Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems

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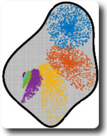
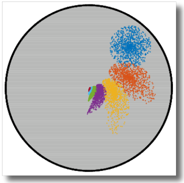


The Problem

- Semialgebraic initial conditions
  $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : g_1^0(\mathbf{x}) \geqslant 0, \dots, g_{m_0}^0(\mathbf{x}) \geqslant 0\}$
- Polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$,
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$
- Set of admissible trajectories
  $\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t\in\mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$
- $\mathbf{X}^\infty = \bigcup_{t\in\mathbb{N}} f^t(\mathbf{X}_0) \subseteq \mathbf{X} \subset \mathbb{R}^n$ (box or ball)
- Tractable approximations of $\mathbf{X}^\infty$?

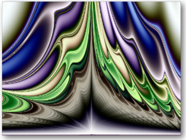


The Problem

- Occurs in several contexts :
  1. program analysis: fixpoint computation

```
toyprogram (x_1,x_2)
requires (0.25 ≤ x_1 ≤ 0.75 && 0.25 ≤ x_2 ≤ 0.75);
while (x_1^2 + x_2^2 ≤ 1){
        x_1 = x_1 + 2x_1x_2;
            x_2 = 0.5(x_2 - 2x_1^3);
      }
```

  2. hybrid systems, biology: Neuron Model, Growth Model
  3. control: integrator, Hénon map



Related work: LP relaxations

1. Contractive methods based on LP relaxations and polyhedra projection [Bertsekas 72]
2. Extension to nonlinear systems [Harwood et al. 16]
3. Bernstein/Krivine-Handelman representations [Ben Sassi et al. 15, Ben Sassi et al. 12]

$\oplus$ LP relaxations $\implies$ scalability

$\ominus$ Convex approximations of nonconvex sets $\implies$ coarse

$\ominus$ No convergence guarantees (very often)

Related work: SDP relaxations

1. Upper bounds of the volume of a semialgebraic set [Henrion et al. 09]

2. Tractable approximations of sets defined with quantifiers $\exists, \forall$ [Lasserre 15]

3. Semidefinite characterization of region of attraction [Henrion-Korda 14]

4. Convex computation of maximum controlled invariant [Korda-Henrion-Jones 13]

Related work: SDP relaxations

5. SDP approximation of polynomial images of semialgebraic sets [M.-Henrion-Lasserre 15]

- $\mathbf{X}_1 := f(\mathbf{X}_0) \subseteq \mathbf{X}$, with $\mathbf{X} \subset \mathbb{R}^n$ a box or a ball
  $\Longrightarrow$ Discrete-time system with a single iteration

- 💡 Approximation of image measure supports
  $\Longrightarrow$ certified SDP over approximations of $\mathbf{X}_1$

- $\mathbf{X}_t := f^t(\mathbf{X}_0)$

  ⊖ $\deg f^t = d \times t \implies$ very expensive computation

  ⊖ Would only approximate $\mathbf{X}_t$ and not $\mathbf{X}^\infty$

Contribution

- General framework to approximate $\mathbf{X}^\infty$
  $\oplus$ **No discretization** is required

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- Infinite-dimensional LP formulation
  💡 support of measures solving Liouville's Equation

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- Infinite-dimensional LP formulation
  💡 support of measures solving Liouville's Equation

- Finite-dimensional SDP relaxations

- $\mathbf{X}^\infty \subseteq \mathbf{X}^r := \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geqslant 1\}$
  $\oplus$ Strong convergence guarantees $\lim_{r\to\infty} \mathrm{vol}(\mathbf{X}^r \backslash \mathbf{X}^\infty) = 0$
  $\oplus$ Compute w_r by solving one **semidefinite program**

The Problem

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for Reachable Sets (RS)

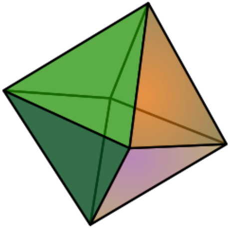
Application Examples

Conclusion

What is Semidefinite Programming?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d} \;.$$



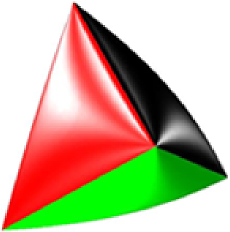
Polyhedron

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij}\, z_j \geqslant d_i$"

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \, .$$



Spectrahedron

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \,, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \,.$$

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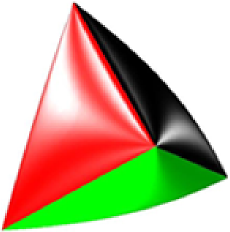

Spectrahedron

Applications of SDP

- Combinatorial optimization
- Control theory
- Matrix completion
- Unique Games Conjecture (Khot '02) :
  "A *single concrete algorithm* provides **optimal guarantees** among all efficient algorithms for a large class of computational problems."
  (Barak and Steurer survey at ICM'14)
- Solving polynomial optimization (Lasserre '01)

Polynomial Optimization

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \dots, g_l(\mathbf{x}) \geqslant 0\}$

- $p^* := \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$: NP hard

- Sums of squares $\Sigma[\mathbf{x}]$
  e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$

- Quadratic module
  $\mathcal{Q}(\mathbf{X}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^{l} \sigma_j(\mathbf{x}) g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$

- $f \in \mathcal{Q}(\mathbf{X}) \implies \forall \mathbf{x} \in \mathbf{X}, f(\mathbf{x}) \geqslant 0$

Infinite LP Reformulation

- Borel σ-algebra $\mathcal{B}(\mathbf{X})$ (generated by the open sets of $\mathbf{X}$)

- $\mathcal{M}_+(\mathbf{X})$: set of Borel measures supported on $\mathbf{X}$.
  If $\mu \in \mathcal{M}_+(\mathbf{X})$ then

  1. $\mu : \mathcal{B} \to [0, \infty)$, $\mu(\emptyset) = 0$

  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any disjoint countable $(B_i) \subset \mathcal{B}(\mathbf{X})$

  3. Lebesgue **Volume** of $B \in \mathcal{B}(\mathbf{X})$

$$\operatorname{vol} B := \int_{\mathbf{X}} \lambda_B \,, \text{ with } \lambda_B(d\mathbf{x}) := \mathbf{1}_B(\mathbf{x})\, d\mathbf{x}$$

- supp μ is the smallest set $\mathbf{X}$ such that $\mu(\mathbb{R}^n \backslash \mathbf{X}) = 0$

Infinite LP Reformulation

$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{X}$ if there exists a finite measure μ supported on $\mathbf{X}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{X}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n .$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{X})$: space of probability measures supported on $\mathbf{X}$
- $\mathcal{Q}(\mathbf{X})$: quadratic module

Polynomial Optimization Problems (POP)

$\mathcal{Q}(\mathbf{X})$ **Archimedean**: $\exists N > 0$ s.t. $N - \|\mathbf{x}\|_2^2 \in \mathcal{Q}(\mathbf{X}) \implies$

$$
\begin{array}{lll}
 & \text{(Primal)} & \\
\inf & \displaystyle\int_{\mathbf{X}} f \, d\mu & \\
\text{s.t.} & \mu \in \mathcal{M}_+(\mathbf{X}) &
\end{array}
\quad = \quad
\begin{array}{ll}
 & \text{(Dual)} \\
\sup & m \\
\text{s.t.} & m \in \mathbb{R} \,, \\
 & f - m \in \mathcal{Q}(\mathbf{X})
\end{array}
$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{X})$
- Truncated quadratic module
  $\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^{l} \sigma_j g_j \,, \deg(g_j\, \sigma_j) \leqslant 2r \,, \sigma_j \in \Sigma[\mathbf{x}] \right\}$

Polynomial Optimization Problems (POP)

$$
\begin{array}{lll}
 & \text{(Moment)} & & & \text{(SOS)} \\
\inf & \sum_{\alpha} f_\alpha \, \mathbf{z}_\alpha & = & \sup & m \\
\text{s.t.} & \mathbf{M}_{r-v_j}(g_j\, \mathbf{z}) \succcurlyeq 0 \,, \quad 0 \leqslant j \leqslant l, & & \text{s.t.} & m \in \mathbb{R} \;, \\
 & \mathbf{z}_1 = 1 & & & f - m \in \mathcal{Q}_r(\mathbf{X})
\end{array}
$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$
(SDP) \begin{cases}
\mathcal{P}: & \inf_{\mathbf{z}} & \sum_\alpha c_\alpha \mathbf{z}_\alpha \\
 & \text{s.t.} & \sum_\alpha F_\alpha \, \mathbf{z}_\alpha - F_0 \succcurlyeq 0 \\
 & & \\
\mathcal{D}: & \sup_{\mathbf{Y}} & \text{Trace}\,(F_0 \, \mathbf{Y}) \\
 & \text{s.t.} & \text{Trace}\,(F_\alpha \, \mathbf{Y}) = c_\alpha \;, \quad \mathbf{Y} \succcurlyeq 0 \;.
\end{cases}
$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

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Application Examples

Conclusion

Occupation Measures and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$

- **Pushforward** $f_\# : \mathcal{M}_+(\mathbf{X}_0) \to \mathcal{M}_+(\mathbf{X})$:

$$f_\# \mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{X}_0 : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{X})$$

- $f_\# \mu_0$ is the **image measure** of μ_0 under f

Occupation Measures and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ and

$$\mu_1 := f_\# \, \mu_0$$
$$\ldots$$
$$\mu_t := f_\# \, \mu_{t-1}$$
$$\nu_t := \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} f_\#^i \, \mu_0$$

- The measures μ_t, ν_t, μ_0 satisfy **Liouville's Equation**:

$$\mu_t + \nu_t = f_\# \, \nu_t + \mu_0$$

Occupation Measures and Liouville's Equation

- Let $\mu_t := \lambda_{\mathbf{X}_t}$: Lebesgue measure on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\mu_t = f^t_{\#}\, \mu_{0,t}$
  $\implies$ μ_t satisfies **Liouville's Equation**.

Occupation Measures and Liouville's Equation

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  $\implies \mu_t$ satisfies **Liouville's Equation**.
- Let $\lambda_{\mathbf{X}^T}$: Lebesgue measure restriction on $\mathbf{X}^T := \bigcup_{t=0}^{T} \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

Occupation Measures and Liouville's Equation

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- Let $\lambda_{\mathbf{X}^T}$: Lebesgue measure restriction on $\mathbf{X}^T := \bigcup_{t=0}^T \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

$$\lambda_{\mathbf{X}^T} + \nu^T = f_\# \nu^T + \mu_0^T$$

ν^T average **occupation measure**: measures time spent in $\mathbf{X}^T$

Infinite Dimensional LP Formulation

Volume Assumption

Define $\mathbf{Y}^0 := \mathbf{X}^0$ and $\mathbf{Y}^t := \mathbf{X}_t \backslash \mathbf{X}^{t-1}$.

$$\lim_{T \to \infty} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t \leqslant \infty.$$

Infinite Dimensional LP Formulation

Volume Assumption

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Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Infinite Dimensional LP Formulation

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Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Proof

- $\lambda_{\mathbf{X}^T} = \sum_{t=0}^{T} \lambda_{\mathbf{Y}^t} \to \lambda_{\mathbf{X}^\infty}$ as $T \to \infty$
- $\mu_t + \nu_t = f_\# \nu_t + \mu_{0,t} \implies \nu^T := \sum_{t=0}^{T} \nu_t$ has mass $\leqslant \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t$

Infinite Primal LP for RS Characterization

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \,, \\
& \mu + \nu = f_\# \nu + \mu_0 \,, \\
& \mu \leqslant \lambda_{\mathbf{X}} \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,, \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X}) \,.
\end{aligned}
$$

Infinite Primal LP for RS Characterization

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p^T := \sup_{\mu_0,\mu,\nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X}\,, \\
& \mu + \nu = f_{\#}\, \nu + \mu_0\,, \\
& \mu \leqslant \lambda_{\mathbf{X}}\,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0)\,, \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})\,.
\end{aligned}
$$

Lemma

1. Optimal solution $\mu^\star := \lambda_{\mathbf{S}^T}$ for some $\mathbf{S}^T$ s.t. $\mathbf{X}^T \subseteq \mathbf{S}^T \subseteq \overline{\mathbf{X}}^\infty$
2. Under Volume Assumption, $\mathbf{S}^T = \overline{\mathbf{X}}^\infty$

LP Primal-dual Conic Formulation

The LP can be cast as follows:

$$\begin{aligned} p^T = \sup_{x} \quad & \langle x, c \rangle_1 \\ \text{s.t.} \quad & \mathcal{A}\, x = b\,, \\ & x \in E_1^+\,, \end{aligned}$$

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with

- $E_1 := \mathcal{M}(\mathbf{X}_0) \times \mathcal{M}(\mathbf{X})^3 \quad F_1 := \mathcal{C}(\mathbf{X}_0) \times \mathcal{C}(\mathbf{X})^3$
- $x = (\mu_0, \mu, \hat{\mu}, \nu, a) \quad c = (0, 1, 0, 0, 0) \quad b = (T \operatorname{vol} \mathbf{X}, 0, \lambda_{\mathbf{X}})$
- the linear operator $\mathcal{A} : E_1 \to E_2$ given by

$$\mathcal{A}\,(\mu_0, \mu, \hat{\mu}, \nu, a) := \begin{bmatrix} \int_{\mathbf{X}} \nu + a \\ \mu + \nu - f_{\#}\, \nu - \mu_0 \\ \mu + \hat{\mu} \end{bmatrix}.$$

LP Primal-dual Conic Formulation

Primal LP

$$
\begin{aligned}
p^* = \sup_{x} \quad & \langle x, c \rangle_1 \\
\text{s.t.} \quad & \mathcal{A}\, x = b\,, \\
& x \in E_1^+\,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^T = \inf_{y} \quad & \langle b, y \rangle_2 \\
\text{s.t.} \quad & \mathcal{A}'\, y - c \in F_1^+\,.
\end{aligned}
$$

with

- $y := (u, v, w) \in \mathbb{R} \times \mathcal{C}(\mathbf{X})^2$
- $\mathcal{A}'\,(u, v, w) := \begin{bmatrix} -v \\ w + v \\ w \\ u + v - v \circ f \\ -v \end{bmatrix}.$

LP Primal-dual Conic Formulation

Primal LP

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p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \,, \\
& \mu + \nu = f_{\#} \nu + \mu_0 \,, \\
& \mu \leqslant \lambda_{\mathbf{X}} \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,, \\
& \mu, \nu \in \mathcal{M}_+(\mathbf{X}) \,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^T := \inf_{u, v, w} \quad & \left(\int w(\mathbf{x}) + T u \right) \lambda_{\mathbf{X}}(d\mathbf{x}) \\
\text{s.t.} \quad & v \in \mathcal{C}_+(\mathbf{X}_0) \,, \\
& w - v - 1 \in \mathcal{C}_+(\mathbf{X}) \,, \\
& w \in \mathcal{C}_+(\mathbf{X}) \,, \\
& u + v \circ f - v \in \mathcal{C}_+(\mathbf{X}) \,, \\
& u \geqslant 0 \,, \\
& u \in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X}) \,.
\end{aligned}
$$

Zero Duality Gap

Lemma

1. $p^T = d^T$ and $\exists$ minimizing sequence (u_k, v_k, w_k) for dual LP.
2. $u_k = 0 \implies$ Volume Assumption $\implies p^T = d^T = \operatorname{vol} \overline{\mathbf{X}}^{\infty}$

Strong Convergence Properties

Strengthening of the dual LP:

$$
\begin{aligned}
d_r^T := \inf_{u,v,w} \quad & \sum_{\beta \in \mathbb{N}_{2r}^n} w_\beta z_\beta^{\mathbf{X}} + uTz_0^X \\
\text{s.t.} \quad & v \in \mathcal{Q}_r(\mathbf{X}_0)\,, \\
& w - v - 1 \in \mathcal{Q}_r(\mathbf{X})\,, \\
& u + v \circ f - v \in \mathcal{Q}_{rd}(\mathbf{X})\,, \\
& w \in \mathcal{Q}_r(\mathbf{X})\,, \\
& u \in \mathbb{R}^+\,.
\end{aligned}
$$

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{S}^T, \mathbf{X} \backslash \mathbf{S}^T$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{S}^T, \mathbf{X} \backslash \mathbf{S}^T$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.
2. Dual SDP has optimal solution (u_r, v_r, w_r):

$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r + u_r T - \mathbf{1}_{\mathbf{S}^T}| = 0\,.$$

Strong Convergence Properties

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3. Let $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{S}^T, \mathbf{X} \setminus \mathbf{S}^T$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.
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$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r + u_r T - \mathbf{1}_{\mathbf{S}^T}| = 0 \,.$$

3. Let $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.
4. $u_r = 0 \Rightarrow$ Volume Assumption $\Rightarrow \lim_{r \to \infty} \mathrm{vol}(\mathbf{X}_r^\infty \setminus \mathbf{X}^\infty) = 0$.

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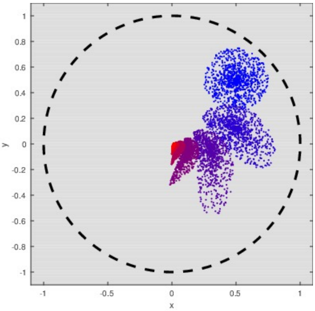
Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

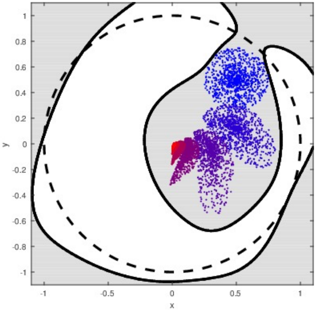


$\mathbf{X}_2^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

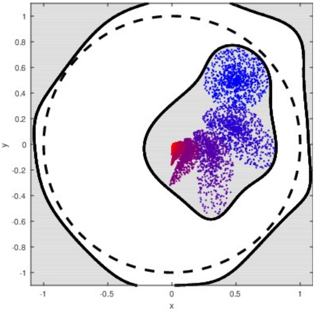


$\mathbf{X}_3^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$



$\mathbf{X}_4^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
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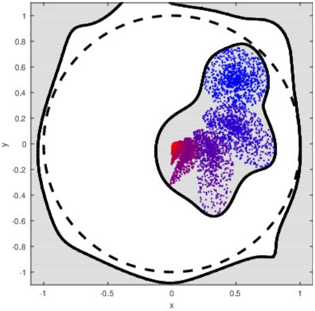

$\mathbf{X}_5^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
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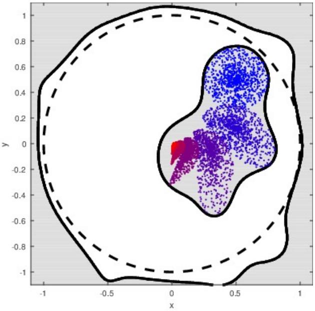
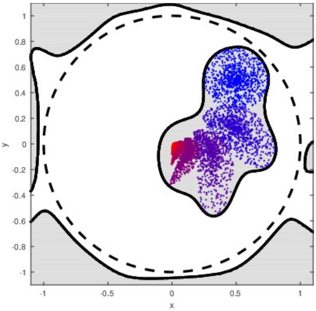


$\mathbf{X}_6^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

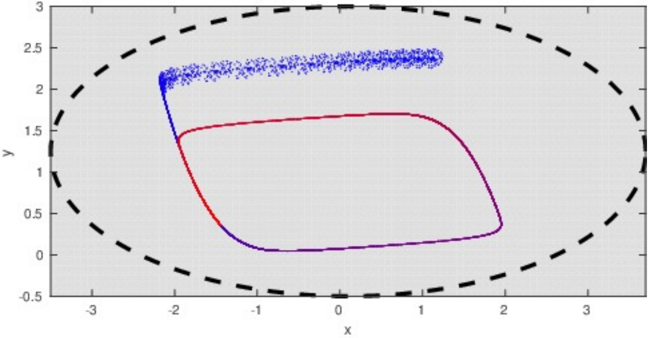


$\mathbf{X}_7^{\infty}$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

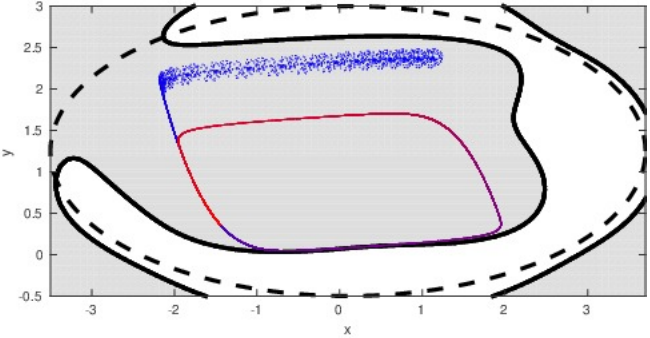


$\mathbf{X}_2^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

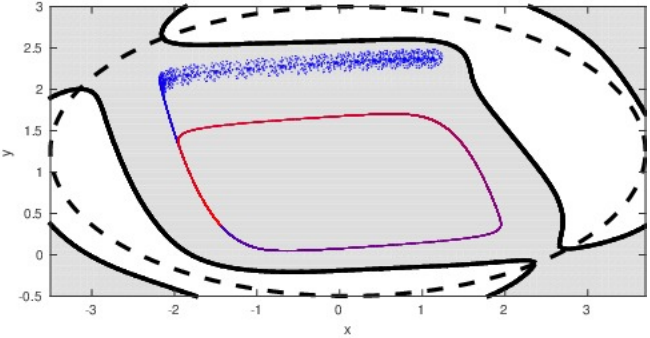


$\mathbf{X}_3^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

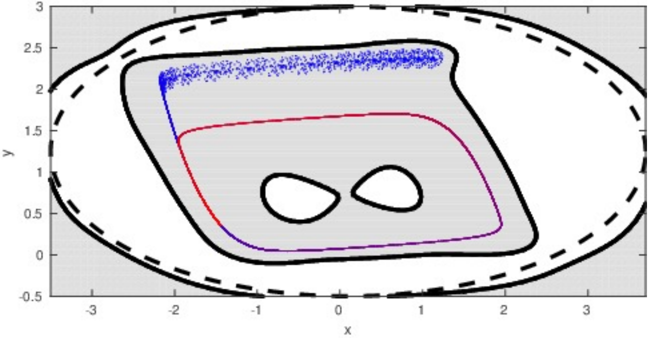


$\mathbf{X}_4^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

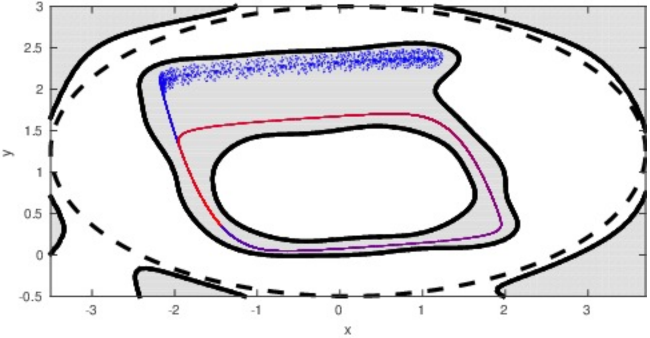


$\mathbf{X}_5^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$



$\mathbf{X}_6^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
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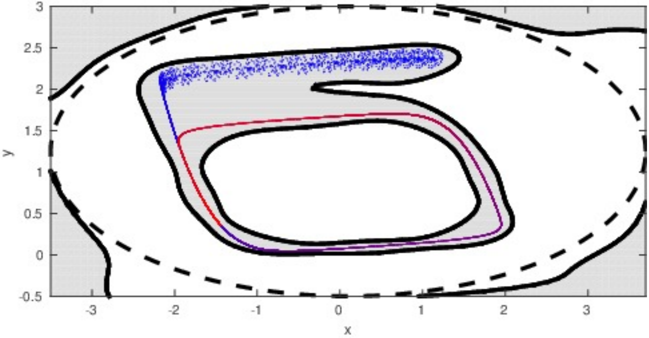
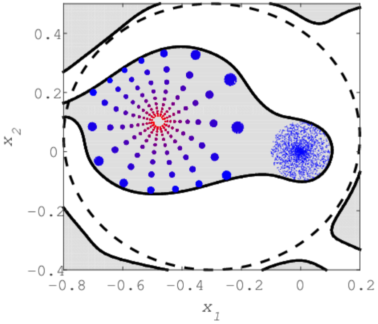


$\mathbf{X}_7^\infty$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
$$x_2^+ := 2x_1x_2 + c_2 \,,$$



$\mathbf{X}_5^\infty$ with $c_1 = -0.7$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1\,,$$
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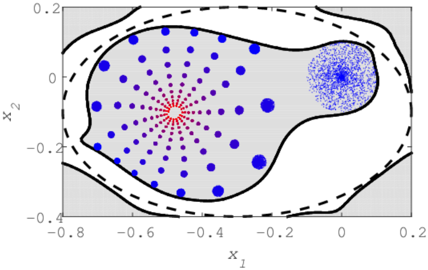

$\mathbf{X}_5^\infty$ with $c_1 = -0.7$ and $c_2 = -0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
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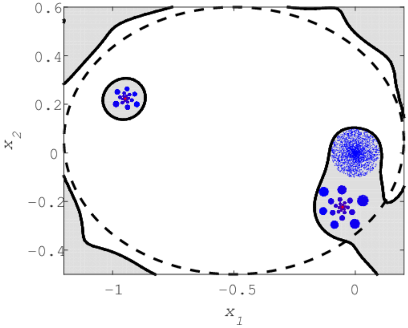
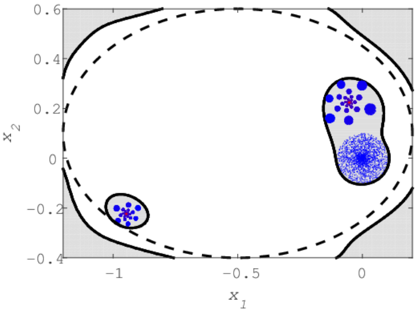


$\mathbf{X}_5^\infty$ with $c_1 = -0.9$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
$$x_2^+ := 2x_1 x_2 + c_2 \,,$$



$\mathbf{X}_5^\infty$ with $c_1 = -0.9$ and $c_2 = -0.2$

The Problem

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for Reachable Sets (RS)

Application Examples

Conclusion

Conclusion

$\oplus$ Certified Approximation of the **whole reachable set** $\mathbf{X}^\infty$

$\ominus$ Computational complexity: $\binom{n+2rd}{n}$ SDP variables

$\oplus$ **Structure sparsity** can be exploited

Conclusion

Further research:

- Volume Assumption: $\lim_{T \to \infty} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t \leqslant \infty$ always true?

- finite time, continuous setting?
  💡Use previous framework approximating:

  1. region of attraction
  2. maximum controlled invariant

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End

Thank you for your attention!

- V. Magron, P.-L. Garoche, D. Henrion and X. Thirioux. Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems. `arxiv.org/abs/1703.05085`

`http://www-verimag.imag.fr/~magron`