Certification of inequalities involving transcendental functions using SDP

Joint Work with B. Werner, S. Gaubert and X. Allamigeon

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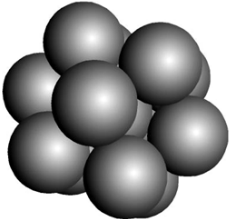
Kepler Conjecture (1611):

The maximal density of sphere packings in 3-space is $\dfrac{\pi}{\sqrt{18}}$

- It corresponds to the way people would intuitively stack oranges, as a pyramid shape
- The proof of T. Hales (1998) consists of thousands of non-linear inequalities
- Many recent efforts have been done to give a formal proof of these inequalities: Flyspeck Project
- Motivation: get positivity certificates and check them with Proof assistants like COQ



Inequalities issued from Flyspeck non-linear part involve:

1. Semi-Algebraic functions algebra $\mathcal{A}$: composition of polynomials with $|\cdot|$, $(\cdot)^{\frac{1}{p}} (p \in \mathbb{N}_0), +, -, \times, /, \sup, \inf$
2. Transcendental functions $\mathcal{T}$: composition of semi-algebraic functions with $\mathtt{arctan}$, $\mathtt{arcos}$, $\mathtt{arcsin}$, $\mathtt{exp}$, $\mathtt{log}$, $|\cdot|$, $(\cdot)^{\frac{1}{p}} (p \in \mathbb{N}_0), +, -, \times, /, \sup, \inf$

Lemma$_{9922699028}$ from Flyspeck

$K := [4; 6.3504]^3 \times [6.3504; 8] \times [4; 6.3504]^2 \qquad P, Q \in \mathbb{R}[X]$

$\forall x \in K, -\frac{\pi}{2} + \mathtt{arctan} \frac{P(x)}{\sqrt{Q(x)}} + 1.6294 - 0.2213 (\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913 (\sqrt{x_4} - 2.52) + 0.728 (\sqrt{x_1} - 2.0) \geq 0.$

Tight inequality: global optimum $= 1.7 \times 10^{-4}$

General Framework

Given K a compact set, and f a transcendental function, bound from below $f^* = \inf_{x \in K} f(x)$ and prove $f^* \geq 0$

1. f is underestimated by a semi-algebraic function f_{sa} on a compact set $K_{sa} \supset K$
2. Reduce the problem $\inf_{x \in K_{sa}} f_{sa}(x)$ to a polynomial optimization problem (POP) in a lifted space K_{pop}
3. Solve classicaly the POP problem $\inf_{x \in K_{pop}} f_{pop}(x)$ using a sparse variant hierarchy of SDP relaxations by Lasserre

$$f^* \geq f_{sa}^* \geq f_{pop}^* \underbrace{\geq 0}_{\text{If the relaxations are accurate enough}}$$

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Transcendental Functions Underestimators

- Let $f \in \mathcal{T}$ be a transcendental univariate elementary function such as $arctan$, exp, ..., defined on a real interval I.
- Basic convexity/semiconvexity properties and monotonicity of f are used to find lower and upper semi-algebraic bounds.

Example with $arctan$:

- $arctan$ is semiconvex on I: $\exists c < 0$ such that $arctan - \frac{c}{2}(\cdot)^2$ is convex on I
- $\forall a \in I = [m; M]$, $arctan(a) \geq \max_{i \in C} \{par_{a_i}^{-}(a)\}$ where C define an index collection of parabola tangent to the function curve and underestimating f.

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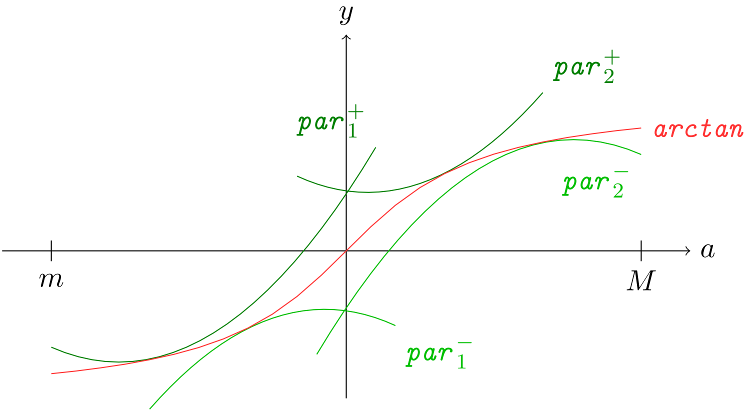
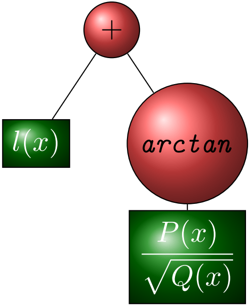
Transcendental Functions Underestimators

Example with arctan:



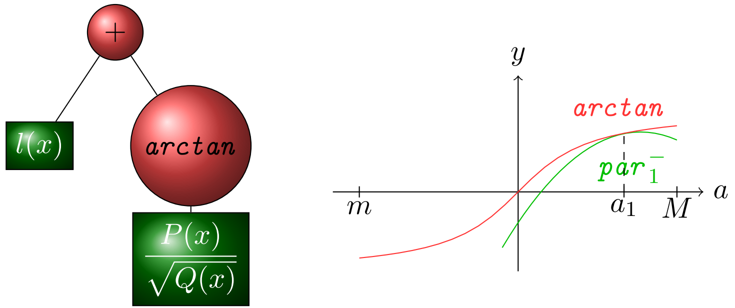
Adaptative Semi-algebraic Approximations Algorithm

- The first step is to build the abstract syntax tree from an inequality, where leaves are semi-algebraic functions and nodes are univariate transcendental functions (arctan, exp, ...) or basic operations ($+$, $\times$, $-$, $/$).
- With $l := -\frac{\pi}{2} + 1.6294 - 0.2213\,(\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913\,(\sqrt{x_4} - 2.52) + 0.728\,(\sqrt{x_1} - 2.0)$, the tree is:



Adaptative Semi-algebraic Approximations

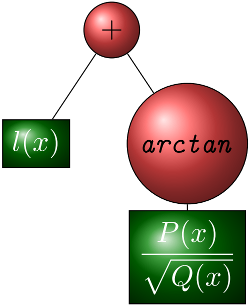
$\texttt{algo}_{\texttt{iter}}$ First iteration:

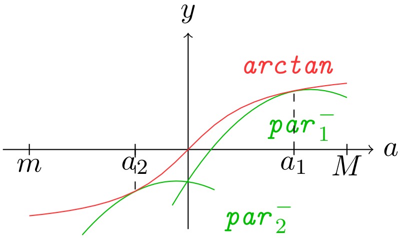


1. Evaluate f with `randeval` and obtain a minimizer guess x_1.
   Compute $a_1 := \frac{P(x_1)}{\sqrt{Q(x_1)}} = 0.84460$
2. Get the equation of par_1^-
3. Compute $m_1 \leq \min_{x \in K} \{ l(x) + par_1^- (\frac{P(x)}{\sqrt{Q(x)}}) \}$

Adaptative Semi-algebraic Approximations

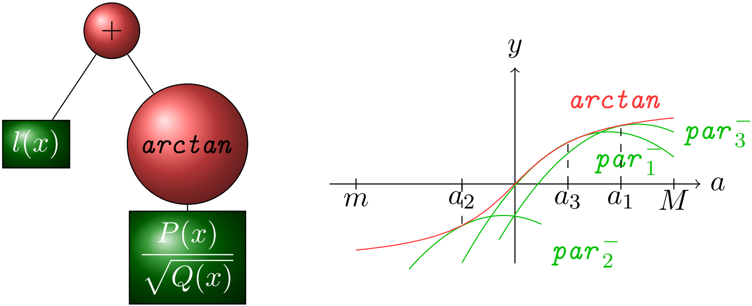
algo$_{\text{iter}}$ Second iteration:





1. $m_1 = -0.746 < 0$, obtain a new minimizer x_2.
2. Compute $a_2 := \dfrac{P(x_2)}{\sqrt{Q(x_2)}} = -0.374$ and par_2^-
3. Compute $m_2 \leq \min_{x \in K} \{ l(x) + \max_{i \in \{1,2\}} \{ par_i^- (\dfrac{P(x)}{\sqrt{Q(x)}}) \} \}$

$\texttt{algo}_{\texttt{iter}}$ Third iteration:



1. $m_2 = -0.112 < 0$, obtain a new minimizer x_3.
2. Compute $a_3 := \dfrac{P(x_3)}{\sqrt{Q(x_3)}} = 0.357$ and $\texttt{par}^{-}_{3}$
3. Compute $m_3 \leq \min\limits_{x \in K}\{l(x) + \max\limits_{i \in \{1,2,3\}} \{\texttt{par}^{-}_{i}\,(\dfrac{P(x)}{\sqrt{Q(x)}})\}\}$

Adaptative Semi-algebraic Approximations

- $m_3 = -0.0333 < 0$, obtain a new minimizer x_4 and iterate again...

Theorem: Convergence of Semi-algebraic underestimators

Let $f \in \mathcal{T}$ and $(x_p^{opt})_{p \in \mathbb{N}}$ be a sequence of control points obtained to define the hierarchy of f-underestimators in the previous algorithm $\mathtt{algo}_{\mathtt{iter}}$ and x^* be an accumulation point of $(x_p^{opt})_{p \in \mathbb{N}}$. Then, x^* is a global minimizer of f on K.

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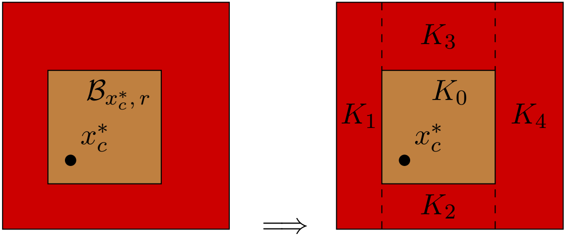
Local Solutions to Global Issues

- Instead of increasing both relaxation orders, fix the SDP relaxation order $k \leq 3$ and the number of control points p.
- If $\texttt{algo}_{\texttt{iter}}$ returns a negative lower bound then cut the initial box K in several boxes $(K_i)_{1 \leq i \leq c}$ and solve the inequality on each K_i.



End

Thank you for your attention!