Semidefinite Characterization of Invariant Measures for Polynomial Systems

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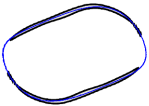

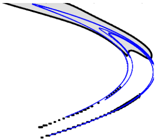

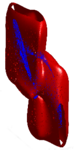

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Invariant Measures

Problem statement

- Polynomial transition map
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f^1(\mathbf{x}), \ldots, f^n(\mathbf{x})) \in \mathbb{R}^n[\mathbf{x}]$
- Semialgebraic state constraints
  $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \ldots, g_l(\mathbf{x}) \geq 0\}$
- **Discrete**-time systems:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t)\,, \quad \mathbf{x}_t \in \mathbf{X}\,, \quad t \in \mathbb{N}$$

  or **Continuous**-time systems:

$$\dot{\mathbf{x}} = f(\mathbf{x})\,, \quad \mathbf{x} \in \mathbf{X}\,, \quad t \in [0, \infty)$$

Problem statement

- Borel σ-algebra $\mathcal{B}(\mathbf{X})$ (generated by the open sets of $\mathbf{X}$)
- $\mu \in \mathcal{M}_+(\mathbf{X})$: set of Borel measures supported on $\mathbf{X}$:
  1. Non-negative **2.** Countably additive

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- Moments of the Lebesgue measure on $\mathbf{A} \subseteq \mathbf{X}$

$$z_\beta^{\mathbf{A}} := \int \mathbf{x}^\beta \lambda_{\mathbf{A}}(d\mathbf{x}) \in \mathbb{R}\,, \quad \beta \in \mathbb{N}^n$$

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- Invariant set: $f(\mathbf{I}) = \mathbf{I}$
- Invariant measure: $\mu(\mathbf{B}) = \mu(f^{-1}(\mathbf{B}))\ \forall \mathbf{B} \in \mathcal{B}(\mathbf{X}) \qquad \mu = f_\# \mu$

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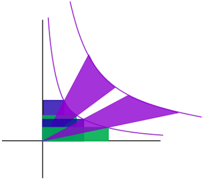
Squeeze mapping: $(x, y) \mapsto f(x, y) := (ax, y/a)$
$\mathbf{X} := [0, 1]^2$
$\lambda_{\mathbf{X}} = f_\# \lambda_{\mathbf{X}}$



The Problem

How to characterize the *invariant measures*?

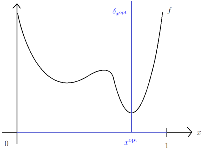
Related work: approximating invariant sets and measures

- Determine **long term dynamical behaviors**.
  - Numerical **integration** under some initial conditions
    $\longrightarrow$ Resulting trajectory could exhibit chaotic behaviors!
  - Alternative: approximate the *densities* directly.
- **Subdivision** techniques [Dellnitz et al 1997, Aston-Junge 2014]
- **Markov chain** based approximation of dynamical behavior. GAIO [Dellnitz-Froyland-Junge 2001]
- **Multilevel** subdivision scheme for uncertain ODEs [Dellnitz-Hohmann-Ziessler 2017]

Related work: Lasserre hierarchy

Cast a polynomial optimization problem as an *infinite-dimensional* LP over measures [Lasserre 2001]

$$f^\star := \inf_{\mathbf{x}\in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f(\mathbf{x}) d\mu$$



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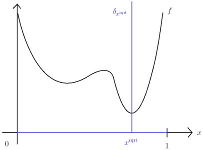
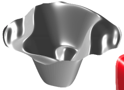
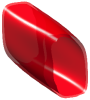
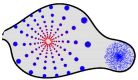






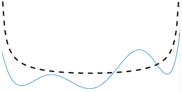


⇝ Regions of attraction [Henrion-Korda 14]

⇝ Maximum invariants [Korda et al 13]

⇝ Reachable sets [Magron et al 17]

⇝ Invariant 1D densities [Henrion 2012]



Contribution

- **Extension** of Lasserre's hierarchy of semidefinite relaxations

  $\rightsquigarrow$ no time/space discretization required.
- Approximate **as close as desired**:
  - Invariant densities.
  - Support of singular invariant measures.
- Relies on a hierarchy of **finite-dimensional** SDPs.

Overview

SDP for Polynomial Optimization (Reminder)

Invariant Densities

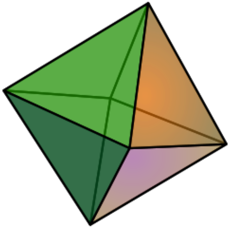
Singular Invariant Measures

SDP for Polynomial Optimization (Reminder)

What is Semidefinite Programming?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geq \mathbf{d} \quad .$$



Polyhedron

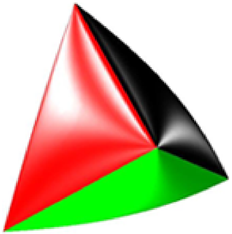
- Linear cost $\mathbf{c}$
- Linear inequalities "$\sum_i A_{ij}\, z_j \geq d_i$"

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succeq \mathbf{F}_0 \ .$$

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)



Spectrahedron

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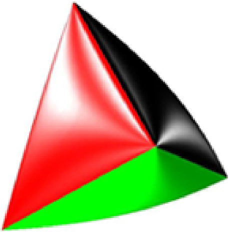

Spectrahedron

- Linear cost $\mathbf{c}$
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SDP for Polynomial Optimization

- Prove **polynomial inequalities** with SDP:

$$f(a,b) := a^2 - 2ab + b^2 \geq 0 \ .$$

- Find $\mathbf{z}$ s.t. $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succeq 0} \begin{pmatrix} a \\ b \end{pmatrix} \ .$

- Find $\mathbf{z}$ s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F_1}} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F_2}} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F_3}} z_3 \succeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F_0}}$

SDP for Polynomial Optimization

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve:

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i \, z_i \succeq \mathbf{F}_0 \,, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \,. \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succeq 0 \quad$ (eigenvalues 0 and 2)
- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a-b)^2 \,.$
- Solving **SDP** $\Longrightarrow$ Finding **Sums of Squares** certificates

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \ldots, g_l(\mathbf{x}) \geq 0\}$

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 $:= [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geq 0, \quad x_2(1 - x_2) \geq 0\}$

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$$\overbrace{x_1 x_2}^{f} + \frac{1}{8} = \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Sums of squares (SOS) σ_i

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- Sums of squares (SOS) σ_i
- Bounded degree:
  $\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^{l} \sigma_j g_j, \ \text{ with } \deg \sigma_j \, g_j \leq 2r \right\}$

SDP for Polynomial Optimization

- **Hierarchy of SDP relaxations**:
  $m_r := \sup_m \Big\{ m : f - m \in \mathcal{Q}_r(\mathbf{X}) \Big\}$
- Convergence guarantees $m_r \uparrow f^*$ [Lasserre 01]
- Can be computed with SDP solvers (csdp, sdpa)
- **"No Free Lunch" Rule**: $\binom{n+2r}{n}$ SDP variables
- Extension to semialgebraic functions
  $\rightsquigarrow$ $r(\mathbf{x}) = p(\mathbf{x})/\sqrt{q(\mathbf{x})}$ [Lasserre-Putinar 10]

Primal-dual Moment-SOS [Lasserre 01]

$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{X}$ if there exists a finite measure μ supported on $\mathbf{X}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{X}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n .$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{X})$: space of probability measures supported on $\mathbf{X}$
- $\mathcal{Q}(\mathbf{X})$: quadratic module

Polynomial Optimization Problems (POP)

(Primal) $\qquad\qquad$ (Dual)

$$\begin{aligned} \inf \quad & \int_{\mathbf{X}} f \, d\mu \\ \text{s.t.} \quad & \mu \in \mathcal{M}_+(\mathbf{X}) \end{aligned} \quad = \quad \begin{aligned} \sup \quad & m \\ \text{s.t.} \quad & m \in \mathbb{R} \ , \\ & f - m \in \mathcal{Q}(\mathbf{X}) \end{aligned}$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{X})$
- Truncated quadratic module $\mathcal{Q}_r(\mathbf{X})$

Polynomial Optimization Problems (POP)

$$
\begin{array}{llcll}
 & \text{(Moment)} & & & \text{(SOS)} \\
\inf & \sum_\alpha f_\alpha \, \mathbf{z}_\alpha & = & \sup & m \\
\text{s.t.} & \mathbf{M}_{r-r_j}(g_j \, \mathbf{z}) \succcurlyeq 0 \,, \quad 0 \leq j \leq l, & & \text{s.t.} & m \in \mathbb{R} \,, \\
 & z_0 = 1 & & & f - m \in \mathcal{Q}_r(\mathbf{X})
\end{array}
$$

Invariant Densities

Invariant Densities in Lebesgue Spaces

Discrete systems: $f_{\#}\mu - \mu = 0$

Continuous systems: $\sum_{i=1}^{n} \frac{\partial(f^i \mu)}{\partial x_i} = 0$

$\rightsquigarrow$ μ is invariant when $\mathcal{L}_f(\mu) = 0$

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Lebesgue decomposition: $\mu = \nu + \psi$ with

- Absolute continuity $\nu \ll \lambda$: $\lambda(\mathbf{A}) = 0 \implies \nu(\mathbf{A}) = 0$
- Singular $\psi \perp \lambda$: $\lambda(\mathbf{A}) = \nu(\mathbf{B}) = 0$ for disjoint $\mathbf{A}$, $\mathbf{B}$

Focus on the case when $\psi = 0$ and $\mu = \nu$

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Focus on the case when $\psi = 0$ and $\mu = \nu$

$\mu \ll \lambda \implies \exists$ measurable h with $d\mu = h d\lambda$

Notation abuse $\|\mu\|_p = \|h\|_p = (\int_{\mathbf{X}} |h(\mathbf{x})|^p d\mathbf{x})^{1/p}$, for $p \geq 1$

Compute or approximate $h \in L_p(\mathbf{X})$?

Invariant Densities in Lebesgue Spaces

p and q conjugate: $1/p + 1/q = 1$, for $p \geq 1$

Reminder: $\ell_{\mathbf{y}}(g) = \sum_\alpha y_\alpha g_\alpha$ for $g \in \mathbb{R}[\mathbf{x}]$

Invariant Densities in Lebesgue Spaces

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Reminder: $\ell_{\mathbf{y}}(g) = \sum_\alpha y_\alpha g_\alpha$ for $g \in \mathbb{R}[\mathbf{x}]$

Theorem ([Yosida]+ [Riesz-Haviland])

Let $\mathbf{y} = (y_\alpha)$ *TFAE:*

(i) $\mathbf{y}$ *is represented by* $\mu \in L_p(\mathbf{X})$ *with* $\|\mu\|_p \leq \gamma < \infty$ *for some* $\gamma \geq 0$.

(ii) $\exists \gamma \geq 0$ *s.t.* $\forall g \in \mathbb{R}[\mathbf{x}]$*:*

$$|\ell_{\mathbf{y}}(g)| \leq \gamma \ell_{\mathbf{z}}(|g|^q)^{1/q}$$

and $\ell_{\mathbf{y}}(g) \geq 0$ *for all* $g \in \mathbb{R}[\mathbf{x}]$ *nonnegative on* $\mathbf{X}$.

Invariant Densities in $L_2(\mathbf{X})$

When $p = q = 2$, we rewrite $|\ell_{\mathbf{y}}(g)| \leq \gamma \ell_{\mathbf{z}}(|g|^q)^{1/q}$:

$$\mathbf{C}_2^r(\mathbf{y}) := \begin{pmatrix} \mathbf{M}_r(\mathbf{z}) & \mathbf{y} \\ \mathbf{y}^T & \gamma^2 \end{pmatrix} \succeq 0 \,, \quad \forall r \in \mathbb{N} \,.$$

💡SDP!

Entry (α, β) of $\mathbf{M}_r(\mathbf{z}) = \ell_{\mathbf{z}}(\mathbf{x}^{\alpha+\beta}) = z_{\alpha+\beta}$

Invariant Densities in $L_\infty(\mathbf{X})$

When $p = \infty$ and $q = 1$, we rewrite $|\ell_{\mathbf{y}}(g)| \leq \gamma \ell_{\mathbf{z}}(|g|^q)^{1/q}$:

$$\mathbf{C}_\infty^r(\mathbf{y}) := \gamma \mathbf{M}_r(\mathbf{z}) - \mathbf{M}_r(\mathbf{y}) \succeq 0 \quad \forall r \in \mathbb{N} \, .$$

SDP!

Entry (α, β) of $\mathbf{M}_r(\mathbf{y}) = \ell_{\mathbf{y}}(\mathbf{x}^{\alpha+\beta}) = y_{\alpha+\beta}$

Infinite-dimensional Conic Formulation

$$
\begin{aligned}
\rho_{\mathrm{ac}}^{\star} := \sup_{\mu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \mathcal{L}_f(\mu) = 0\,, \\
& \|\mu\|_p \leq 1\,, \\
& \mu \in L_p(\mathbf{X})_+\,.
\end{aligned} \tag{1}
$$

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Theorem

$\exists!$ *invariant prob. meas.* $\mu_{ac} \in L_p(\mathbf{X}) \implies$ *CONIC* (1) *has the unique optimal solution* $\mu^\star_{ac} := \rho^\star_{ac}\,\mu_{ac}$.

Invariant density $\mu \implies \gamma = 1$ ok!

A Hierarchy of SDP Relaxations

$$
\begin{aligned}
\rho_{\mathrm{ac}}^{r} := \sup_{\mathbf{y}} \quad & y_0 \\
\text{s.t.} \quad & \mathscr{I}_{\mathbf{y}}(\mathbf{x}^{\alpha}) = 0\,, \quad \forall \alpha \in \mathbb{N}^n_{2r}\,, \\
& \mathbf{C}_p^r(\mathbf{y}) \succeq 0\,, \\
& \mathbf{M}_{r-r_j}(g_j\,\mathbf{y}) \succeq 0, \quad j = 0, \ldots, l\,.
\end{aligned}
$$

A Hierarchy of SDP Relaxations

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Theorem

- ***Existence*** *of an optimal solution* $\mathbf{y}^r$.
- ***Convergence:***

$$\lim_{r \to +\infty} y^r_\alpha = y^\star_\alpha, \qquad \lim_{r \to +\infty} \rho^r_{ac} = \rho^\star_{ac},$$

and $\mathbf{y}^\star$ *is the moment sequence of the invariant measure* μ_{ac}.

Approximations of Invariant Densities in $L_2(\mathbf{X})$

Define $h^r \in \mathbb{R}_{2r}[\mathbf{x}]$ with coefficient vector

$$\mathbf{h}^r := \mathbf{M}_r(\mathbf{z})^{-1}\mathbf{y}^r.$$

Theorem ([Henrion et. al])

Strong convergence w.r.t. the L_2-norm:

$$\lim_{r \to \infty} \|h^r - h^\star\|_2 = 0\,.$$

Approximations of Invariant Densities in $L_2(\mathbf{X})$

Define $h^r \in \mathbb{R}_{2r}[\mathbf{x}]$ with coefficient vector

$$\mathbf{h}^r := \mathbf{M}_r(\mathbf{z})^{-1}\mathbf{y}^r.$$

Theorem ([Henrion et. al])

Strong convergence w.r.t. the L_2-norm:

$$\lim_{r\to\infty} \|h^r - h^\star\|_2 = 0\,.$$

Proof.

- $\mathbf{X}$ compact $\implies \mathbb{R}[\mathbf{x}]$ dense in $L_2(\mathbf{X})$
  $\implies \exists(u^r) \subset \mathbb{R}[\mathbf{x}]$ with $\|u^r - h^\star\|_2 \to 0$
- h^r optimal solution for $\min_u \|u - h^\star\|_2$ □

Numerical Experiments I

Extension to Piecewise Polynomial Systems

$$\mathbf{x}_{t+1} = f_i(\mathbf{x}_t) \quad \text{for } \mathbf{x} \in \mathbf{X}_i\,, \quad i \in I\,, \quad t \in \mathbb{N}\,,$$
$$\dot{\mathbf{x}} = f_i(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbf{X}_i\,, \quad i \in I\,, \quad t \in [0, \infty)\,.$$

- Use the piecewise structure of the dynamics f_i.
- Decompose the **global** invariant $\mu = \sum_{i \in I} \mu_i$ with **local** $\mu_i \in \mathcal{M}_+(\mathbf{X}_i)$ invariant w.r.t. f_i

$$\begin{aligned} \sup_{\mu_i} \quad & \sum_{i \in I} \int_{\mathbf{X}_i} \mu_i \\ \text{s.t.} \quad & \mathcal{L}_{f_i}(\mu_i) = 0\,, \quad i \in I\,, \\ & \sum_{i \in I} \|\mu_i\|_p \leq 1\,, \\ & \mu_i \in L_p(\mathbf{X}_{i+})\,, \quad i \in I\,. \end{aligned}$$

Square Integrable Invariant Density

$t^+ = T(t) := t + w \mod 1$

on $\mathbf{T} := [0, 1]$, $w \in \mathbb{R} \backslash \mathbb{Q} \implies \lambda_{\mathbf{T}}$ is invariant

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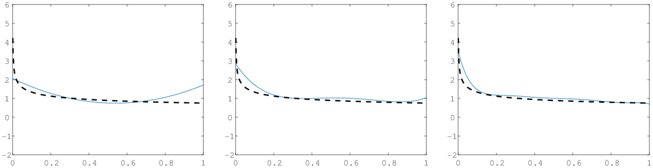
$$x^+ = F^{-1} \circ T \circ F(x)$$

on $\mathbf{X} := [0,1] \implies h^\star$ is invariant



$r = 4$ $r = 6$ $r = 8$

Approximation h_2^r (solid) of exact density $h^\star$ (dashed)

Rotational Flow Map

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{cases} \qquad (x_1, x_2) \in \mathbf{X} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 \leq 1\}.$$

- $\lambda_{\mathbf{X}}$ is invariant.

  Proof. Use Green-Ostrogradski's formula.
- Result: for $r = 2, p = \infty$, we obtain $h_p^r(\mathbf{x}) = 1 = h^\star(\mathbf{x})$.

Piecewise Affine Map

$$x^+ := \begin{cases} 2x & \text{if } x \in \mathbf{X}_1 := [0, \frac{1}{2}] \\ 1 + \frac{3}{2}(\frac{1}{2} - x) & \text{if } x \in \mathbf{X}_2 := [\frac{1}{2}, 1] \end{cases}$$

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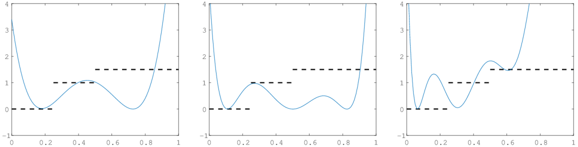
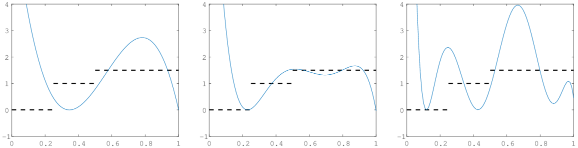


$r = 4, p = 2$ $\quad$ $r = 6, p = 2$ $\quad$ $r = 8, p = 2$



$r = 4, p = \infty$ $\quad$ $r = 6, p = \infty$ $\quad$ $r = 8, p = \infty$

Approximation h_p^r of exact density $h^\star = \mathbf{1}_{[\frac{1}{4}, \frac{1}{2}]} + \frac{3}{2}\mathbf{1}_{[\frac{1}{2}, 1]}$

Singular Invariant Measures

Infinite-Dimensional LP Formulation

We know the **moments** of $\lambda_{\mathbf{X}}$, unknown μ is **invariant** proba.

$$\begin{aligned} \rho^\star_{\text{sing}} = \sup_{\mu,\nu,\hat{\nu},\psi} \quad & \int_{\mathbf{X}} \nu \\ \text{s.t.} \quad & \int_{\mathbf{X}} \mu = 1\,, \quad \mathcal{L}_f(\mu) = 0\,, \\ & \nu + \psi = \mu\,, \quad \nu + \hat{\nu} = \lambda_{\mathbf{X}}\,, \\ & \mu, \nu, \hat{\nu}, \psi \in \mathcal{M}_+(\mathbf{X})\,. \end{aligned} \tag{2}$$

Theorem

$\exists!$ *invariant prob. meas.* $\mu^\star_{\text{sing}} \implies$ *LP* (2) *has the unique optimal solution* $(\mu^\star_{\text{sing}}, 0, \lambda_{\mathbf{X}}, \mu^\star_{\text{sing}})$ *and* $\rho^\star_{\text{sing}} = 0$.

A Hierarchy of SDPs

$$
\begin{aligned}
\rho^r_{\text{sing}} := \sup_{\mathbf{u},\mathbf{v},\hat{\mathbf{v}},\mathbf{y}} \quad & v_0 \\
\text{s.t.} \quad & u_0 = 1\,, \quad \mathscr{I}_{\mathbf{u}}(\mathbf{x}^\alpha) = 0\,, \quad \forall \alpha \in \mathbb{N}^n_{2r}\,, \\
& v_\alpha + y_\alpha = u_\alpha\,, \quad v_\alpha + \hat{v}_\alpha = z_\alpha, \quad \forall \alpha \in \mathbb{N}^n_{2r}\,, \\
& \mathbf{M}_{r-r_j}(g_j\,\mathbf{u})\,, \mathbf{M}_{r-r_j}(g_j\,\mathbf{v}) \succeq 0\,, \quad j = 0, \dots, l\,, \\
& \mathbf{M}_{r-r_j}(g_j\,\hat{\mathbf{v}})\,, \mathbf{M}_{r-r_j}(g_j\,\mathbf{y}) \succeq 0\,, \quad j = 0, \dots, l\,.
\end{aligned}
$$

Theorem

- ***Existence*** *of an optimal solution* $\mathbf{u}^r$.
- ***Convergence:***

$$
\lim_{r \to +\infty} u^r_\alpha = u^\star_\alpha, \qquad \lim_{r \to +\infty} \rho^r_{sing} = \rho^\star_{sing},
$$

and $\mathbf{u}^\star$ *is the moment sequence of the invariant measure* $\mu^\star_{sing}$.

Approximation of Supports

How to **approximate the support** $\mathbf{S}$ of $\mu^\star_{\mathsf{sing}}$ with the SDP solution $\mathbf{u}^r$?

Approximation of Supports

How to **approximate the support S** of $\mu^\star_{\text{sing}}$ with the SDP solution $\mathbf{u}^r$?

Christoffel polynomial $\kappa_r(\mathbf{x}) := \mathbf{m}_r(\mathbf{x})^T \mathbf{M}_r(\mathbf{u})^{-1} \mathbf{m}_r(\mathbf{x})$

with $\mathbf{m}_r(\mathbf{x}) = (1, x_1, \dots, x_n, x_1^2, \dots, x_n^r)$

Approximation of Supports

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$$\mathbf{S}^r := \left\{ \mathbf{x} \in \mathbf{X} : \binom{d_r + n}{n} \geq C_r \kappa_r(\mathbf{x}) \right\}.$$

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Theorem

sup dist convergence: $\lim_{r \to \infty} \sup_{\mathbf{S}^r} \mathrm{dist}(\mathbf{x}, \mathbf{S}) = 0$

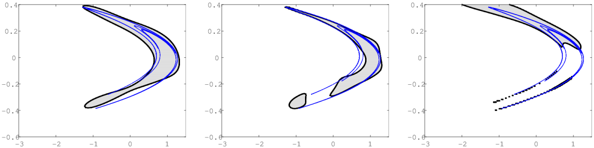
Numerical experiments II

Hénon Map

$$\begin{cases} x_1^+ &= 1 - a x_1^2 + x_2 \\ x_2^+ &= b x_1 \end{cases} \qquad (x_1, x_2) \in \mathbf{X} := [-3, 1.5] \times [-0.6, 0.4].$$

Hénon Map

$$\begin{cases} x_1^+ &= 1 - a x_1^2 + x_2 \\ x_2^+ &= b x_1 \end{cases} \qquad (x_1, x_2) \in \mathbf{X} := [-3, 1.5] \times [-0.6, 0.4].$$



$r = 4$ $r = 6$ $r = 8$

Hénon attractor (blue) and support approximations $\mathbf{S}^r$ (light gray) for $a = 1.4$ and $b = 0.3$.

Van der Pol Oscillator

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a(1 - x_1^2)x_2 - x_1 \end{cases} \qquad \mathbf{X} := [-3, 3] \times [-4, 4].$$

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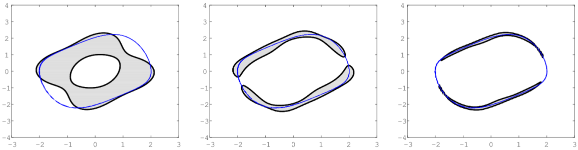

$r = 4$ $\quad r = 6$ $\quad r = 8$

Van der Pol attractor and support approximations $\mathbf{S}^r$ for $a = 0.5$.

Arneodo-Coullet System

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -ax_1 - bx_2 - x_3 + cx_1^3 \end{cases}$$

$$\mathbf{X} := [-4, 4] \times [-8, 8] \times [-12, 12].$$

Arneodo-Coullet System

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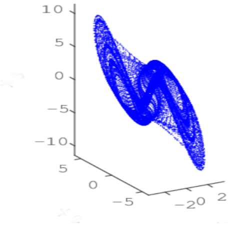
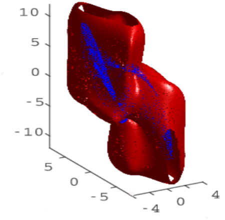


Attractor



$r = 4$

Arneodo-Coullet attractor and support approximations (red) for $a = -5.5$, $b = 3.5$ and $c = -1$.

Conclusion and Perspectives

Conclusion

- 2 methods to characterize invariant measures:
  1. Approximate invariant densities in L_2 and L_∞
     $\Longrightarrow$ **strong convergence**: $\|h^\star - h^r\|_2 \to 0$
  2. Approximate support $\mathbf{S}$ of singular measures with Christoffel
     $\Longrightarrow$ **sup dist convergence**: $\sup_{\mathbf{S}^r} \operatorname{dist}(\mathbf{x}, \mathbf{S}) \to 0$
- $\rightsquigarrow$ Extension to piecewise polynomial systems.

Perspectives

- Exploit sparsity or symmetry for large size systems.
- Improve accuracy of results:
    - Alternative bases: Chebyshev or rational functions
    - Ill-conditioning nature of the moment matrix
- **Open Question:** Extension of Hausdorff convergence of Christoffel [Lasserre-Pauwels 2017] to singular measures?