Sparse (Non)commutative Polynomial Optimization

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Joint commutative work with J.-B. Lasserre, H. N. A. Mai & J. Wang

Joint Noncommutative work with I. Klep & J. Povh

Real Algebraic Geometry with a View Toward Hyperbolic Programming and Free Probability, Oberwolfach

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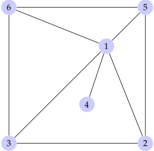
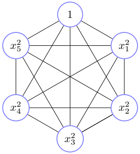




The Moment-Sums of Squares Hierarchy

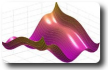
NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Theory

(Primal)

$\inf \int f \, d\mu$

with μ proba $\Rightarrow$



INFINITE LP

(Dual)

$\sup \lambda$

$\Leftarrow$ with $f - \lambda \geqslant 0$

The Moment-Sums of Squares Hierarchy

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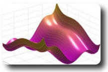
Practice

(Primal **Relaxation**)



(Dual **Strengthening**)

moments $\int \mathbf{x}^\alpha \, d\mu$

$f - \lambda =$ **sum of squares**

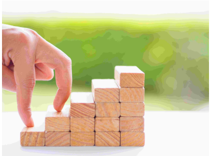
finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f^*$

[Lasserre/Parrilo 01]

degree d & n vars $\Longrightarrow \binom{n+2d}{n}$ **SDP** VARIABLES

Numeric solvers $\Longrightarrow$ **Approx** Certificate



Exploiting correlative sparsity

Exploiting term sparsity

Exploiting correlative sparsity

Exploiting term sparsity

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

- Semialgebraic set
  $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Sums of squares (SOS) σ_j

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- Sums of squares (SOS) σ_j
- Bounded degree:
  $\mathcal{M}(\mathbf{K})_d := \left\{ \sigma_0 + \sum_{j=1}^{l} \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2d \right\}$

SDP for Polynomial Optimization

- **Hierarchy of SDP relaxations**:
$$\lambda_d := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{M}(\mathbf{K})_d \right\}$$

- Convergence guarantees $\lambda_d \uparrow f^*$ [Lasserre 01]

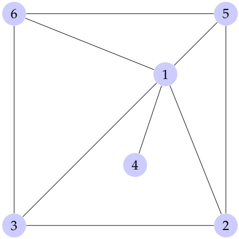
- Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

- **"No Free Lunch" Rule**: $\binom{n+2d}{n}$ SDP variables

Sparse polynomial optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$



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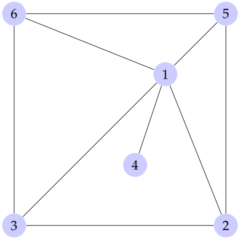

1. Index sets $I_1, \dots, I_p$
2. Average size $\kappa \rightsquigarrow \binom{\kappa+2d}{\kappa}$ vars

$I_1 = \{1, 4\}$
$I_2 = \{1, 2, 3, 5\}$
$I_3 = \{1, 3, 5, 6\}$
Dense SDP: 210 vars
Sparse SDP: 115 vars

Sparse polynomial optimization [Waki, Lasserre 06]

Sparse $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{R}[x, I_k]$

Sparse $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$ with $g_j \in \mathbb{R}[x, I_{k(j)}]$ for some $k(j)$

Additional constraints $n_k - \sum_{i \in I_k} x_i^2 \geqslant 0$ in $\mathbf{K}$

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$$\forall k = 2, \ldots, p \quad I_k \cap \bigcup_{j<k} I_j \subseteq I_i \quad \text{for some } i < k$$

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Theorem: Sparse Putinar's representation [Lasserre 06]

$f > 0$ on $\mathbf{K}$ + RIP $\implies$ $\boxed{f = \sigma_{01} + \cdots + \sigma_{0p} + \sum_{j=1}^{m} \sigma_j g_j}$

with $\sigma_{0k} \in \Sigma[x, I_k]$, $\sigma_j \in \Sigma[x, I_{k(j)}]$

Sparse examples

- Chained singular function:

$$f_{\mathrm{cs}} = \sum_{i \in J} ((x_i + 10x_{i+1})^2 + 5(x_{i+2} - x_{i+3})^2 + (x_{i+1} - 2x_{i+2})^4 + 10(x_i - x_{i+3})^4)$$

where $J = \{1, 3, 4, \ldots, n-3\}$ and n is a multiple of 4

$I_k = \{k, k+1, k+2, k+3\}$

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- Generalized Rosenbrock function:

$$f_{\mathsf{gR}} = 1 + \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i^2)^2 + (1 - x_{i+1})^2\right)$$

💡 $I_k = \{k, k+1\}$

Roundoff Errors

- Exact:

$$f(\mathbf{x}) := x_1 x_2 + x_3 x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1 x_2 (1 + e_1) + x_3 x_4 (1 + e_2)](1 + e_3)$$

- $\mathbf{x} \in \mathbf{X}\,, \quad \mid e_i \mid \leqslant 2^{-\delta} \quad \delta = 24$ (single) or 53 (double)

Roundoff Errors

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \mathbf{e})$
Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\alpha} r_{\alpha}(\mathbf{e}) \mathbf{x}^{\alpha}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = \ell(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$, ℓ **linear** in $\mathbf{e}$

3: Bound $h(\mathbf{x}, \mathbf{e})$ with interval arithmetic

4: Bound $\ell(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

Exploiting sparsity for roundoff error bounds

$l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^{m} s_i(\mathbf{x}) e_i$

$I_1, \ldots, I_m$ correspond to $\{\mathbf{x}, e_1\}, \ldots, \{\mathbf{x}, e_m\}$

Dense relaxation: $\binom{n+m+2d}{n+m}$ SDP variables

Sparse relaxation: $m\binom{n+1+2d}{n+1}$ SDP variables

Simple comparison [M.-Constantinides-Donaldson 17]

$$f(\mathbf{x}) := x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

$$\mathbf{x} \in [4.00, 6.36]^6, \quad \mathbf{e} \in [-\epsilon, \epsilon]^{15}, \quad \epsilon = 2^{-53}$$

- Dense SDP: $\binom{6+15+4}{6+15} = 12650$ variables $\rightsquigarrow$ **Out of memory**

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- Sparse SDP `Real2Float` tool: $15\binom{6+1+4}{6+1} = 4950 \leadsto 759\epsilon$

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- Symbolic Taylor `FPTaylor` tool: 721ϵ (21 $\times$ more CPU)

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- Interval arithmetic: 922ϵ (10 $\times$ less CPU)
- Symbolic Taylor `FPTaylor` tool: 721ϵ (21 $\times$ more CPU)
- SMT-based `rosa` tool: 762ϵ (19 $\times$ more CPU)

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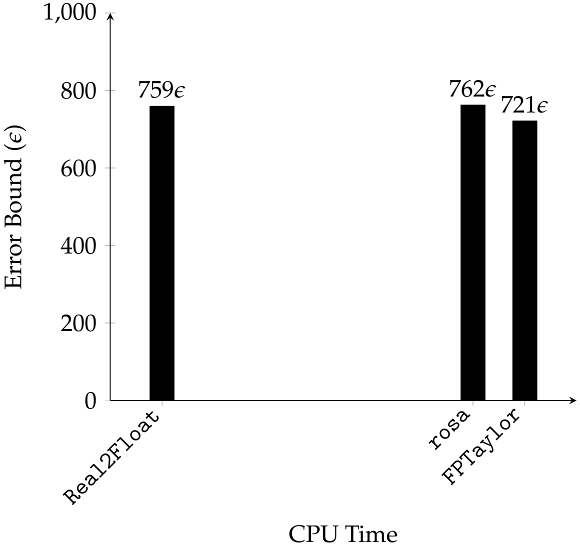
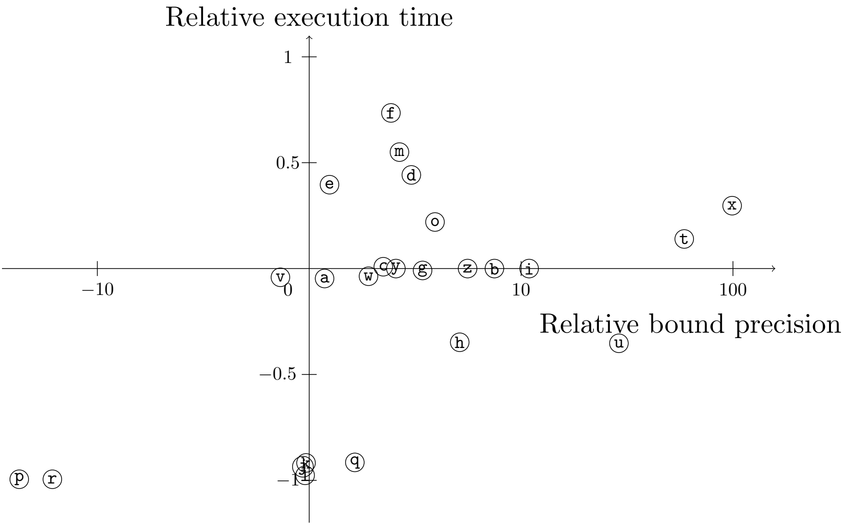


SOS versus `rosa` [M.-Constantinides-Donaldson 17]



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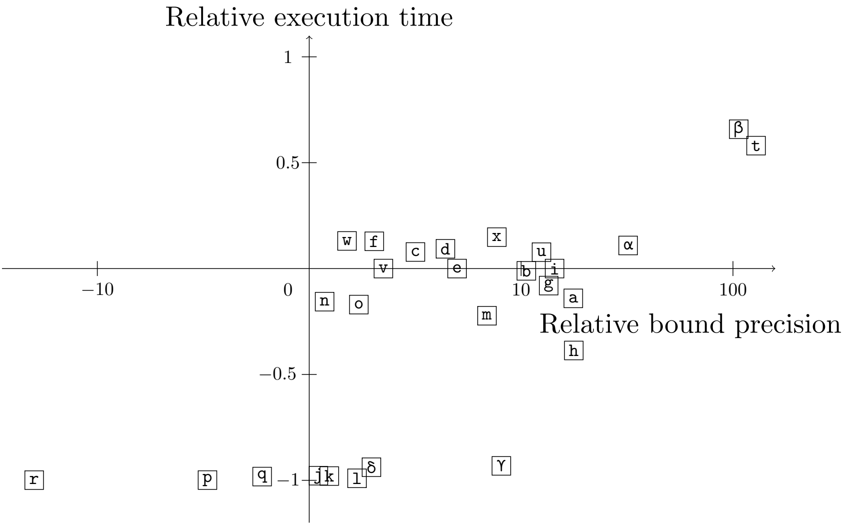

Noncommutative (NC) polynomials

Symmetric **Matrix** variables X_i, Y_j

$f = X_1(Y_1 + Y_2 + Y_3) + X_2(Y_1 + Y_2 - Y_3) + X_3(Y_1 - Y_2) - X_1 - 2Y_1 - Y_2$

with $X_1 X_2 \neq X_2 X_1$, **involution** $(X_1 Y_3)^\star = Y_3 X_1$

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Constraints $\mathbf{K} = \{(X, Y) : X_i, Y_j \succcurlyeq 0, X_i^2 = X_i, Y_j^2 = Y_j, X_i Y_j = Y_j X_i\}$

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MINIMAL EIGENVALUE OPTIMIZATION

$$\lambda_{\min} = \inf \{ \langle f(X, Y)\mathbf{v}, \mathbf{v} \rangle : (X, Y) \in \mathbf{K}, \|\mathbf{v}\| = 1 \}$$

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MINIMAL EIGENVALUE OPTIMIZATION

$$\begin{aligned}
\lambda_{\min} &= \inf \{\langle f(X,Y)\mathbf{v}, \mathbf{v}\rangle : (X,Y) \in \mathbf{K}, \|\mathbf{v}\| = 1\} \\
&= \sup \quad \lambda \\
&\quad \text{s.t.} \quad f(X,Y) - \lambda \mathbf{I} \succcurlyeq 0, \quad \forall (X,Y) \in \mathbf{K}
\end{aligned}$$

Putinar's representation

"**Archimedean**" constraint in $\mathbf{K} = \{X : g_j(X) \succcurlyeq 0\}$:
$N - \sum_i X_i^2 \succcurlyeq 0$

Theorem: NC Putinar's representation [Helton-McCullough 02]

$f \succ 0$ on $\mathbf{K} \implies \boxed{f = \sum_i s_i^\star s_i + \sum_j \sum_i t_{ji}^\star g_j t_{ji}}$ with s_i, $t_{ji} \in \mathbb{R}\langle X \rangle$

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NC variant of Lasserre's Hierarchy for $\lambda_{\min}$:

💡 replace "$f - \lambda \mathbf{I} \succcurlyeq 0$ on $\mathbf{K}$" by $f - \lambda \mathbf{I} = \sum_i s_i^\star s_i + \sum_j \sum_i t_{ji}^\star g_j t_{ji}$
with s_i, t_{ji} of **bounded** degrees

Sparse Putinar's representation [Klep-M.-Povh 19]

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with $s_{ki} \in \mathbb{R}\langle X, I_k \rangle$, $t_{ji} \in \mathbb{R}\langle X, I_{k(j)} \rangle$

Proof outline: by contradiction

$f \succ 0$ on $\mathbf{K}$ and

$$f \notin \underbrace{\{\sum_k \sum_i s_{ki}^\star s_{ki} + \sum_j \sum_i t_{ji}{}^\star g_j t_{ji} : s_{ki} \in \mathbb{R}\langle X, I_k \rangle, t_{ji} \in \mathbb{R}\langle X, I_{k(j)} \rangle\}}_{\mathcal{M}(\mathbf{K})^{\text{sparse}}}$$

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💡 Separation argument $\implies$ there exists $L : \mathbb{R}\langle X \rangle \to \mathbb{R}$ with $L(f) \leqslant 0$ and $L(\mathcal{M}(\mathbf{K})^{\text{sparse}}) \subseteq \mathbb{R}^{\geqslant 0}$

Proof outline: by contradiction

$f \succ 0$ on $\mathbf{K}$ and

$$f \notin \underbrace{\{\sum_k \sum_i s_{ki}^\star s_{ki} + \sum_j \sum_i t_{ji}{}^\star g_j t_{ji} : s_{ki} \in \mathbb{R}\langle X, I_k \rangle, t_{ji} \in \mathbb{R}\langle X, I_{k(j)} \rangle\}}_{\mathcal{M}(\mathbf{K})^{\text{sparse}}}$$

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Gelfand-Naimark-Segal (GNS) construction:

Find $A \in \mathbf{K}$ and $\mathbf{v} \neq 0$ such that

$$L(f) = \langle f(A)\mathbf{v}, \mathbf{v} \rangle$$

Proof outline: induction on p & amalgamation

- $p = 1 \rightarrow$ NC Putinar's representation of f

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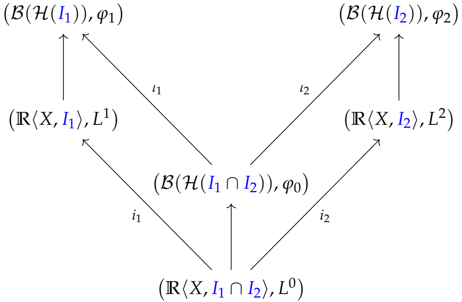
GNS construction on
$L^k = L|_{\mathbb{R}\langle X, I_k \rangle}$
$\Downarrow$
$L^k(g) = \langle g(A^k)\mathbf{w}^k, \mathbf{w}^k \rangle$ for

- Hilbert space $\mathcal{H}(I_k)$
- A^k ($\times$ operator)
- unit vector $\mathbf{w}^k \in \mathcal{H}(I_k)$



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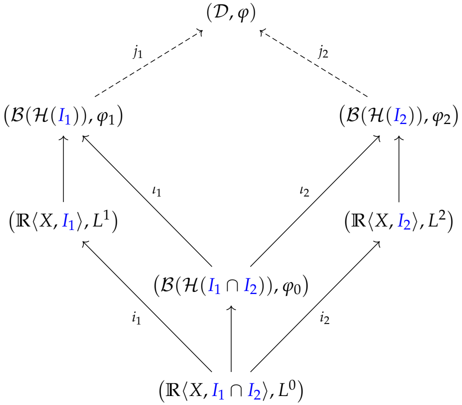

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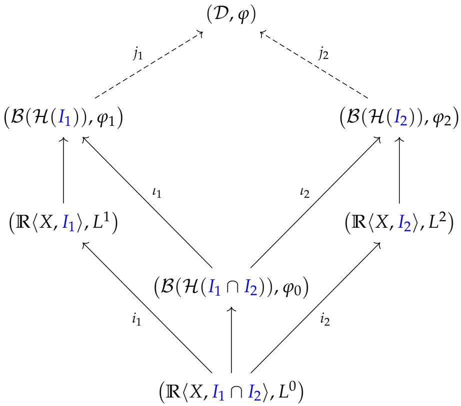

GNS on $\mathcal{D} \implies A \in \mathbf{K}$ and $\mathbf{v} \neq 0$

Sparsity & no RIP $\Rightarrow$ no sparse representation

$f = (X_1 + X_2 + X_3)^2 = f_1 + f_2 + f_3$
$f_1 = \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + X_1X_2 + X_2X_1 \in \mathbb{R}\langle X_1, X_2\rangle$
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$\mathbf{K} = \{X : 1 - X_i^2 \succcurlyeq 0\}$
$f \succcurlyeq 0$ on $\mathbf{K}$ but $f + \lambda \in \mathcal{M}(\mathbf{K})^{\text{sparse}} \Leftrightarrow \lambda \geqslant 3$

Sparsity & no RIP ⇒ no sparse representation

$$f + 3 = (X_1 + X_2)^2 + (X_1 + X_3)^2 + (X_2 + X_3)^2 + \sum_i (1 - X_i^2) \in \mathcal{M}(\mathbf{K})^{\text{sparse}}$$

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Duality argument: find $L_k : \mathbb{R}\langle X, I_k \rangle \to \mathbb{R}$ such that

1. $L_k(1) = 1$
2. $L_k(h^\star h) \geqslant 0 \qquad L_k(h^\star (1 - X_k^2) h) \geqslant 0 \qquad \forall h \in \mathbb{R}\langle X, I_k \rangle$
3. $\sum_k L_k(f_k) = -3$
4. $L_j = L_k$ on $\mathbb{R}\langle X, I_j \cap I_k \rangle$

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Answer:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = -A$$

$$L_k(g) = \text{tr}\, g(A, B) \qquad \text{for } g \in \mathbb{R}\langle X, I_k \rangle$$

A sparse Helton-McCullough's theorem?

Theorem [Helton-McCullough 02]

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Lemma [Klep-M.-Povh 19]

There is no sparse analog of the Helton-McCullough's theorem

A counter-example

$$v = \begin{bmatrix} X_1 & X_1X_2 & X_2 & X_3 & X_3X_2 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & -1 & -1 & 0 & \alpha \\ -1 & 2 & 0 & -\alpha & 0 \\ -1 & 0 & 3 & -1 & 9 \\ 0 & -\alpha & -1 & 6 & -27 \\ \alpha & 0 & 9 & -27 & 142 \end{bmatrix} \qquad \alpha \in \mathbb{R}$$

$$\begin{aligned} f &= vGv^\star \\ &= X_1^2 - X_1X_2 - X_2X_1 + 3X_2^2 - 2X_1X_2X_1 + 2X_1X_2^2X_1 - X_2X_3 \\ &\quad - X_3X_2 + 6X_3^2 + 9X_2^2X_3 + 9X_3X_2^2 - 54X_3X_2X_3 + 142X_3X_2^2X_3 \end{aligned}$$

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RIP holds but ...

$\ldots G \succcurlyeq 0 \Leftrightarrow 0.27 \lesssim \alpha \lesssim 1.11 \Rightarrow \boxed{f \notin \Sigma\langle X_1, X_2 \rangle + \Sigma\langle X_2, X_3 \rangle}$

Sparse positive definite forms [Mai-Lasserre-M. 20]

Thm: Reznick's representation [Reznick 95]

pd form $f \implies \boxed{f = \frac{\sigma}{\|\mathbf{x}\|_2^{2d}}}$ with $\sigma \in \Sigma[\mathbf{x}]$, $d \in \mathbb{N}$

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Sparse $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{R}[\mathbf{x}, I_k]$

RUNNING INTERSECTION PROPERTY (RIP)

$$\boxed{\forall k = 2, \ldots, p \quad \underbrace{I_k \cap \bigcup_{j<k} I_j}_{\hat{I}_k} \subseteq I_{s_k} \quad \text{for some } s_k < k}$$

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Thm: Sparse Reznick's representation [Mai-Lasserre-M. 20]

pd form f + RIP $\implies \boxed{f = \sum_k \frac{\sigma_k}{H_k{}^d}}$ with $\sigma_k \in \Sigma[\mathbf{x}, I_k]$, $d \in \mathbb{N}$

Uniform H_k involve products $\|\mathbf{x}(I)\|^2$ for $I \in \{I_k, \hat{I}_k, \hat{I}_i : s_i = k\}$

Proof ingredients

1. Induction on p as in [Grimm-Netzer-Schweighofer 07] for sparse positive polynomials on compact sets

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2. Similar ingredients used for generalized Schmüdgen's Positivstellensatz [Berr-Wörmann 01, Schweighofer 03]:

   $\mathbb{R}$-algebra fin. generated by $x_1, \ldots, x_n$ & **fractions** $\frac{\mathbf{x}(I_k)^\alpha}{\|\mathbf{x}(I_k)\|_2^{2d_k}}$

The need for denominators

Commutative version of previous example:

$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 18x_2^2x_3 + 88x_2^2x_3^2$

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Sparse example: I_{3322} Bell Inequality

Entanglement in quantum mechanics

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$$f = X_1(Y_1 + Y_2 + Y_3) + X_2(Y_1 + Y_2 - Y_3) + X_3(Y_1 - Y_2) - X_1 - 2Y_1 - Y_2$$

$$\mathbf{K} = \{(X, Y) : X_i, Y_j \succcurlyeq 0, X_i^2 = X_i, Y_j^2 = Y_j, X_i Y_j = Y_j X_i\}$$

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level	sparse	dense [Pál-Vértesi 18]
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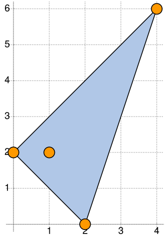
Exploiting correlative sparsity

Exploiting term sparsity

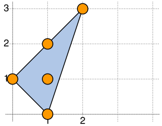
Exploiting sparsity via Newton polytope

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$



Newton polytope $\mathcal{P} = \operatorname{conv}(\text{spt}(f))$

Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



$$f = \begin{pmatrix} x & y & xy & xy^2 & x^2y^3 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} x \\ y \\ xy \\ xy^2 \\ x^2y^3 \end{pmatrix}$$

Exploiting term sparsity [Wang-M.-Lasserre 20]

$$f = x^2 - 2xy + 3y^2 - 2x^2y + 2x^2y^2 - 2yz + 6z^2 + 18y^2z - 54yz^2 + 142y^2z^2$$

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$\rightsquigarrow \frac{6\times 7}{2} = 28$ "unknown" entries in Q

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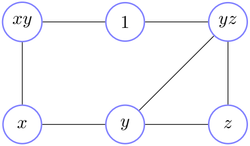
💡 **Term** sparsity pattern graph G



Exploiting term sparsity [Wang-M.-Lasserre 20]

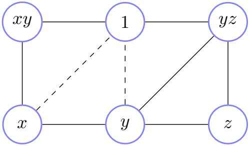
$$f = x^2 - 2xy + 3y^2 - 2x^2y + 2x^2y^2 - 2yz + 6z^2 + 18y^2z - 54yz^2 + 142y^2z^2$$

Newton polytope $\rightarrow f = \begin{pmatrix} 1 & x & y & z & xy & yz \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} 1 \\ x \\ y \\ z \\ xy \\ yz \end{pmatrix}$

$\rightsquigarrow \frac{6 \times 7}{2} = 28$ “unknown” entries in Q

💡 **Term** sparsity pattern graph G

Chordal extension



Replace Q by Q_G with nonzero entries at edges of G

$\rightsquigarrow$ 6 + 9 = 15 “unknown” entries in Q_G

Lyapunov functions from networked systems

$$f = \sum_{i=1}^{N} a_i (x_i^2 + x_i^4) - \sum_{i,k=1}^{N} b_{ik} x_i^2 x_k^2 \quad a_i \in [1,2] \quad b_{ik} \in [\frac{0.5}{N}, \frac{1.5}{N}]$$

$\rightsquigarrow \binom{N+2}{2}(\binom{N+2}{2}+1)/2$ "unknown" entries in $Q = 231$ for $N = 5$

Lyapunov functions from networked systems

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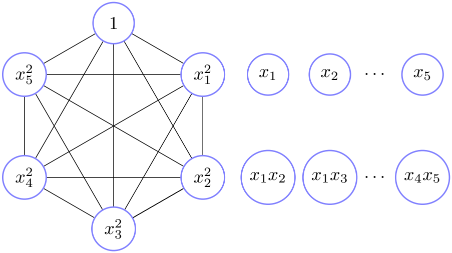
$\leadsto \binom{N+2}{2}(\binom{N+2}{2}+1)/2$ "unknown" entries in $Q = 231$ for $N = 5$

tsp graph G



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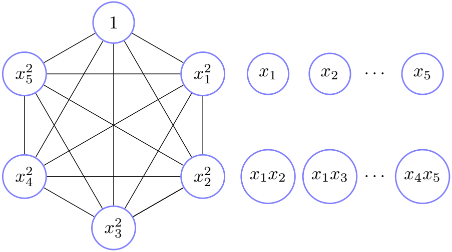

$\rightsquigarrow (N+1)^2$ “unknown” entries in $Q_G = 36$ for $N = 5$

Proof that $f \geqslant 0$ for $N = 80$ in ~ 10 seconds!

Conclusion and Perspectives

- Quest for efficiency

Commutative: **Roundoff error/Networked systems** $n \simeq \mathbf{10^2}$

Noncommutative: **Minimal eigenvalue** $n \simeq \mathbf{20 - 30}$

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DEGREE BOUNDS of Sparse Reznick's representations?

SPARSE RATIONAL SOS representations for sparse nonnegative polynomials?

SYMMETRIC noncommutative problems?

Combine CORRELATIVE and TERM sparsity

Conclusion and Perspectives

APPLICATIONS IN QUANTUM PHYSICS

- Quantum games: number of mutually unbiased bases in dim 6, OPEN FOR SEVERAL DECADES!! **symmetric**
- Ground state energy of hamiltonians **symmetric** & **sparse**
- Inflation for quantum correlations **symmetric** & **sparse**

APPLICATIONS IN DYNAMICAL SYSTEMS

- learning, feedback design, decentralized stabilization **sparse**
- Quantum Optimal Control **sparse**
- Optimal power-flow **sparse**

APPLICATIONS IN MACHINE LEARNING

- robustness of deep networks **sparse**
- Lipschitz global/local constants of deep networks **sparse**

Thank you for your attention!

https://homepages.laas.fr/vmagron

- Chen, Lasserre, Magron & Pauwels. Polynomial Optimization for Bounding Lipschitz Constants of Deep Networks, arxiv:2002.03657
- Mai, Lasserre & Magron. A sparse version of Reznick's Positivstellensatz, arxiv:2002.05101
- Wang, Magron & Lasserre. TSSOS: a moment-SOS hierarchy that exploits term sparsity, arxiv:1912.08899 TSSOS
- Klep, Magron & Povh. Sparse Noncommutative Polynomial Optimization, arxiv:1909.00569 NCSOStools
- Magron, Constantinides & Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming, *TOMS*. arxiv:1507.03331 Real2Float