO-PLAYER GAME

- ↪ Univariate optimization [M.-Safey El Din-Schweighofer 18]
- ↪ Multivariate optimization [M.-Safey El Din 18]



💡 **Hybrid** Symbolic/Numeric Algorithms

sum of squares of  $f - \epsilon$ ?



Error Compensation

$$\simeq \rightarrow =$$

$\simeq$  Output!





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# End

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Thank you for your attention!

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<https://homepages.laas.fr/vmagron/>