The Moment-Sums of Squares Hierarchy for Polynomial Optimization

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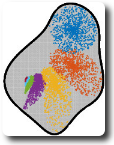
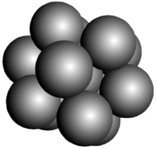
TU Chemnitz





Outline

- Semidefinite Optimization

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- Representation of positive polynomials

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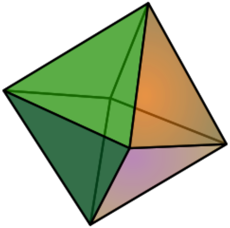
- Representation of positive polynomials
- The $\mathbf{K}$-moment problem

What is Semidefinite Optimization?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d} \; .$$



Polyhedron

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij} z_j \geqslant d_i$"

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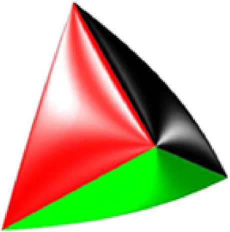

Spectrahedron

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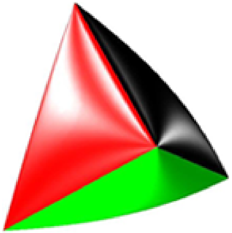

Spectrahedron

Example

$$\mathbf{P}: \quad \min_{\mathbf{z}} \quad \left\{ z_1 + z_2 \ : \right. \\ \text{s.t.} \quad \left. \begin{bmatrix} 3 + 2z_1 + z_2 & z_1 - 5 \\ z_1 - 5 & z_1 - 2z_2 \end{bmatrix} \succcurlyeq 0 \right\}$$

or, equivalently

$$\mathbf{P}: \min_{\mathbf{z}} \quad \left\{ z_1 + z_2 \ : \right. \\ \text{s.t.} \quad \left. \begin{bmatrix} 3 & -5 \\ -5 & 0 \end{bmatrix} + z_1 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + z_2 \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \succcurlyeq 0 \right\}$$

Its DUAL is the convex optimization problem:

$$\mathbf{P}^* : \max \{ \langle \mathbf{F}_0 , \mathbf{Y} \rangle \mid \quad \mathbf{Y} \succcurlyeq 0; \langle \mathbf{F}_i, \mathbf{Y} \rangle = c_i, \quad i = 1, \ldots, n \}$$

Example (continued)

The dual of

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is the semidefinite program

$$\mathbf{P}^* : \quad \max_{\mathbf{Y} \succcurlyeq 0} \left\{ \langle \mathbf{Y}, - \begin{bmatrix} 3 & -5 \\ -5 & 0 \end{bmatrix} \rangle : \langle \mathbf{Y}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \rangle = 1; \langle \mathbf{Y}, \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \rangle = 1 \right\}$$

or, equivalently

$$\mathbf{P}^* : \quad \max \left\{ -3y_1 + 10y_2 \ : \ 2y_1 + 2y_2 + y_3 = 1; y_1 - 2y_3 = 1; \mathbf{Y} = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \succcurlyeq 0 \right\}$$

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Several academic SDP software packages exist, (e.g. MATLAB "LMI toolbox", SeduMi, SDPT3, ...). However, so far, size limitation is more severe than for LP software packages.
Pioneer contributions by A. Nemirovsky, Y. Nesterov, N.Z. Shor, B.D. Yudin,...

Consider the polynomial optimization problem:

$$\mathbf{P}: \quad f^* = \min\{ f(\mathbf{x}) : \quad g_j(\mathbf{x}) \geqslant 0, \, j = 1, \ldots, m \}$$

for some polynomials f, $g_j \in \mathbb{R}[\mathbf{x}]$.

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Why Polynomial Optimization?

After all ... $\mathbf{P}$ is just a particular case of Non Linear Programming (NLP)!

True!

... if one is interested with a LOCAL optimum only!!

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The fact that f and g_j are POLYNOMIALS does not help much!

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Remember that for the GLOBAL minimum f^*:

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and so to compute f^* ...
one needs to handle EFFICIENTLY the difficult constraint

$$f(\mathbf{x}) - \lambda \geqslant 0 \quad \forall \mathbf{x} \in \mathbf{K}$$

i.e. one needs
TRACTABLE CERTIFICATES of POSITIVITY on $\mathbf{K}$
for the polynomial $\mathbf{x} \mapsto f(\mathbf{x}) - \lambda$!

Global Optimization

Consider the GLOBAL optimization problem

$$(\mathbf{P}) \quad f^* := \min_x \{ f(\mathbf{x}) \mid \quad g_j(\mathbf{x}) \geqslant 0, j = 1, \ldots, m\}$$

where f, $g_j(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ are all **real-valued functions.** Let

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n \mid \quad g_j(\mathbf{x}) \geqslant 0, j = 1, \ldots, m\}$$

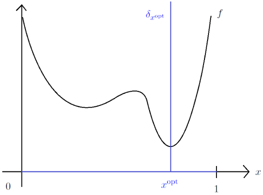
be the feasible set.

Two dual points of view

I: THE PRIMAL SIDE

f^* is a **global minimum if and only if**

$$(*) \quad f^* = \min_{\mu \in M(\mathbf{K})} \left\{ \int_{\mathbf{K}} f \, d\mu \mid \quad \mu(\mathbf{K}) = 1 \right\}$$



i.e. one optimizes over the set $\mathcal{P}(\mathbf{K}) \subset M(\mathbf{K})$ of probability measures with support contained in $\mathbf{K}$.

$$f \geqslant f^* \Rightarrow \int_{\mathbf{K}} f \, d\mu \geqslant \int_{\mathbf{K}} f^* \, d\mu = f^*, \forall \mu \in \mathcal{P}(\mathbf{K})$$

On the other hand, with $\mathbf{x} \in \mathbf{K}$ and $\mu := \delta_{\mathbf{x}}$, $\int f \, d\mu = f(\mathbf{x})$

Two dual points of view (continued)

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f^* is a **global minimum** **if and only if:**

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and in particular, $\mathbf{x} \mapsto f(\mathbf{x}) - f^*$ is nonnegative on $\mathbf{K}$.

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Observe that these two characterizations are proper to the global optimum f^*, and are NOT valid for a local minimum!

BUT ... this is just LP-Duality

The optimization problem

$$f^* = \min_{\mu \in M(\mathbf{K})} \left\{ \int_{\mathbf{K}} f \, d\mu \quad | \quad \mu(\mathbf{K}) = 1 \right\}$$

is the **infinite-dimensional** LP

$$f^* = \min_{\mu \in \mathcal{M}(\mathbf{K})} \{ \langle f, \mu \rangle \mid \quad \langle 1, \mu \rangle = 1; \; \mu \geqslant 0 \}$$

where :

- $\mathcal{M}(\mathbf{K})$ is the **space** of **finite** *signed* **Borel measures** on $\mathbf{K}$, and
- $\langle \cdot, \cdot \rangle$ is the duality bracket between $\mathcal{C}(\mathbf{K})$ and $\mathcal{M}(\mathbf{K})$:

$$\langle f, \mu \rangle = \int_{\mathbf{K}} f \, d\mu, \qquad \forall \, f \in \mathcal{C}(\mathbf{K}), \; \mu \in \mathcal{M}(\mathbf{K})$$

As in the finite dimensional case ...., the **dual** LP reads:

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or, equivalently:

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☞ Not possible in general .... BUT ...

Concerning the primal LP, NOTICE that if f is a polynomial, i.e.,

$$f(\mathbf{x}) = \sum_{\alpha} f_{\alpha}\, \mathbf{x}^{\alpha} \quad \left(= \sum_{\alpha} f_{\alpha}\, x_1^{\alpha_1} \cdots x_1^{\alpha_n} \right)$$

then $$\int_{\mathbf{K}} f\, d\mu = \sum_{\alpha} f_{\alpha} \underbrace{\int_{\mathbf{K}} \mathbf{x}^{\alpha}\, d\mu}_{= y_{\alpha}} = \sum_{\alpha} f_{\alpha}\, y_{\alpha},$$

and so the primal LP reads:

$$f^* = \min \left\{ \sum_{\alpha} f_{\alpha}\, y_{\alpha} \, : \, y_0 = 1; \quad y \in \Delta \right\},$$

where $\Delta = \{ y = (y_{\alpha}) \, : \, \exists\, \mu \in \mathcal{M}(\mathbf{K})$ s.t. $y_{\alpha} = \int_{\mathbf{K}} \mathbf{x}^{\alpha}\, d\mu, \, \forall \alpha \}$.

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$\rightarrow$ a FINITE-DIMENSIONAL CONVEX optimization problem!!

SIMILARLY for the dual LP, NOTICE that if f is a polynomial of degree d, then the constraint of the dual LP

$$f(\mathbf{x}) - \gamma \geqslant 0 \quad \forall \mathbf{x} \in \mathbf{K}$$

is the same as stating that the polynomial $f - \gamma 1$ belongs to the FINITE-DIMENSIONAL convex cone

$$\Theta_d = \{g \in \mathbb{R}[\mathbf{x}]_d : g \geqslant 0 \text{ on } \mathbf{K}\}.$$

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$$f^* = \max \{\gamma : f - \gamma 1 \in \Theta_d\},$$

$\to$ a FINITE-DIMENSIONAL CONVEX optimization problem!!

First good news ...

When:

- $\mathbf{K} \subset \mathbb{R}^n$ is the compact semi-algebraic set

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n \mid \quad g_j(\mathbf{x}) \geqslant 0, \quad j = 1, \ldots, m \}$$

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- a CHARACTERIZATION of real sequences $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, such that

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$\rightarrow$ PRACTICAL COMPUTATION is possible!

Other and (not s.o.s. based) representations of positive polynomials are available (Krivine, Handelman, Vasilescu).

† They lead to Linear Inequalities instead of LMIs and so

... to LP-relaxations instead of SDP-relaxations

.. but less efficient and ill-behaved ... despite so far, LP software packages are more powerful than SDP packages!!

Putinar's Positivstellensatze

Let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set:

$$\mathbf{K} := \{ x \in \mathbb{R}^n \, : \, g_j(\mathbf{x}) \geqslant 0, \quad j = 1, \ldots, m \}$$

for some polynomials $\{g_j\} \subset \mathbb{R}[\mathbf{x}]$.

and with g_0 being the constant polynomial 1,

define the quadratic module

$$\mathcal{Q}(g_1, \ldots, g_m) = \{g \in \mathbb{R}[X] : g = \sum_{j=0}^{m} \sigma_j \, g_j\},$$

where the $(\sigma_j)_{j=0}^{m}$ are s.o.s. polynomials.

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- It requires $m+1$ s.o.s. weights σ_j.

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Theorem (Putinar-Jacobi-Prestel)

Let $\mathbf{K}$ be compact and Assumption 1 hold. Then

$$[\, f \in \mathbb{R}[\mathbf{x}] \quad \textit{and} \quad f > 0 \textit{ on } \mathbf{K} \,] \Rightarrow f \in \mathcal{Q}(g_1, \ldots, g_m), \;\; i.e.,$$

$$f(\mathbf{x}) \; = \; \sigma_0(\mathbf{x}) + \sum_{j=1}^{m} \sigma_j(\mathbf{x}) \, g_j(\mathbf{x}), \qquad \forall x \in \mathbb{R}^n$$

for some s.o.s. polynomials $\{\sigma_j\}_{j=0}^{m}$.

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• **Moreover, Assumption 1 holds true if e.g. :**

- all the g_j's are linear (hence $\mathbf{K}$ is a polytope), or if
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• **If** $\mathbf{x} \in \mathbf{K} \Rightarrow \|\mathbf{x}\| \leqslant M$ **for some (known)** M, then it suffices to add the redundant quadratic constraint $M^2 - \|\mathbf{x}\|^2 \geqslant 0$, in the definition of $\mathbf{K}$

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Indeed, let

$$v_d(\mathbf{x}) = (\mathbf{x}^\alpha), \qquad |\alpha| := \sum_i \alpha_i \leqslant d$$

be a basis of $\mathbb{R}[\mathbf{x}]_d$ (polynomials of degree at most d)

Let $f \in \mathbb{R}[\mathbf{x}]_{2d}$ be a s.o.s. polynomial, that is, $f = \sum_{k=1}^s q_k(\mathbf{x})^2$, for some polynomials $\{q_k\}_{k=1}^s \subset \mathbb{R}[\mathbf{x}]_d$.

Denote also $q_k = \{q_{k\alpha}\}_{\alpha \in \mathbb{N}^n}$, the vector of coefficients of the polynomial q_k, in the basis $v_d(\mathbf{x})$, that is,

$$q_k(\mathbf{x}) = \langle q_k, v_d(\mathbf{x}) \rangle = \sum_{|\alpha| \leqslant r} q_{k\alpha} \mathbf{x}^\alpha$$

and define the real symmetric matrix $Q := \sum_{k=1}^s q_k q_k^T \succcurlyeq 0$.

$$\langle v_d(\mathbf{x}), Q\, v_d(\mathbf{x}) \rangle = \sum_{k=1}^s \langle q_k, v_d(\mathbf{x}) \rangle^2 = \sum_{k=1}^s q_k(\mathbf{x})^2 = f(\mathbf{x})$$

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and define the real symmetric matrix $Q := \sum_{k=1}^s q_k q_k^T \succcurlyeq 0$.

$$\langle v_d(\mathbf{x}), Q\, v_d(\mathbf{x}) \rangle = \sum_{k=1}^s \langle q_k, v_d(\mathbf{x}) \rangle^2 = \sum_{k=1}^s q_k(\mathbf{x})^2 = f(\mathbf{x})$$

Conversely, let $Q \succcurlyeq 0$ be a real $s(d) \times s(d)$ positive semidefinite symmetric matrix ($s(d)$ is the dimension of the vector space $\mathbb{R}[\mathbf{x}]_d$). As $Q \succcurlyeq 0$, write $Q = \sum_{k=1}^s q_k q_k^T$, so that

$$f(\mathbf{x}) = \langle v_d(\mathbf{x}), Q\, v_d(\mathbf{x}) \rangle = \sum_{k=1}^s \langle q_k, v_d(\mathbf{x}) \rangle^2 = \sum_{k=1}^s q_k(\mathbf{x})^2$$

is s.o.s.

Next, write the matrix $v_d(\mathbf{x})\, v_d(\mathbf{x})^T$ as:

$$v_d(\mathbf{x})\, v_d(\mathbf{x})^T \;=\; \sum_{\alpha \in \mathbb{N}^n_{2d}} \mathbf{B}_\alpha\, \mathbf{x}^\alpha,$$

so that checking whether

$$f(\mathbf{x}) = \langle v_d(\mathbf{x}), Q\, v_d(\mathbf{x}) \rangle \;=\; \langle Q, v_d(\mathbf{x})\, v_d(\mathbf{x})^T \rangle,$$

for some $Q \succcurlyeq 0$ reduces to checking the LMI

$$\begin{cases} \langle \mathbf{B}_\alpha, Q \rangle &=& f_\alpha, \quad \alpha \in \mathbb{N}^n,\ |\alpha| \leqslant 2d \\ Q &\succcurlyeq& 0 \end{cases}.$$

Example

Let $t \mapsto f(t) = 6 + 4t + 9t^2 - 4t^3 + 6t^4$. Is f an SOS? Do we have

$$f(t) = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}^T \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$$

Example

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We must have:

$$a = 6\,;\, 2\,b = 4;\, d + 2\,c = 9;\, 2\,e = -4;\, f = 6.$$

And so we must find a scalar c such that

$$Q = \begin{bmatrix} 6 & 2 & c \\ 2 & 9 - 2c & -2 \\ c & -2 & 6 \end{bmatrix} \succcurlyeq 0.$$

With $c = -4$ we have

$$Q = \begin{bmatrix} 6 & 2 & -4 \\ 2 & 17 & -2 \\ -4 & -2 & 6 \end{bmatrix} \succcurlyeq 0.$$

and

$$Q = 2 \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}' + 9 \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}'$$

$$+18 \begin{bmatrix} 1/\sqrt{(}18) \\ 4/\sqrt{(}18) \\ -1/\sqrt{(}18) \end{bmatrix} \begin{bmatrix} 1/\sqrt{(}18) \\ 4/\sqrt{(}18) \\ -1/\sqrt{(}18) \end{bmatrix}'$$

and so

$$f(t) = (1 + t^2)^2 + (2 - t - 2t^2)^2 + (1 + 4t - t^2)^2$$

II. DUAL side: The K-moment problem

Let $\{\mathbf{x}^\alpha\}$ be a canonical basis for $\mathbb{R}[\mathbf{x}]$, and let $y := \{y_\alpha\}$ be a given sequence indexed in that basis.

The K-moment problem

Given $\mathbf{K} \subset \mathbb{R}^n$, **does there exist a measure** μ on $\mathbf{K}$, such that

$$y_\alpha = \int_{\mathbf{K}} \mathbf{x}^\alpha \, d\mu, \qquad \forall \alpha \in \mathbb{N}^n \quad ?$$

(where $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$).

Given $y = \{y_\alpha\}$, let $L_y : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, be the linear functional

$$f \left(= \sum_\alpha f_\alpha \, \mathbf{x}^\alpha\right) \quad \mapsto \quad L_y(f) := \sum_{\alpha \in \mathbb{N}^n} f_\alpha \, y_\alpha .$$

Moment matrix $M_d(y)$

with rows and columns also indexed in the basis $\{\mathbf{x}^\alpha\}$.

$$M_d(y)(\alpha, \beta) := L_y(\mathbf{x}^{\alpha+\beta}) = y_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{N}^n, \quad |\alpha|, |\beta| \leqslant d.$$

For instance in $\mathbb{R}^2$: $\quad M_1(y) = \begin{bmatrix} \overbrace{y_{00}}^{1} & | & \overbrace{y_{10}}^{X_1} & \overbrace{y_{01}}^{X_2} \\ - & & - & - \\ y_{10} & | & y_{20} & y_{11} \\ y_{01} & | & y_{11} & y_{02} \end{bmatrix}$

Importantly ...

$$M_d(y) \succcurlyeq 0 \iff L_y(h^2) \geqslant 0, \qquad \forall h \in \mathbb{R}[X]_d$$

Localizing matrix

The "Localizing matrix" $M_d(\theta y)$ w.r.t. a polynomial $\theta \in \mathbb{R}[X]$
with $\mathbf{x} \mapsto \theta(\mathbf{x}) = \sum_\gamma \theta_\gamma \, \mathbf{x}^\gamma$, has its rows and columns also indexed in the basis $\{X^\alpha\}$ of $\mathbb{R}[\mathbf{x}]_d$, and with entries:

$$\begin{aligned} M_d(\theta\, y)(\alpha, \beta) &= L_y(\theta\, \mathbf{x}^{\alpha+\beta}) \\ &= \sum_{\gamma \in \mathbb{N}^n} \theta_\gamma \, y_{\alpha+\beta+\gamma}, \qquad \begin{cases} \alpha, \beta \in \mathbb{N}^n \\ |\alpha|, |\beta| \leqslant d. \end{cases} \end{aligned}$$

For instance, in $\mathbb{R}^2$, and with $X \mapsto \theta(\mathbf{x}) := 1 - x_1^2 - x_2^2$,

$$M_1(\theta\, y) = \begin{bmatrix} \overbrace{y_{00} - y_{20} - y_{02}}^{1} & \overbrace{y_{10} - y_{30} - y_{12}}^{x_1} & \overbrace{y_{01} - y_{21} - y_{03}}^{x_2} \\ y_{10} - y_{30} - y_{12} & y_{20} - y_{40} - y_{22} & y_{11} - y_{21} - y_{12} \\ y_{01} - y_{21} - y_{03} & y_{11} - y_{21} - y_{12} & y_{02} - y_{22} - y_{04} \end{bmatrix}$$

Importantly ...

$$M_d(\theta\, y) \succcurlyeq 0 \quad \Longleftrightarrow \quad L_y(h^2\, \theta) \geqslant 0, \qquad \forall h \in \mathbb{R}[\mathbf{x}]_d$$

Putinar's dual conditions

Again $\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n \mid \quad g_j(\mathbf{x}) \geqslant 0, j = 1, \ldots, m\}$.

Assumption 1:

For some $M > 0$, the quadratic polynomial

$M - \|\mathbf{x}\|^2$ is in the quadratic module $\mathcal{Q}(g_1, \ldots, g_m)$

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For some $M > 0$, the quadratic polynomial $M - \|\mathbf{x}\|^2$ is in the quadratic module $\mathcal{Q}(g_1, \ldots, g_m)$

Theorem (Putinar: dual side)

Let $\mathbf{K}$ be compact, and Assumption 1 hold. Then $y = \{y_\alpha\}$ has a representing measure μ on $\mathbf{K}$ if and only if

$$(**) \quad L_y(f^2) \geqslant 0; \quad L_y(f^2 g_j) \geqslant 0, \quad \forall j = 1, \ldots, m; \quad \forall f \in \mathbb{R}[\mathbf{x}].$$

Checking whether (**) holds for all $f \in \mathbb{R}[\mathbf{x}]$ with degree $\leqslant d$

reduces to checking whether $M_d(y) \succcurlyeq 0$ and $M_d(g_j\, y) \succcurlyeq 0$, for all $j = 1, \ldots, m$!

$\rightarrow m + 1$ LMI conditions to verify!

A Hierarchy of SDP Relaxations

Recall the PRIMAL LP

$$f^* = \min_{\mu \in \mathcal{M}(\mathbf{K})} \left\{ \int_{\mathbf{K}} f \, d\mu \; : \; \mu(\mathbf{K}) = 1 \right\}$$

where $\mathcal{M}(\mathbf{K})$ is the space of Borel measures on $\mathbf{K}$

A Hierarchy of SDP Relaxations

Let $\deg g_j = 2v_j$ or $2v_j - 1$. The SDP-relaxation $\mathbf{Q}_d$, $d \in \mathbb{N}$, reads:

$$\mathbf{Q}_d \begin{cases} \rho_d = \min\limits_{y} \quad L_y(f) & (\rightarrow \text{think of } \int f \, d\mu) \\ \text{s.t.} \quad \underbrace{M_{d-v_j}(g_j\, y) \succcurlyeq 0, \quad j = 0, \ldots m}_{\text{necessary conditions for } y_\alpha = \int \mathbf{x}^\alpha d\mu} & . \\ L_y(1) \ = \ 1 & (\rightarrow \mu(\mathbf{K}) = 1) \end{cases}$$

A Hierarchy of SDP Relaxations

... whose dual is the SDP

$$\mathbf{Q}_d^* \begin{cases} \rho_d^* = \max\limits_{\lambda, \{\sigma_j\}} \quad \lambda \\ \qquad \text{s.t.} \quad f - \lambda = \sum\limits_{k=0}^{m} \sigma_k \, g_k \\ \qquad \qquad \{\sigma_k\} \text{ are s.o.s.}; \quad \deg \sigma_0, \deg \sigma_k \, g_k \leqslant 2d \end{cases}$$

A Hierarchy of SDP Relaxations

Recall that $\mathbf{K} \subset \mathbb{R}^n$ is the basic semi-algebraic set

$$\mathbf{K} := \{ \ x \in \mathbb{R}^n \mid \quad g_j(\mathbf{x}) \geqslant 0, \, j = 1, \ldots, m \}.$$

Assumption 1:

For some $M > 0$, $M - \|\mathbf{x}\|^2$ is in $\mathcal{Q}(g_1, \ldots, g_m)$

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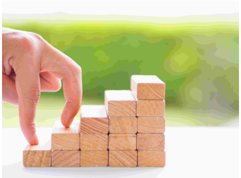
Assumption 1:

For some $M > 0$, $M - \|\mathbf{x}\|^2$ is in $\mathcal{Q}(g_1, \ldots, g_m)$

Theorem (Lasserre 01)

Let $\mathbf{K}$ be compact, and let Assumption 1 holds. Then:

- $\rho_d^* \leqslant \rho_d \leqslant f^*$ *for all* d *and* $\rho_d^*, \rho_d \uparrow f^*$ *as* $d \to \infty$*.*
- *If* $\operatorname{int} \mathbf{K} \neq \emptyset$*, then* $\rho_d = \rho_d^*$ *and the "sup" is attained.*



Global optimality check & extracting solutions

Exactness of a particular SDP-relaxation

Let y be an optimal solution of $\mathbf{Q}_d$ and let $2v \geqslant \max_j \deg g_j$. If

$$\operatorname{rank} M_d(y) = \operatorname{rank} M_{d-v}(y) \; (=: \; s)$$

then $\rho_d = f^*$

and one may extract s GLOBAL MINIMIZERS

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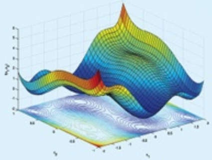

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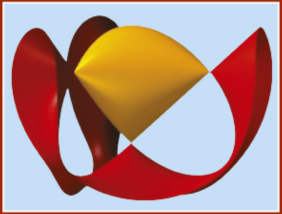

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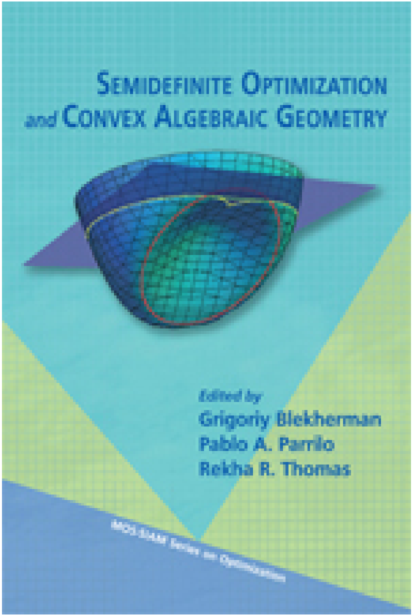
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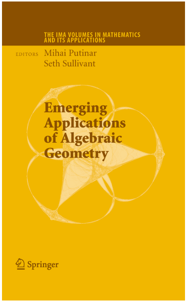

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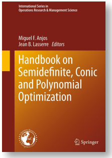

2012, 2012, XI, 957 p. 57 illus.

M.F. Anjos, Polytechnique Montréal, QC, Canada; **J.B. Lasserre**, LAAS, Toulouse Cedex 4, France (Eds.)

Handbook on Semidefinite, Conic and Polynomial Optimization

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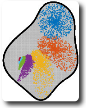

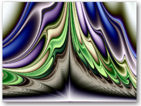

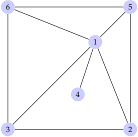

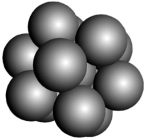

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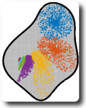

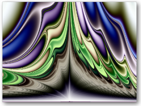

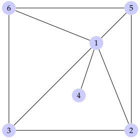

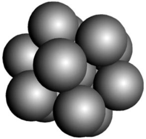

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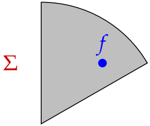

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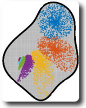
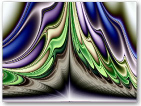
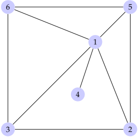

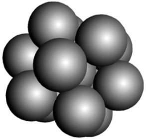

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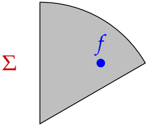

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