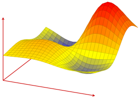
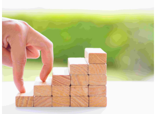
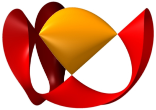
Tractable semidefinite bounds of positive maximal singular values

Victor Magron POP team, CNRS LAAS

Joint work with N. H. A. Mai, Y. Ebihara, and H. Waki

Moment Problems, Convex Algebraic Geometry, and Semidefinite Relaxations at **MTNS 2022**, Bayreuth

September 15, 2022

Positive maximal singular values

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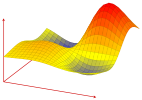
💡 Also a **polynomial optimization problem**

Polynomial optimization

NP-hard NON CONVEX Problem $f^\star = \sup f(x)$

Theory

(Primal)



(Dual)

$\sup \quad \int f \, d\mu \qquad\qquad\qquad\qquad \inf \quad \lambda$

with $\quad \mu$ proba $\Rightarrow$ **infinite LP** $\Leftarrow$ with $\quad \lambda - f \geq 0$

Polynomial optimization

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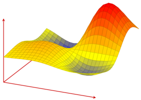
Practice

(Primal **Relaxation**)



(Dual **Strengthening**)

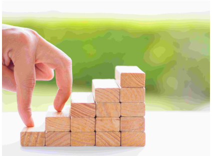
moments $\int x^\alpha \, d\mu$

$\lambda - f =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

Lasserre's hierarchy of **CONVEX Problems** $\downarrow f^*$
[Lasserre/Parrilo 01]

degree k & n vars $\Longrightarrow \binom{n+2k}{n}$ **SDP** variables



Sparse polynomial optimization

💡 Exploiting sparsity

few terms [Reznick '78] or few correlations [Lasserre, Waki et al. '06]

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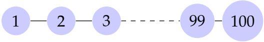
$\rightsquigarrow x_1x_2 + x_2x_3 + \dots x_{99}x_{100}$



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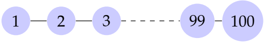

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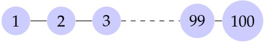

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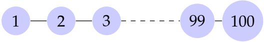

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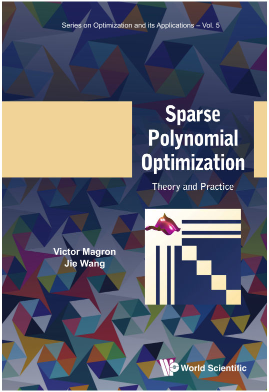
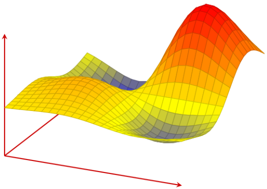
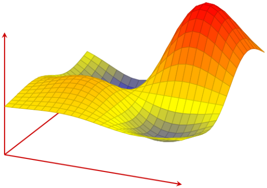


Polynomial optimization on the nonnegative orthant



Roadmap:

Polynomial optimization on the nonnegative orthant



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1. Make use of denominators in certain representations of positive polynomials

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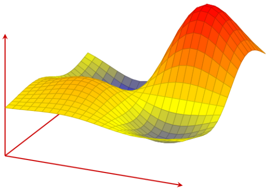

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1. Make use of denominators in certain representations of positive polynomials
2. Define sums of s-nomial squares

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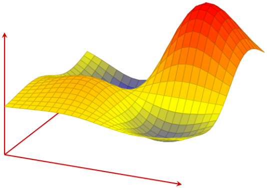

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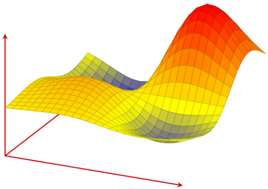

Roadmap:

1. Make use of denominators in certain representations of positive polynomials
2. Define sums of s-nomial squares
3. Combine 1. and 2. to speed-up the resolution of the corresponding convex relaxations
4. Apply this to positive maximal singular values

Polynomial optimization on the nonnegative orthant

$$f^\star = \sup_{x \in S} f(x)$$

where $S = \{x \in \mathbb{R}^n \,:\, x_i \geq 0\,,\, g_j(x) \geq 0\}$

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Positive maximal singular value:
$\check{f} = (x^2)^T (M^T M) x^2$, $g_1 = 1$, $g_2 = 1 - \sum x_i^4$

Extension of Pólya's theorem

[Pólya, 1928]

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+ "dehomogenize" $\|x\|_2^2 \to 1 + \|x\|_2^2$

Extension of Pólya's theorem

[Dickinson and Povh, 2015, Mai et al., 2022]

Let $\check{f}, \check{g}_j$ be even polynomials such that

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Then there exist $\bar{c}, c > 0$ depending on $\check{f}, \check{g}_j$ s.t. $\forall \varepsilon > 0$

$$k \geq \bar{c}\varepsilon^{-c} \implies (1 + \|x\|_2^2)^k(\check{f} + \varepsilon) = \sum_{j=1}^m \sigma_j \check{g}_j$$

for some SOS of monomials σ_j, $\deg(\sigma_j \check{g}_j) \leq 2(k + d)$.

Examples

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- There do not exist SOS of monomials σ_0, σ_1 s.t. $f = \sigma_0 + \sigma_1(1 - x^2)$
- With a uniform denominator, we obtain

$$(1 + x^2)^2 f = \bar{\sigma}_0 + \bar{\sigma}_1(1 - x^2)\,,$$

where $\bar{\sigma}_0 = x^8$, $\bar{\sigma}_1 = x^4 + \frac{15}{4}x^2 + \frac{9}{4}$ are SOS of monomials.

Sums of even s-nomial squares

An even s-nomial square is a polynomial which can be written as

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For any $s \in \mathbb{N}\backslash\{0\}$, one has the following obvious inclusions

$$\Sigma_1 \subset \cdots \subset \Sigma_s \subset \Sigma_{s+1} \subset \ldots .$$

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SDP reformulation with maximal block size s

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For fixed s, $\rho_k^{(s)} \downarrow f^\star$ as $k \to \infty$ with rate $\mathcal{O}(\varepsilon^{-c})$ under mild condition

Application to Positive maximal singular values

Linear time invariant discrete system:

$$\begin{cases} x(t+1) = Ax(t) + Bw(t), x(0) = 0 \\ z(t) = Cx(t) + Dw(t) \end{cases}$$

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Strategy from [Ebihara et al., 2021], take r time steps

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💡 Certify stability by estimating

$$\sigma_+(M)^2 = \max_{\mathrm{x} \in \mathbb{R}^n_+} \{\mathrm{x}^\top (M^\top M)\mathrm{x} \ : \ \|x\|_2^2 \leq 1\}$$

Numerical example

25 variables

k: relaxation order

s: upper bound on the block size

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1	168.4450	0.04	0	26	**91.28158**	0.7
2	**91.28158**	877				

the same bound up to more than 1250 times faster

Performance  vs  **Accuracy**

Take-away

1. The uniform denominators (in Pólya's representation) allow us to control the size of the SDP relaxations (using sums of even s-nomial squares)

Take-away

1. The uniform denominators (in Pólya's representation) allow us to control the size of the SDP relaxations (using sums of even s-nomial squares)
2. Our method is a powerful & accurate MODELING tool for POPs on the nonnegative orthant (e.g., positive maximal singular values)

Many thanks for your attention!

https://homepages.laas.fr/vmagron

github:InterRelax

Mai, Lasserre, Magron & Toh. Tractable hierarchies of convex relaxations for polynomial optimization on the nonnegative orthant arXiv:2209.06175

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