Moment polynomials for nonlinear Bell inequalities

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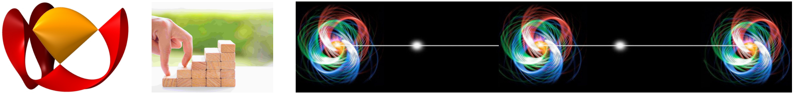
Joint work with Igor Klep & Jurij Volčič

ICFO Seminar 30 October 2023



Motivation: Bell inequalities

Moment polynomials

A complete NPA hierarchy

Back to Bell inequalities

Covariance Bell inequalities [Pozsgay et al. 17]

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$$\begin{aligned}
\mathrm{cov}_{3322} = &\mathrm{cov}(A_1, B_1) + \mathrm{cov}(A_1, B_2) + \mathrm{cov}(A_1, B_3) \\
&+ \mathrm{cov}(A_2, B_1) + \mathrm{cov}(A_2, B_2) - \mathrm{cov}(A_2, B_3) \\
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$$\Pi = \frac{3}{8}(+++/+++) + \frac{3}{8}(--+/--+) + \frac{1}{4}(-+-/-+-)$$

$(A_1 A_2 A_3 / B_1 B_2 B_3)$: strategy where Alice and Bob deterministically output A_x and B_y for inputs x and y

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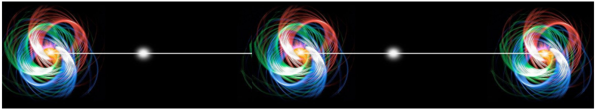
How to get upper bounds?

Bilocal Bell inequality [Tavakoli et al. 21-22]



Alice $\leftsquigarrow$ Source 1 $\rightsquigarrow$ Bob $\leftsquigarrow$ Source 2 $\rightsquigarrow$ Charlie

Observers hold particles from different sources and therefore a priori share no correlations

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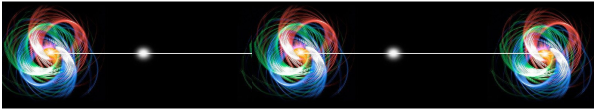

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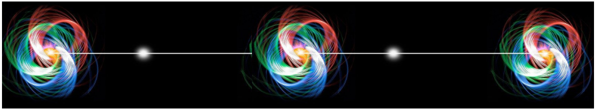

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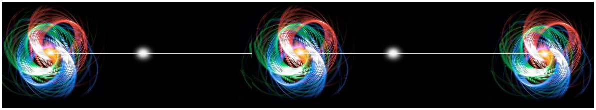

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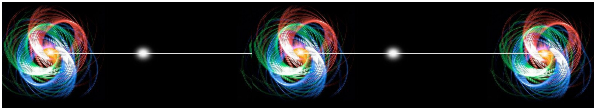

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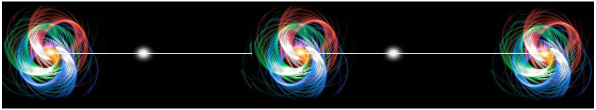

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💡 Classical model

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Binary random variables A_i, B_j, C_k

$$\frac{1}{3}\sum_{i\in\{1,2,3\}}\Big(E(B_iC_i) - E(A_iB_i)\Big) - \sum_{\{i,j,k\}=\{1,2,3\}} E(A_iB_jC_k)$$

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💡 Tetrahedral symmetry yields 3 What is the max?

Moment polynomials

Elements of $\mathscr{M}[x]$

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at a proba μ on $\mathbb{R}^2$ with fourth order moments and a pair $X = (X_1, X_2) \in \mathbb{R}^2$, f evaluates as

$$f(\mu, X) = X_1 X_2 \int x_1 x_2^3 \, \mathrm{d}\mu - X_2^2 \left(\int x_1^2 \, \mathrm{d}\mu \right)^3 + X_2 - 2$$

Related business

Trace polynomials $\mathrm{tr}(x_1^2)\,\mathrm{tr}(x_2) + \mathrm{tr}(x_2)$ with $\mathrm{tr}(uv) = \mathrm{tr}(vu)$

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💡 Proba is a state $\Rightarrow$ moment polynomials are state polynomials

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for Bell $f = \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\mathtt{m}(b_i c_i) - \mathtt{m}(a_i b_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} \mathtt{m}(a_i b_j c_k) \in \mathscr{M}$

Two types of constraints:

- $s_1(X) \geqslant 0$ with $s_1 \in \mathbb{R}[x]$

$$X \in K(S_1)$$

for Bell $a_i^2 = b_j^2 = c_k^2 = 1$

- $s_2(\mu) \geqslant 0$ with $s_2 \in \mathscr{M}$ $\qquad$ μ proba on $K(S_1)$

$$\mu \in \mathcal{K}(S_1, S_2)$$

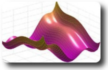
for Bell $\mathtt{m}(a_1 a_2 a_3 \; c_1 c_2 c_3) = \mathtt{m}(a_1 a_2 a_3) \; \mathtt{m}(c_1 c_2 c_3)$

💡 NPA hierarchy to optimize over $\mathscr{M}[x]$

Hierarchies for polynomial optimization

NP-hard NON CONVEX problem $f_{\min} = \inf f(X)$

Theory



(Primal)

$\inf \quad \int f \, d\mu$

with μ proba $\Rightarrow$

INFINITE LP

(Dual)

$\sup \quad \lambda$

$\Leftarrow$ with $f - \lambda \geqslant 0$

Hierarchies for polynomial optimization

NP-hard NON CONVEX problem $f_{\min} = \inf f(X)$

Practice

(Primal **Relaxation**)

moments $\int X^{\alpha} \, d\mu$

finite number $\Rightarrow$

SDP

(Dual **Strengthening**)

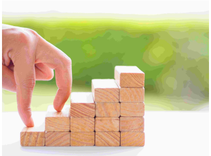
$f - \lambda =$ **sum of squares**

$\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$

[Lasserre '01]

degree r & n vars $\Longrightarrow \binom{n+2r}{n}$ **SDP** VARIABLES



A simple example

$f_{\min} = \min f(X)$ over $K(S)$

Semialgebraic set $K(S) = \{X \in \mathbb{R}^n : s(X) \geqslant 0, \quad s \in S\}$

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$$\overbrace{X_1 X_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(X_1 + X_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{X_1(1 - X_1)}^{s_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{X_2(1 - X_2)}^{s_2}$$

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Sums of squares (SOS) σ_j

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Sums of squares (SOS) σ_j

Quadratic module: $\mathrm{QM}(S)_r = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \ \deg \sigma_j \, s_j \leqslant 2r \right\}$

Hierarchies for polynomial optimization

$$f_{\min} = \min_{X \in K(S)} f(X)$$

- $\mathcal{P}(K(S))$: proba on $K(S)$
- quadratic module $\mathrm{QM}(S) = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \text{ with } \sigma_j \text{ SOS} \right\}$

Infinite-dimensional linear programs (LP)

(Primal) (Dual)

$$\begin{aligned} \inf \quad & \int_{K(S)} f \, d\mu & = \quad & \sup \quad \lambda \\ \text{s.t.} \quad & \mu \in \mathcal{P}(K(S)) & & \text{s.t.} \quad \lambda \in \mathbb{R} \\ & & & \qquad f - \lambda \in \mathrm{QM}(S) \end{aligned}$$

Hierarchies for polynomial optimization

$$f_{\min} = \min_{X \in K(S)} f(X)$$

- Pseudo-moment sequences $\mathbf{y}$ up to order r
- Truncated quadratic module $\mathrm{QM}(S)_r$

Finite-dimensional semidefinite programs (SDP)

$$\begin{array}{rllcll} & & \text{(Moment)} & & & \text{(SOS)} \\ f_r = \inf & \sum_\alpha f_\alpha \, y_\alpha & & = & \sup & \lambda \\ \text{s.t.} & \mathbf{M}_{r-r_j}(s_j \, \mathbf{y}) \succcurlyeq 0 & & & \text{s.t.} & \lambda \in \mathbb{R} \\ & y_0 = 1 & & & & f - \lambda \in \mathrm{QM}(S)_r \end{array}$$

Hierarchies for polynomial optimization

- Moment matrices are indexed by monomials

$$\mathbf{M}_1(\mathbf{y}) = \begin{array}{cc} & \begin{array}{cccc} 1 & & x_1 & x_2 \end{array} \\ \begin{array}{c} 1 \\ \\ x_1 \\ x_2 \end{array} & \left(\begin{array}{cccc} 1 & | & y_{10} & y_{01} \\ & - & - & - \\ y_{10} & | & y_{20} & y_{11} \\ y_{01} & | & y_{11} & y_{02} \end{array}\right) \end{array}$$

Hierarchies for polynomial optimization

Theorem [Putinar 93, Lasserre 01]: positive polynomials

For $f \in \mathbb{R}[x]$, $S \subseteq \mathbb{R}[x]$, if $\underbrace{N}_{>0} - \sum_i x_i^2 \in \mathrm{QM}(S)$ then

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✓ Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

A complete NPA hierarchy

Objective function $f \in \mathscr{M}[x]$

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💡 moment matrices & quadratic modules

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$$f_{\min} = \min_{X \in K(S_1), \mu \in \mathcal{K}(S_1, S_2)} f(\mu)$$

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Moment matrices are (slightly) more complicated than in $\mathbb{R}[x]$

$$\mathbf{M}_1(\mathbf{y}) = \begin{array}{ccccccc} & 1 & & x_1 & x_2 & \mathtt{m}_{10} & \mathtt{m}_{01} \\ 1 & 1 & | & y_{1000} & y_{0100} & y_{0010} & y_{0001} \\ & & - & - & - & - & - \\ x_1 & y_{1000} & | & y_{2000} & y_{1100} & y_{1010} & y_{1001} \\ x_2 & y_{0100} & | & y_{1100} & y_{0200} & y_{0110} & y_{0101} \\ \mathtt{m}_{10} & y_{0010} & | & y_{1010} & y_{0110} & y_{0020} & y_{0011} \\ \mathtt{m}_{01} & y_{0001} & | & y_{1001} & y_{0101} & y_{0011} & y_{0002} \end{array}$$

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\hline
1 & 1 & | & y_{1000} & y_{0100} & y_{0010} & y_{0001} \\
 & & - & - & - & - & - \\
x_1 & \star & | & y_{2000} & y_{1100} & y_{1010} & y_{1001} \\
x_2 & \star & | & \star & y_{0200} & y_{0110} & y_{0101} \\
\mathtt{m}_{10} & \star & | & \star & \star & y_{0020} & y_{0011} \\
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Quadratic module $\mathrm{QM}(S_1, S_2)$ is also more complicated

$$\begin{aligned} \sum \mathtt{m}(p^2 s): &\quad s \in \{1\} \cup S_1 \quad p \in \mathscr{M}[x] \\ \sum q^2 s: &\quad s \in \{1\} \cup S_2 \quad q \in \mathscr{M} \end{aligned}$$

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Theorem [Klep-M.-Volcic 23]: positive moment polynomials

For $f \in \mathscr{M}$, $S_1 \subseteq \mathbb{R}[x]$, $S_2 \subseteq \mathscr{M}$, if $\underbrace{N}_{>0} - \sum_i x_i^2 \in \mathrm{QM}(S_1)$ then

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Back to Bell inequalities

Binary A_i, B_j

$$\begin{aligned}\mathrm{cov}_{3322} = {} & \mathrm{cov}(A_1, B_1) + \mathrm{cov}(A_1, B_2) + \mathrm{cov}(A_1, B_3) \\ & + \mathrm{cov}(A_2, B_1) + \mathrm{cov}(A_2, B_2) - \mathrm{cov}(A_2, B_3) \\ & + \mathrm{cov}(A_3, B_1) - \mathrm{cov}(A_3, B_2)\end{aligned}$$

Back to Bell inequalities

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$$f = \mathtt{m}_{100100} - \mathtt{m}_{100000}\,\mathtt{m}_{000100} + \mathtt{m}_{100010} - \mathtt{m}_{100000}\,\mathtt{m}_{000010} + \cdots$$

$S_1 = \{\pm(1 - a_i^2), \pm(1 - b_j^2)\}$ $\qquad$ $S_2 = \emptyset$

Back to Bell inequalities

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$r = 2$: SDP with $4\,146$ variables $f_2 = 4.5$

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Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3}\sum_{i\in\{1,2,3\}}\Big(E(B_iC_i)-E(A_iB_i)\Big)-\sum_{\{i,j,k\}=\{1,2,3\}}E(A_iB_jC_k)$$

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satisfying bilocality constraints

$$E(A_1A_2A_3\;C_1C_2C_3)=E(A_1A_2A_3)\;E(C_1C_2C_3)$$

+ similar factorization constraints

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(E(B_i C_i) - E(A_i B_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} E(A_i B_j C_k)$$

satisfying bilocality constraints

$$E(A_1 A_2 A_3 \; C_1 C_2 C_3) = E(A_1 A_2 A_3) \; E(C_1 C_2 C_3)$$

+ similar factorization constraints & vanishing constraints

$$\begin{aligned} E(A_i) = E(B_i) = E(C_i) = 0 \quad & \text{for } i \in \{1,2,3\} \\ E(A_i B_j) = E(B_i C_j) = 0 \quad & \text{for } i \neq j \\ E(A_i B_j C_k) = 0 \quad & \text{for } |\{i,j,k\}| \leq 2 \end{aligned}$$

Back to Bell inequalities

[Tavakoli et al. 21-22] Tetrahedral symmetry yields 3

Back to Bell inequalities

[Tavakoli et al. 21-22] Tetrahedral symmetry yields 3

$$\sup \ \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\mathtt{m}(b_i c_i) - \mathtt{m}(a_i b_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} \mathtt{m}(a_i b_j c_k)$$

s.t.

$\mathtt{m}(a_1 a_2 a_3 \ c_1 c_2 c_3) = \mathtt{m}(a_1 a_2 a_3) \ \mathtt{m}(c_1 c_2 c_3)$

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$r = 3$: SDP with $31\,017$ variables $f_3 = 4$

Back to Bell inequalities

[Tavakoli et al. 21-22] Tetrahedral symmetry yields 3

$$\sup\ \frac{1}{3}\sum_{i\in\{1,2,3\}}\Big(\mathtt{m}(b_ic_i)-\mathtt{m}(a_ib_i)\Big)-\sum_{\{i,j,k\}=\{1,2,3\}}\mathtt{m}(a_ib_jc_k)$$

s.t.

$$\mathtt{m}(a_1a_2a_3\ c_1c_2c_3)=\mathtt{m}(a_1a_2a_3)\,\mathtt{m}(c_1c_2c_3)$$

$a_i^2=b_j^2=c_k^2=1$ and $\mathtt{m}(a_i)=\mathtt{m}(b_j)=\mathtt{m}(c_k)=0$

$$\mathtt{m}(a_ib_j)=\mathtt{m}(b_jc_k)=0$$

$\mathtt{m}(a_ib_jc_k)=0$ for $|\{i,j,k\}|\leq 2$

$r=3$: SDP with $31\,017$ variables $f_3=4=f_{\max}$ attained with

$$\eta_0=(1\ 1\ 1\ 1)\quad \eta_1=(1\ 1\ \text{-1}\ \text{-1})\quad \eta_2=(1\ \text{-1}\ 1\ \text{-1})\quad \eta_3=(1\ \text{-1}\ \text{-1}\ 1)$$

Back to Bell inequalities

[Tavakoli et al. 21-22] Tetrahedral symmetry yields 3

$$\sup \ \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\mathtt{m}(b_i c_i) - \mathtt{m}(a_i b_i) \Big) - \sum_{\{i,j,k\}=\{1,2,3\}} \mathtt{m}(a_i b_j c_k)$$

s.t.

$\mathtt{m}(a_1 a_2 a_3 \ c_1 c_2 c_3) = \mathtt{m}(a_1 a_2 a_3) \, \mathtt{m}(c_1 c_2 c_3)$

$a_i^2 = b_j^2 = c_k^2 = 1$ and $\mathtt{m}(a_i) = \mathtt{m}(b_j) = \mathtt{m}(c_k) = 0$

$\mathtt{m}(a_i b_j) = \mathtt{m}(b_j c_k) = 0$

$\mathtt{m}(a_i b_j c_k) = 0$ for $|\{i, j, k\}| \leq 2$

$r = 3$: SDP with $31\,017$ variables $f_3 = 4 = f_{\max}$ attained with

$$\eta_0 = (1 \ 1 \ 1 \ 1) \quad \eta_1 = (1 \ 1 \ \text{-}1 \ \text{-}1) \quad \eta_2 = (1 \ \text{-}1 \ 1 \ \text{-}1) \quad \eta_3 = (1 \ \text{-}1 \ \text{-}1 \ 1)$$

$$A_i = \eta_0 \otimes \eta_i \qquad B_j = \left(\eta_0 \otimes \eta_0 - 2 \sum_{i=1}^{4} e_i \otimes e_i \right) \cdot \eta_i \otimes \eta_0 \qquad C_k = \eta_k \otimes \eta_0$$

Upcoming PhD positions

TENORS
Tensor modEliNg, geOmetRy and optimiSation
Marie Skłodowska-Curie Doctoral Network
2024-2027

Tensors are nowadays ubiquitous in many domains of applied mathematics, computer science, signal processing, data processing, machine learning and in the emerging area of quantum computing. TENORS aims at fostering cutting-edge research in tensor sciences, stimulating interdisciplinary and intersectoriality knowledge developments between algebraists, geometers, computer scientists, numerical analysts, data analysts, physicists, quantum scientists, and industrial actors facing real-life tensor-based problems.

Partners:

1. Inria, Sophia Antipolis, France (B. Mourrain, A. Mantzaflaris)
2. CNRS, LAAS, Toulouse, France (D. Henrion, V. Magron, M. Skomra)
3. NWO-I/CWI, Amsterdam, the Netherlands (M. Laurent)
4. Univ. Konstanz, Germany (M. Schweighofer, S. Kuhlmann, M. Michałek)
5. MPI, Leipzig, Germany (B. Sturmfels, S. Telen)
6. Univ. Tromsoe, Norway (C. Riener, C. Bordin, H. Munthe-Kaas)
7. Univ. degli Studi di Firenze, Italy (G. Ottaviani)
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15 PhD positions (2024-2027)

(recruitment expected around Oct. 2024)

Scientific coord: B. Mourrain
Adm. manager: Linh Nguyen

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