

Semidefinite Approximations of Projections and Polynomial Images of Semialgebraic Sets

Victor Magron, CNRS VERIMAG

joint work with Didier Henrion and Jean-Bernard Lasserre (LAAS)

CompACS Meeting
18 January 2016

The Problem

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_l(\mathbf{x}) \geq 0\}$
- A polynomial map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$,
 $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$
- $\deg f = d := \max\{\deg f_1, \dots, \deg f_m\}$
- $\mathbf{F} := f(\mathbf{S}) \subseteq \mathbf{B}$, with $\mathbf{B} \subset \mathbb{R}^m$ a box or a ball
- Tractable approximations of \mathbf{F} ?

The Problem

- Includes important special cases:

- 1 $m = 1$: **polynomial optimization**

$$\mathbf{F} \subseteq [\inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}), \sup_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})]$$

- 2 Approximate **projections** of \mathbf{S} when $f(\mathbf{x}) := (x_1, \dots, x_m)$

- 3 **Pareto curve** approximations

For f_1, f_2 two conflicting criteria: $(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) f_2(\mathbf{x}))^\top \right\}$

The Problem

3 **Pareto curve:** set of *weakly Edgeworth-Pareto optimal points*

$$(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) \ f_2(\mathbf{x}))^\top \right\}$$

Definition

A point $\bar{\mathbf{x}} \in \mathbf{S}$ is called a *weakly Edgeworth-Pareto (EP) optimal point* of Problem \mathbf{P} , when there is no $\mathbf{x} \in \mathbf{S}$ such that $f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}})$, $j = 1, 2$.

The Problem

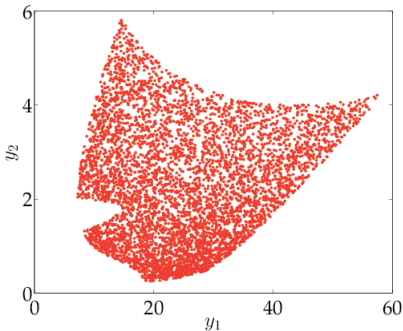
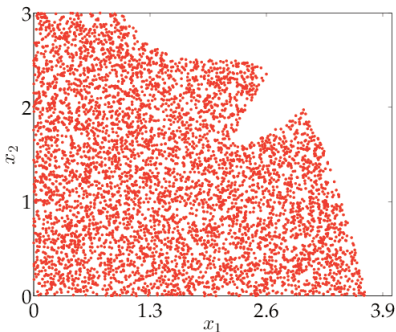
$$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 ,$$

$$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 ,$$

$$\mathbf{S} := \{ \mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geq 0, g_2(\mathbf{x}) \geq 0 \} .$$

$$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 ,$$

$$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 .$$



- 1 Exact description of projections with computer algebra
 - Real quantifier elimination (QE) [Tarski 51, Collins 74, Bochnak-Coste-Roy 98]
 - CAD: computational complexity $(sd)^{2^{O(n)}}$ for a finite set of s polynomials
 - Variant QE under radicality, equidimensionality [Hong-Safey 12]

- 2 Scalarization methods for computing Pareto curve
 - Numerical discretization schemes: modified Polak method [Pol 76]
 - Iterative Eichfelder-Polak algorithm [Eich 09]
 - Normal-boundary intersection method to find uniform spread of points [Das Dennis 98]

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications
- **No discretization** is required

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications
- **No discretization** is required
- Two different methods:
 - 1 Existential QE: $\mathbf{F} \subseteq \mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\}$
 - 2 Image measure supports: $\mathbf{F} \subseteq \mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geq 1\}$

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications
- **No discretization** is required
- Two different methods:
 - 1 Existential QE: $\mathbf{F} \subseteq \mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\}$
 - 2 Image measure supports: $\mathbf{F} \subseteq \mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geq 1\}$
- Strong convergence guarantees

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications
- **No discretization** is required
- Two different methods:
 - 1 Existential QE: $\mathbf{F} \subseteq \mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\}$
 - 2 Image measure supports: $\mathbf{F} \subseteq \mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geq 1\}$
- Strong convergence guarantees
- Compute q_k or w_k with **Semidefinite programming (SDP)**

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Polynomial Optimization

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_l(\mathbf{x}) \geq 0\}$
- $p^* := \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})$: NP hard
- Sums of squares $\Sigma[\mathbf{x}]$
e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$
- $\mathcal{Q}(\mathbf{S}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^l \sigma_j(\mathbf{x})g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$
- **REMEMBER:** $f \in \mathcal{Q}(\mathbf{S}) \implies \forall \mathbf{x} \in \mathbf{S}, f(\mathbf{x}) \geq 0$

Problem reformulation

- Borel σ -algebra \mathcal{B} (generated by the open sets of \mathbb{R}^n)
- $\mathcal{M}_+(\mathbf{S})$: set of probability measures supported on \mathbf{S} .
If $\mu \in \mathcal{M}_+(\mathbf{S})$ then
 - 1 $\mu : \mathcal{B} \rightarrow [0, 1], \mu(\emptyset) = 0$
 - 2 $\mu(\cup_i B_i) = \sum_i \mu(B_i)$, for any countable $(B_i) \subset \mathcal{B}$
 - 3 $\int_{\mathbf{S}} \mu(dx) = 1$
- $\text{supp}(\mu)$ is the smallest set \mathbf{S} such that $\mu(\mathbb{R}^n \setminus \mathbf{S}) = 0$

Problem reformulation

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \int_{\mathbf{S}} f d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence \mathbf{z} has a representing measure on \mathbf{S} if there exists a finite measure μ supported on \mathbf{S} such that

$$\mathbf{z}_\alpha = \int_{\mathbf{S}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n.$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{S})$: space of probability measures supported on \mathbf{S}
- $\mathcal{Q}(\mathbf{S})$: quadratic module

Polynomial Optimization Problems (POP)

$$\begin{array}{ll} \text{(Primal)} & \text{(Dual)} \\ \inf \int_{\mathbf{S}} f d\mu & = \sup \lambda \\ \text{s.t. } \mu \in \mathcal{M}_+(\mathbf{S}) & \text{s.t. } \lambda \in \mathbb{R}, \\ & f - \lambda \in \mathcal{Q}(\mathbf{S}) \end{array}$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences \mathbf{z} of measures in $\mathcal{M}_+(\mathbf{S})$
- Truncated quadratic module $\mathcal{Q}_k(\mathbf{S}) := \mathcal{Q}(\mathbf{S}) \cap \mathbb{R}_{2k}[\mathbf{x}]$

Polynomial Optimization Problems (POP)

(Moment)		(SOS)
$\inf \sum_{\alpha} f_{\alpha} \mathbf{z}_{\alpha}$	=	$\sup \lambda$
s.t. $\mathbf{M}_{k-v_j}(g_j \mathbf{z}) \succcurlyeq 0, \quad 0 \leq j \leq l,$		s.t. $\lambda \in \mathbb{R},$
$\mathbf{z}_1 = 1$		$f - \lambda \in \mathcal{Q}_k(\mathbf{S})$

Lasserre's Hierarchy of SDP relaxations

$$\ell_{\mathbf{z}}(q) : q \in \mathbb{R}[\mathbf{x}] \mapsto \sum_{\alpha} q_{\alpha} \mathbf{z}_{\alpha}$$

- Moment matrix

$$\mathbf{M}(\mathbf{z})_{\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}} := \ell_{\mathbf{z}}(\mathbf{x}^{\alpha} \mathbf{x}^{\beta}) = \mathbf{z}_{\alpha+\beta}$$

- Localizing matrix $\mathbf{M}(\mathbf{g}_j \mathbf{z})$ associated with \mathbf{g}_j

$$\mathbf{M}(\mathbf{g}_j \mathbf{z})_{\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}} := \ell_{\mathbf{z}}(\mathbf{g}_j \mathbf{x}^{\alpha} \mathbf{x}^{\beta}) = \sum_{\gamma} \mathbf{g}_{j, \gamma} \mathbf{z}_{\alpha+\beta+\gamma}$$

Lasserre's Hierarchy of SDP relaxations

■ $\mathbf{M}_k(\mathbf{z})$ contains $\binom{n+2k}{n}$ variables, has size $\binom{n+k}{n}$

■ Truncated matrix of order $k = 2$ with variables x_1, x_2 :

$$\mathbf{M}_2(\mathbf{z}) = \begin{array}{c} 1 \\ - \\ x_1 \\ x_2 \\ - \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \begin{pmatrix} 1 & | & x_1 & x_2 & | & x_1^2 & x_1x_2 & x_2^2 \\ 1 & | & z_{1,0} & z_{0,1} & | & z_{2,0} & z_{1,1} & z_{0,2} \\ - & - & - & - & - & - & - & - \\ z_{1,0} & | & z_{2,0} & z_{1,1} & | & z_{3,0} & z_{2,1} & z_{1,2} \\ z_{0,1} & | & z_{1,1} & z_{0,2} & | & z_{2,1} & z_{1,2} & z_{0,3} \\ - & - & - & - & - & - & - & - \\ z_{2,0} & | & z_{3,0} & z_{2,1} & | & z_{4,0} & z_{3,1} & z_{2,2} \\ z_{1,1} & | & z_{2,1} & z_{1,2} & | & z_{3,1} & z_{2,2} & z_{1,3} \\ z_{0,2} & | & z_{1,2} & z_{0,3} & | & z_{2,2} & z_{1,3} & z_{0,4} \end{pmatrix}$$

Lasserre's Hierarchy of SDP relaxations

- Consider $g_1(\mathbf{x}) := 2 - x_1^2 - x_2^2$. Then $v_1 = \lceil \deg g_1 / 2 \rceil = 1$.

$$\mathbf{M}_1(g_1 \mathbf{z}) = \begin{matrix} & \begin{matrix} 1 & x_1 & x_2 \end{matrix} \\ \begin{matrix} 1 \\ x_1 \\ x_2 \end{matrix} & \begin{pmatrix} 2 - z_{2,0} - z_{0,2} & 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{0,1} - z_{2,1} - z_{0,3} \\ 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{2,0} - z_{4,0} - z_{2,2} & 2z_{1,1} - z_{3,1} - z_{1,3} \\ 2z_{0,1} - z_{2,1} - z_{0,3} & 2z_{1,1} - z_{3,1} - z_{1,3} & 2z_{0,2} - z_{2,2} - z_{0,4} \end{pmatrix} \end{matrix}$$

$$\begin{aligned} \mathbf{M}_1(g_1 \mathbf{z})(3,3) &= \ell(g_1(\mathbf{x}) \cdot x_2 \cdot x_2) = \ell(2x_2^2 - x_1^2x_2^2 - x_2^4) \\ &= 2z_{0,2} - z_{2,2} - z_{0,4} \end{aligned}$$

Lasserre's Hierarchy of SDP relaxations

- Truncation with moments of order at most $2k$
- $v_j := \lceil \deg g_j / 2 \rceil$
- Hierarchy of semidefinite relaxations:

$$\left\{ \begin{array}{l} \inf_{\mathbf{z}} \ell_{\mathbf{z}}(f) = \sum_{\alpha} \int_{\mathbf{S}} f_{\alpha} \mathbf{x}^{\alpha} \mu(d\mathbf{x}) = \sum_{\alpha} f_{\alpha} \mathbf{z}_{\alpha} \\ \mathbf{M}_k(\mathbf{z}) \succeq 0, \\ \mathbf{M}_{k-v_j}(g_j \mathbf{z}) \succeq 0, \quad 1 \leq j \leq l, \\ \mathbf{z}_1 = 1. \end{array} \right.$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$(SDP) \begin{cases} \mathcal{P} : & \inf_{\mathbf{z}} \quad \sum_{\alpha} c_{\alpha} \mathbf{z}_{\alpha} \\ & \text{s.t.} \quad \sum_{\alpha} F_{\alpha} \mathbf{z}_{\alpha} - F_0 \succcurlyeq 0 \\ \\ \mathcal{D} : & \sup_{\mathbf{Y}} \quad \text{Trace} (F_0 \mathbf{Y}) \\ & \text{s.t.} \quad \text{Trace} (F_{\alpha} \mathbf{Y}) = c_{\alpha} , \quad \mathbf{Y} \succcurlyeq 0 . \end{cases}$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Approximation of sets defined with “ \exists ”

Another point of view:

$$\mathbf{F} = \{ \mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } f(\mathbf{x}) = \mathbf{y} \} ,$$

Approximation of sets defined with “ \exists ”

Another point of view:

$$\mathbf{F} = \{ \mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } \|\mathbf{y} - f(\mathbf{x})\|_2^2 = 0 \} ,$$

Approximation of sets defined with “ \exists ”

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } h_f(\mathbf{x}, \mathbf{y}) \geq 0\} ,$$

with

$$h_f(\mathbf{x}, \mathbf{y}) := -\|\mathbf{y} - f(\mathbf{x})\|_2^2 .$$

Approximation of sets defined with “ \exists ”

Existential QE: approximate \mathbf{F} as closely as desired [Lasserre 14]

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\} ,$$

for some polynomials $q_k \in \mathbb{R}_{2k}[\mathbf{y}]$.

A hierarchy of outer approximations of F

■ Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k -truncated quadratic module

■ **REMEMBER:**

$$q - h_f \in \mathcal{Q}_k(\mathbf{K}) \implies \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geq 0$$

A hierarchy of outer approximations of F

- Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k -truncated quadratic module
- **REMEMBER:**
 $q - h_f \in \mathcal{Q}_k(\mathbf{K}) \implies \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geq 0$
- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

A hierarchy of outer approximations of F

- Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k -truncated quadratic module
- **REMEMBER:**
 $q - h_f \in \mathcal{Q}_k(\mathbf{K}) \implies \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geq 0$
- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$
- Hierarchy of Semidefinite programs:

$$\inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K}) \right\} .$$

A hierarchy of outer approximations of F

Assuming the existence of solution q_k , the sublevel sets

$$F_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\} \supseteq F ,$$

provide a sequence of certified outer approximations of F .

A hierarchy of outer approximations of \mathbf{F}

Assuming the existence of solution q_k , the sublevel sets

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\} \supseteq \mathbf{F} ,$$

provide a sequence of certified outer approximations of \mathbf{F} .

It comes from the following:

- q_k feasible solution, $q_k - h_f \in \mathcal{Q}_k(\mathbf{K})$
- $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q_k(\mathbf{y}) \geq h_f(\mathbf{x}, \mathbf{y}) \iff \forall \mathbf{y}, q_k(\mathbf{y}) \geq h(\mathbf{y}) .$

Strong convergence property

Theorem

Assuming that $\overset{\circ}{\mathbf{S}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{K})$ is Archimedean,

- 1 The sequence of optimal solutions (q_k) converges to h w.r.t the $L_1(\mathbf{B})$ -norm:

$$\lim_{k \rightarrow \infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0, (q_k \rightarrow_{L_1} h)$$

Strong convergence property

Theorem

Assuming that $\overset{\circ}{\mathbf{S}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{K})$ is Archimedean,

- 1 The sequence of optimal solutions (q_k) converges to h w.r.t the $L_1(\mathbf{B})$ -norm:

$$\lim_{k \rightarrow \infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0, (q_k \rightarrow_{L_1} h)$$

- 2

$$\lim_{k \rightarrow \infty} \text{vol}(\mathbf{F}_k^1 \setminus \mathbf{F}) = 0.$$

Strong convergence property

Proof of existence

- 1 Existence of optimal q_k by **Slater's condition**

Strong convergence property

Proof of existence

1 Existence of optimal q_k by Slater's condition

■ Dual SDP:

$$\begin{aligned} \rho_k^* &:= \sup_{\mathbf{z}} \ell_{\mathbf{z}}(h_f) \\ \text{s.t. } & \mathbf{M}_k(\mathbf{z}) \succcurlyeq 0, \\ & \mathbf{M}_{k-v_j}(g_j; \mathbf{z}) \succcurlyeq 0, \quad j = 1, \dots, l, \\ & \ell_{\mathbf{z}}(\mathbf{y}^\beta) = \mathbf{z}_\beta^{\mathbf{B}}, \quad \forall \beta \in \mathbb{N}_{2k}^m. \end{aligned}$$

Strong convergence property

Proof of existence

1 Existence of optimal q_k by Slater's condition

■ Dual SDP:

$$\begin{aligned} \rho_k^* &:= \sup_{\mathbf{z}} \ell_{\mathbf{z}}(h_f) \\ \text{s.t. } & \mathbf{M}_k(\mathbf{z}) \succcurlyeq 0, \\ & \mathbf{M}_{k-v_j}(g_j \mathbf{z}) \succcurlyeq 0, \quad j = 1, \dots, l, \\ & \ell_{\mathbf{z}}(\mathbf{y}^\beta) = z_\beta, \quad \forall \beta \in \mathbb{N}_{2k}^m. \end{aligned}$$

■ Strictly feasible \mathbf{z} : moments of Lebesgue measure $\lambda_{\mathbf{K}}$

Strong convergence property

Proof of existence

1 Existence of optimal q_k by Slater's condition

- Dual SDP:

$$\begin{aligned} \rho_k^* &:= \sup_{\mathbf{z}} \ell_{\mathbf{z}}(h_f) \\ \text{s.t. } & \mathbf{M}_k(\mathbf{z}) \succcurlyeq 0, \\ & \mathbf{M}_{k-v_j}(g_j \mathbf{z}) \succcurlyeq 0, \quad j = 1, \dots, l, \\ & \ell_{\mathbf{z}}(\mathbf{y}^\beta) = z_\beta^{\mathbf{B}}, \quad \forall \beta \in \mathbb{N}_{2k}^m. \end{aligned}$$

- Strictly feasible \mathbf{z} : moments of Lebesgue measure $\lambda_{\mathbf{K}}$
- $q = 0$ feasible for Primal SDP:

$$\rho_k := \inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K}) \right\}.$$

Strong convergence property

Proof of convergence

- 1 Approximate h with polynomials:

Strong convergence property

Proof of convergence

1 Approximate h with polynomials:

- h lower semi-continuous, existence of $(f_k) \subset \mathcal{C}(\mathbf{B})$ s.t. $f_k \downarrow h$

Strong convergence property

Proof of convergence

1 Approximate h with polynomials:

- h lower semi-continuous, existence of $(f_k) \subset \mathcal{C}(\mathbf{B})$ s.t. $f_k \downarrow h$
- By Monotone Convergence Theorem, $f_k \rightarrow_{L_1} h$.

Strong convergence property

Proof of convergence

1 Approximate h with polynomials:

- h lower semi-continuous, existence of $(f_k) \subset \mathcal{C}(\mathbf{B})$ s.t. $f_k \downarrow h$
- By Monotone Convergence Theorem, $f_k \rightarrow_{L_1} h$.
- By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \rightarrow_{L_1} h$

Strong convergence property

Proof of convergence

1 Approximate h with polynomials:

- h lower semi-continuous, existence of $(f_k) \subset \mathcal{C}(\mathbf{B})$ s.t. $f_k \downarrow h$
- By Monotone Convergence Theorem, $f_k \rightarrow_{L_1} h$.
- By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \rightarrow_{L_1} h$
- Apply Putinar's Positivstellensatz to $p_k - h_f + \epsilon / \text{vol}(\mathbf{B})$:

$$p_k - h_f + \epsilon / \text{vol}(\mathbf{B}) = \sum_{j=0}^l \sigma_j g_j$$

Strong convergence property

Proof of volume convergence

2 Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geq -1/r\}$

Strong convergence property

Proof of volume convergence

2 Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geq -1/r\}$

■ $\text{vol } \mathbf{F}(r) \rightarrow \text{vol } \mathbf{F}$

Strong convergence property

Proof of volume convergence

2 Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geq -1/r\}$

- $\text{vol } \mathbf{F}(r) \rightarrow \text{vol } \mathbf{F}$
- $\lim_{k \rightarrow \infty} \text{vol } \mathbf{F}_k^1 \leq \text{vol } \mathbf{F}(r)$

Strong convergence property

Proof of volume convergence

2 Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geq -1/r\}$

- $\text{vol } \mathbf{F}(r) \rightarrow \text{vol } \mathbf{F}$
- $\lim_{k \rightarrow \infty} \text{vol } \mathbf{F}_k^1 \leq \text{vol } \mathbf{F}(r)$
- $\text{vol } \mathbf{F} \leq \lim_{k \rightarrow \infty} \text{vol } \mathbf{F}_k^1 \leq \text{vol } \mathbf{F}(r)$

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Infinite dimensional LP formulation

- **Pushforward** $f_{\#} : \mathcal{M}(\mathbf{S}) \rightarrow \mathcal{M}(\mathbf{B})$:

$$f_{\#}\mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{S} : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{B}), \forall \mu_0 \in \mathcal{M}(\mathbf{S})$$

- $f_{\#}\mu_0$ is the **image measure** of μ_0 under f

Infinite dimensional LP formulation

$$p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int_{\mathbf{B}} \mu_1$$

s.t. $\mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}},$
 $\mu_1 = f_{\#}\mu_0,$
 $\mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}).$

Lebesgue measure on \mathbf{B} is $\lambda_{\mathbf{B}}(d\mathbf{y}) := \mathbf{1}_{\mathbf{B}}(\mathbf{y}) d\mathbf{y}$

Infinite dimensional LP formulation

$$p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int_{\mathbf{B}} \mu_1$$

s.t. $\mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}},$
 $\mu_1 = f_{\#} \mu_0,$
 $\mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}).$

Lemma

Let μ_1^* be an optimal solution of the above LP.

Then $\mu_1^* = \lambda_{\mathbf{F}}$ and $p^* = \text{vol } \mathbf{F}.$

LP Primal-dual conic formulation

The LP can be cast as follows:

$$\begin{aligned} p^* &= \sup_x \langle x, c \rangle_1 \\ \text{s.t. } & \mathcal{A}x = b, \\ & x \in E_1^+, \end{aligned}$$

LP Primal-dual conic formulation

The LP can be cast as follows:

$$\begin{aligned} p^* &= \sup_x \langle x, c \rangle_1 \\ \text{s.t. } & \mathcal{A}x = b, \\ & x \in E_1^+, \end{aligned}$$

with

- $E_1 := \mathcal{M}(\mathbf{S}) \times \mathcal{M}(\mathbf{B})^2 \quad F_1 := \mathcal{C}(\mathbf{S}) \times \mathcal{C}(\mathbf{B})^2$
- $x := (\mu_0, \mu_1, \hat{\mu}_1) \quad c := (0, 1, 0) \in F_1 \quad b := (0, \lambda_{\mathbf{B}})$
- the linear operator $\mathcal{A} : E_1 \rightarrow E_2$ given by

$$\mathcal{A}(\mu_0, \mu_1, \hat{\mu}_1) := \begin{bmatrix} -f_{\#}\mu_0 + \mu_1 \\ \mu_1 + \hat{\mu}_1 \end{bmatrix}.$$

LP Primal-dual conic formulation

Primal LP

$$\begin{aligned} p^* &= \sup_x \langle x, c \rangle_1 \\ \text{s.t. } \mathcal{A}x &= b, \\ x &\in E_1^+. \end{aligned}$$

Dual LP

$$\begin{aligned} d^* &= \inf_y \langle b, y \rangle_2 \\ \text{s.t. } \mathcal{A}'y - c &\in \mathcal{C}_+(\mathbf{B})^2. \end{aligned}$$

with

$$\blacksquare y := (v, w) \in \mathcal{M}(\mathbf{B})^2$$

$$\blacksquare \mathcal{A}'(v, w) := \begin{bmatrix} -v \circ f \\ v + w \\ w \end{bmatrix}.$$

LP Primal-dual conic formulation

Primal LP

$$\begin{aligned} p^* &:= \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int \mu_1 \\ \text{s.t. } \mu_1 + \hat{\mu}_1 &= \lambda_{\mathbf{B}}, \\ \mu_1 &= f_{\#} \mu_0, \\ \mu_0 &\in \mathcal{M}_+(\mathbf{S}), \\ \mu_1, \hat{\mu}_1 &\in \mathcal{M}_+(\mathbf{B}). \end{aligned}$$

Dual LP

$$\begin{aligned} d^* &:= \inf_{v, w} \int w(\mathbf{y}) \lambda_{\mathbf{B}}(d\mathbf{y}) \\ \text{s.t. } v(f(\mathbf{x})) &\geq 0, \quad \forall \mathbf{x} \in \mathbf{S}, \\ w(\mathbf{y}) &\geq 1 + v(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbf{B}, \\ w(\mathbf{y}) &\geq 0, \quad \forall \mathbf{y} \in \mathbf{B}, \\ v, w &\in \mathcal{C}(\mathbf{B}). \end{aligned}$$

Zero duality gap

Lemma

$$p^* = d^*$$

Strong convergence property

Strengthening of the dual LP:

$$\begin{aligned} d_k^* &:= \inf_{v, w} \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}} \\ \text{s.t. } & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\ & w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), \\ & w \in \mathcal{Q}_k(\mathbf{B}), \\ & v, w \in \mathbb{R}_{2k}[\mathbf{y}]. \end{aligned}$$

Strong convergence property

Theorem

Assuming that $\mathring{\mathbf{F}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

- 1 The sequence (w_k) converges to $\mathbf{1}_{\mathbf{F}}$ w.r.t the $L_1(\mathbf{B})$ -norm:

$$\lim_{k \rightarrow \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 .$$

Strong convergence property

Theorem

Assuming that $\overset{\circ}{\mathbf{F}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

- 1 The sequence (w_k) converges to $\mathbf{1}_{\mathbf{F}}$ w.r.t the $L_1(\mathbf{B})$ -norm:

$$\lim_{k \rightarrow \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 .$$

- 2 Let $\mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geq 1\}$. Then,

$$\lim_{k \rightarrow \infty} \text{vol}(\mathbf{F}_k^2 \setminus \mathbf{F}) = 0 .$$

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

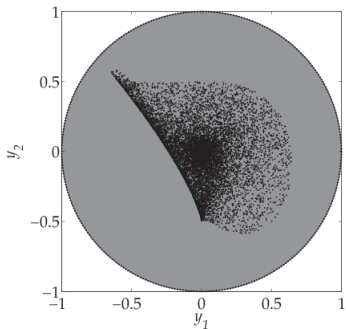
Application examples

Conclusion

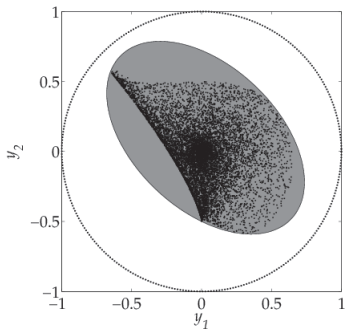
Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1x_2, x_2 - x_1^3)/2$$



F_1^1

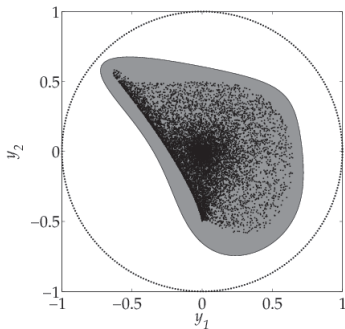


F_1^2

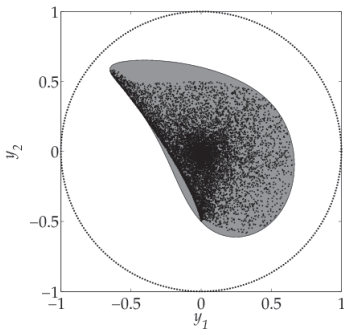
Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1x_2, x_2 - x_1^3)/2$$



F_2^1

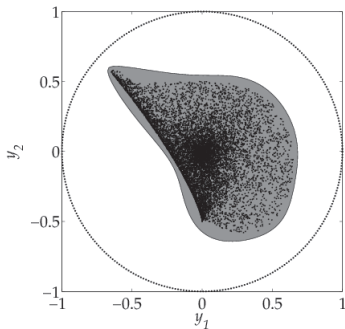


F_2^2

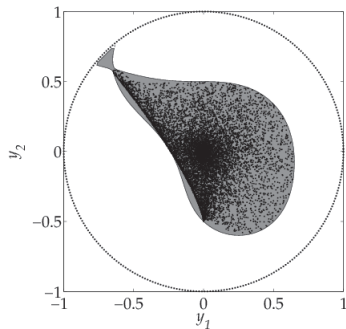
Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1x_2, x_2 - x_1^3)/2$$



F_3^1

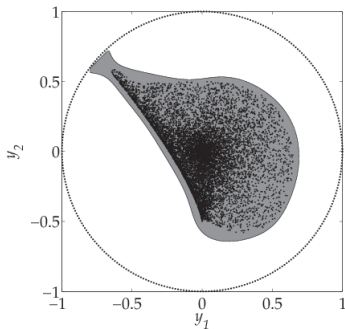


F_3^2

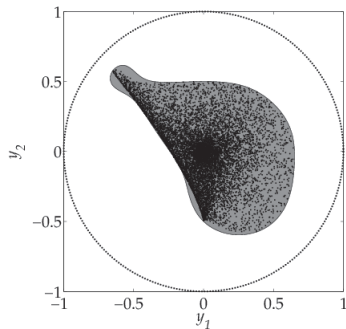
Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1x_2, x_2 - x_1^3)/2$$



F_4^1



F_4^2

Semialgebraic set projections

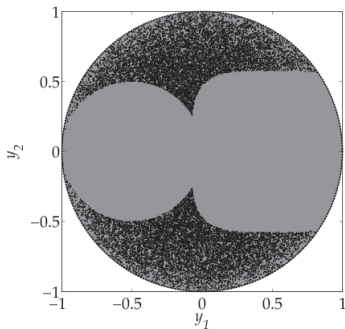
Simpler formulation:

$$\begin{aligned} d_k^* &:= \inf_{v, w} \sum_{\beta \in \mathbb{N}_{2k}^m} w_{\beta} z_{\beta}^{\mathbf{B}} & \inf_w \sum_{\beta \in \mathbb{N}_{2k}^m} w_{\beta} z_{\beta}^{\mathbf{B}} \\ \text{s.t. } & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), & \text{s.t. } w - 1 \in \mathcal{Q}_k(\mathbf{S}), \\ & w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), & w \in \mathcal{Q}_k(\mathbf{B}), \\ & w \in \mathcal{Q}_k(\mathbf{B}), & w \in \mathbb{R}_{2k}[x_1, \dots, x_m]. \\ & v, w \in \mathbb{R}_{2k}[\mathbf{y}]. & \end{aligned}$$

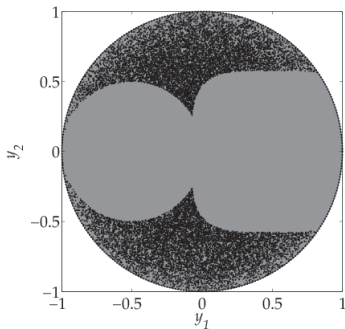
Semialgebraic set projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on \mathbb{R}^2 of the semialgebraic set

$$\mathbf{S} := \{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leq 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geq 0, \\ 1/9 - (x_1 - 1/2)^4 - x_2^4 \geq 0 \}$$



F_2^1

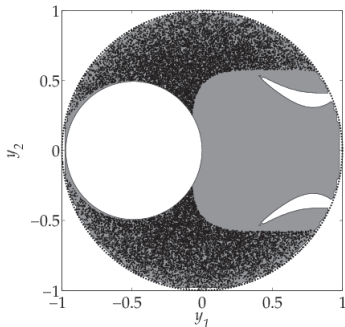


F_2^2

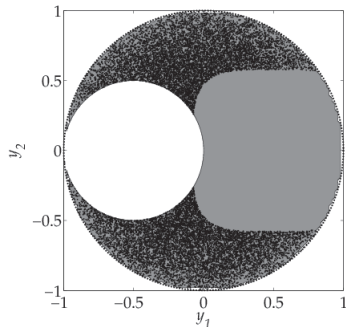
Semialgebraic set projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on \mathbb{R}^2 of the semialgebraic set

$$\mathbf{S} := \{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leq 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geq 0, \\ 1/9 - (x_1 - 1/2)^4 - x_2^4 \geq 0 \}$$



F_3^1

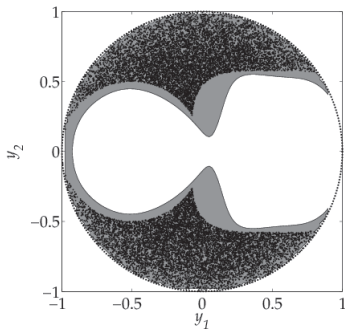


F_3^2

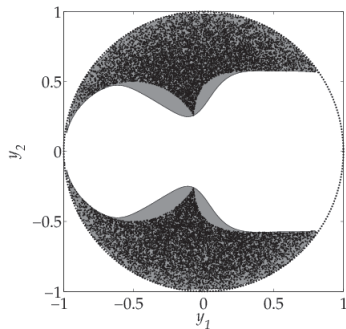
Semialgebraic set projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on \mathbb{R}^2 of the semialgebraic set

$$\mathbf{S} := \{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leq 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geq 0, \\ 1/9 - (x_1 - 1/2)^4 - x_2^4 \geq 0 \}$$



F_4^1



F_4^2

Bicriteria Optimization Problems

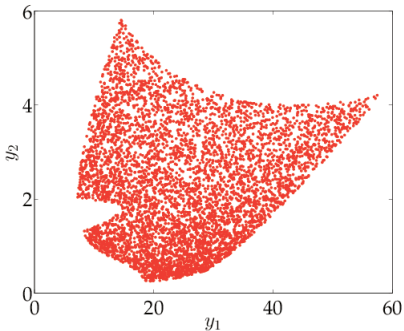
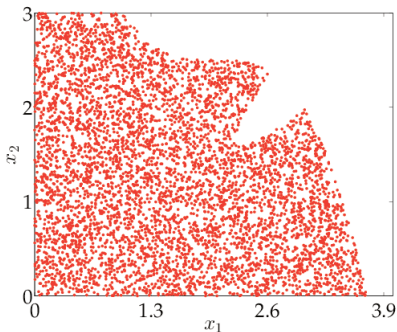
$$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 ,$$

$$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 ,$$

$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geq 0, g_2(\mathbf{x}) \geq 0\} .$$

$$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 ,$$

$$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 .$$



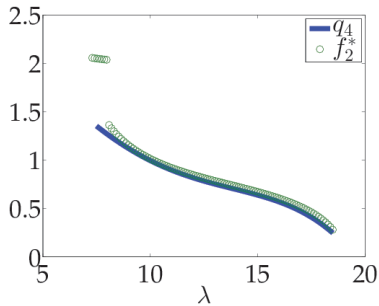
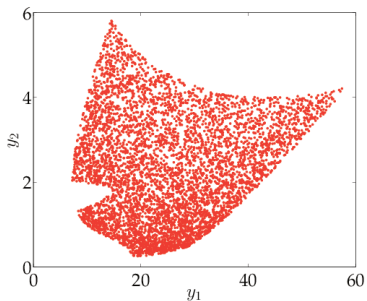
Previous Contributions

- Numerical schemes that **avoid computing finitely many points**.
- Pareto curve approximation with polynomials, **convergence guarantees** in L_1 -norm



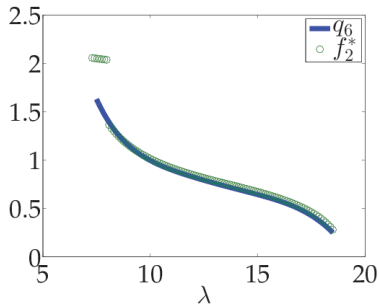
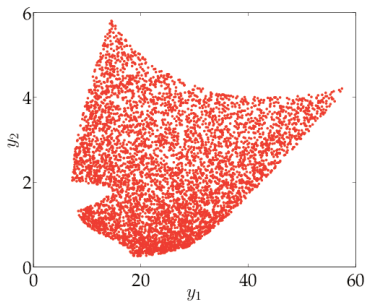
V. Magron, D. Henrion, J.B. Lasserre. Approximating Pareto Curves using Semidefinite Relaxations. *Operations Research Letters*. arxiv:1404.4772, April 2014.

Previous Contributions



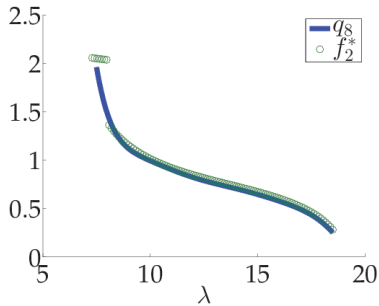
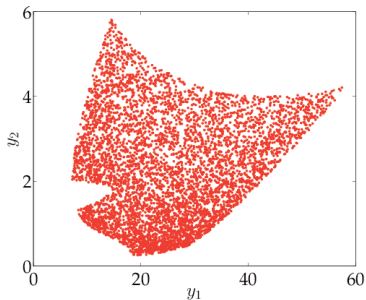
Degree 4

Previous Contributions



Degree 6

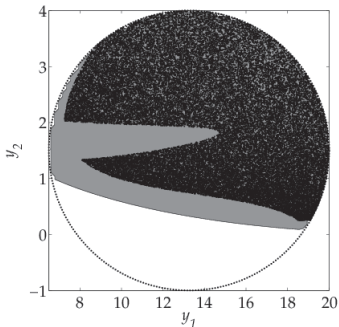
Previous Contributions



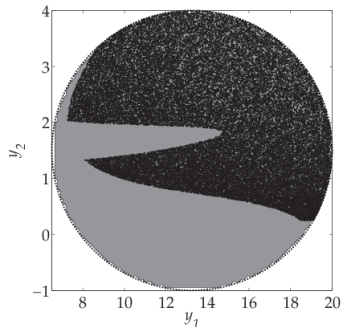
Degree 8

Approximating Pareto curves

Back on our previous nonconvex example:



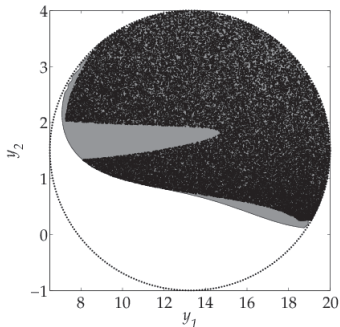
F_1^1



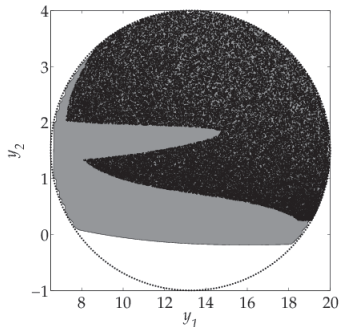
F_1^2

Approximating Pareto curves

Back on our previous nonconvex example:



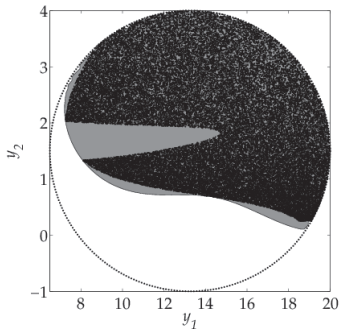
F_2^1



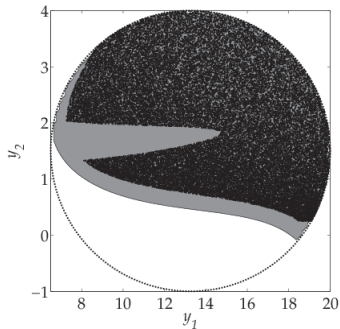
F_2^2

Approximating Pareto curves

Back on our previous nonconvex example:



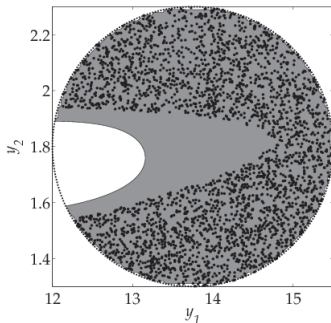
F_3^1



F_3^2

Approximating Pareto curves

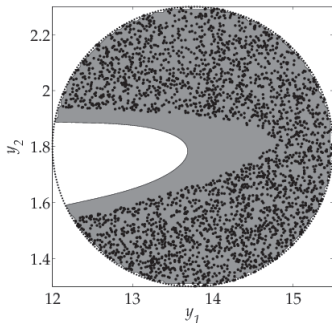
“Zoom” on the region which is hard to approximate:



F_4^1

Approximating Pareto curves

“Zoom” on the region which is hard to approximate:

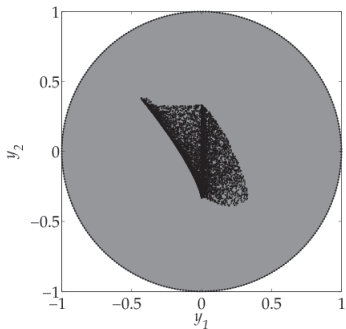


F_5^1

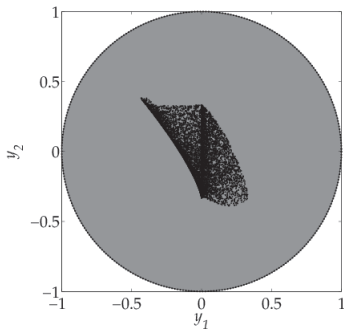
Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1x_2, x_1^2), x_2 - x_1^3)/3$$



\mathbf{F}_1^1

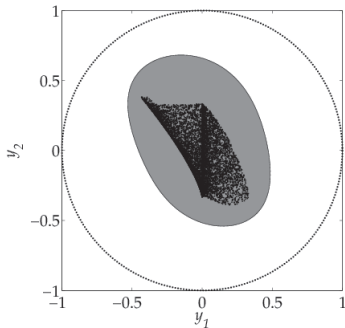


\mathbf{F}_1^2

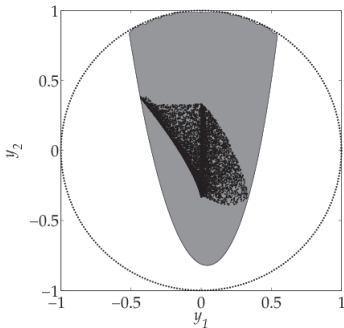
Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1x_2, x_1^2), x_2 - x_1^3)/3$$



F_2^1

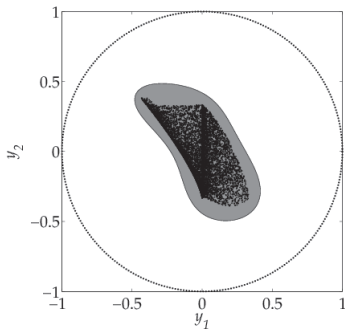


F_2^2

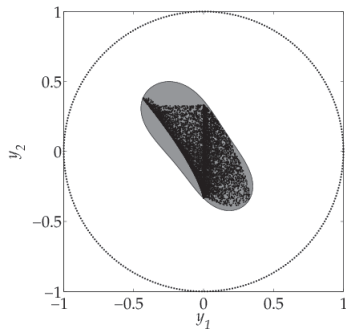
Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1x_2, x_1^2), x_2 - x_1^3)/3$$



F_3^1

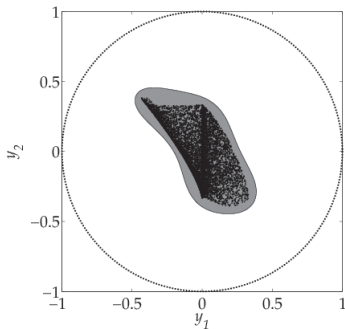


F_3^2

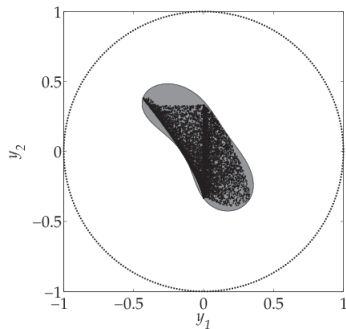
Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1x_2, x_1^2), x_2 - x_1^3)/3$$



F_4^1



F_4^2

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Conclusion

- **Unifying** framework: projections, Pareto curves
- Computational complexity
 - 1 Method 1: $\binom{n+m+2k}{2k}$ SDP variables
 - 2 Method 2: $\binom{n+2kd}{2kd}$ SDP variables
- **Structure sparsity** can be exploited

Conclusion

Further research:

- Alternative positivity certificates LP/SDP
 - 1 Less computationally demanding than SDP
 - 2 More efficient than LP (as generic convergence cannot occur)

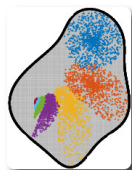
Conclusion

Further research:

Discrete-time polynomial systems: $\mathbf{x}_{k+1} = f(\mathbf{x}_k)$

💡 **certified/convergent** SDP hierarchies

💡 image measure supports 💡 conservation eqs.



End



V. Magron, D. Henrion, J.B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. *SIAM J. Optimization*, 2015.

Thank you for your attention!

<http://www-verimag.imag.fr/~magron>