Semidefinite Approximations of Projections and Polynomial Images of Semialgebraic Sets

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The Problem

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

- A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$,
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$

- $\deg f = d := \max\{\deg f_1, \ldots, \deg f_m\}$

- $\mathbf{F} := f(\mathbf{S}) \subseteq \mathbf{B}$, with $\mathbf{B} \subset \mathbb{R}^m$ a box or a ball

- Tractable approximations of $\mathbf{F}$?

The Problem

- Includes important special cases:

  1. $m = 1$: **polynomial optimization**

  $$\mathbf{F} \subseteq [\inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}), \sup_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})]$$

  2. Approximate **projections** of $\mathbf{S}$ when $f(\mathbf{x}) := (x_1, \ldots, x_m)$

  3. **Pareto curve** approximations
     For f_1, f_2 two conflicting criteria: $(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) \, f_2(\mathbf{x}))^\top \right\}$

The Problem

3 **Pareto curve**: set of *weakly Edgeworth-Pareto optimal points*

$$(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) \, f_2(\mathbf{x}))^\top \right\}$$

Definition

A point $\bar{\mathbf{x}} \in \mathbf{S}$ is called a *weakly Edgeworth-Pareto (EP) optimal point* of Problem **P**, when there is no $\mathbf{x} \in \mathbf{S}$ such that $f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}})$, $j = 1, 2$.
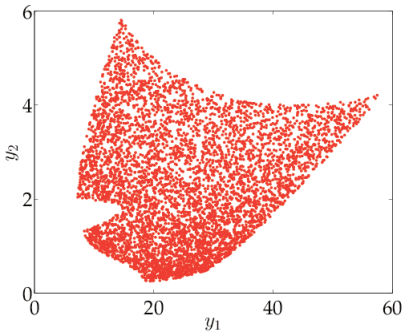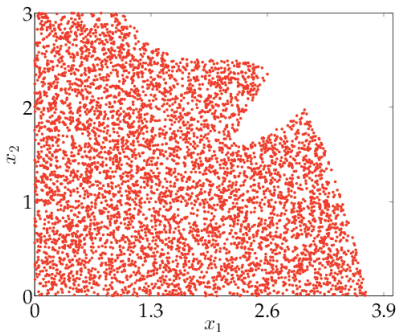
The Problem

$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5$,

$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85$,

$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\}$.

$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2$,

$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4$.

Previous work

1. Exact description of projections with computer algebra

   - Real quantifier elimination (QE) [Tarski 51, Collins 74, Bochnak-Coste-Roy 98]

   - CAD: computational complexity $(sd)^{2^{O(n)}}$ for a finite set of s polynomials

   - Variant QE under radicality, equidimensionality [Hong-Safey 12]

Previous work

2 Scalarization methods for computing Pareto curve

- Numerical discretization schemes: modified Polak method [Pol 76]

- Iterative Eichfelder-Polak algorithm [Eich 09]

- Normal-boundary intersection method to find uniform spread of points [Das Dennis 98]

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications

- **No discretization** is required

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- Two different methods:
  1. Existential QE: $\mathbf{F} \subseteq \mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\}$

  2. Image measure supports: $\mathbf{F} \subseteq \mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$

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- Strong convergence guarantees

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- Strong convergence guarantees

- Compute q_k or w_k with **Semidefinite programming** (SDP)

Polynomial Optimization

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

- $p^* := \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})$: NP hard

- Sums of squares $\Sigma[\mathbf{x}]$
  e.g. $x_1^2 - 2x_1 x_2 + x_2^2 = (x_1 - x_2)^2$

- $\mathcal{Q}(\mathbf{S}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^l \sigma_j(\mathbf{x}) g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$

- **REMEMBER**: $f \in \mathcal{Q}(\mathbf{S}) \implies \forall \mathbf{x} \in \mathbf{S}, f(\mathbf{x}) \geqslant 0$

Problem reformulation

- Borel σ-algebra $\mathcal{B}$ (generated by the open sets of $\mathbb{R}^n$)

- $\mathcal{M}_+(\mathbf{S})$: set of probability measures supported on $\mathbf{S}$.
  If $\mu \in \mathcal{M}_+(\mathbf{S})$ then
  1. $\mu : \mathcal{B} \to [0,1]$, $\mu(\varnothing) = 0$

  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any countable $(B_i) \subset \mathcal{B}$

  3. $\int_{\mathbf{S}} \mu(d\mathbf{x}) = 1$

- $\mathrm{supp}(\mu)$ is the smallest set $\mathbf{S}$ such that $\mu(\mathbb{R}^n \backslash \mathbf{S}) = 0$

Problem reformulation

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \int_{\mathbf{S}} f \, d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{S}$ if there exists a finite measure μ supported on $\mathbf{S}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{S}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \, \alpha \in \mathbb{N}^n.$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{S})$: space of probability measures supported on $\mathbf{S}$

- $\mathcal{Q}(\mathbf{S})$: quadratic module

Polynomial Optimization Problems (POP)

$$
\begin{array}{lll}
\text{(Primal)} & & \text{(Dual)} \\[1em]
\inf \quad \displaystyle\int_{\mathbf{S}} f \, d\mu & = & \sup \quad \lambda \\[1em]
\text{s.t.} \quad \mu \in \mathcal{M}_+(\mathbf{S}) & & \text{s.t.} \quad \lambda \in \mathbb{R} \ , \\[1em]
& & \qquad\quad f - \lambda \in \mathcal{Q}(\mathbf{S})
\end{array}
$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{S})$

- Truncated quadratic module $\mathcal{Q}_k(\mathbf{S}) := \mathcal{Q}(\mathbf{S}) \cap \mathbb{R}_{2k}[\mathbf{x}]$

$$
\begin{array}{ll}
\text{(Moment)} & \qquad\qquad \text{(SOS)} \\[4pt]
\inf \quad \displaystyle\sum_\alpha f_\alpha \, \mathbf{z}_\alpha & = \quad \sup \quad \lambda \\[8pt]
\text{s.t.} \quad \mathbf{M}_{k-v_j}(g_j\,\mathbf{z}) \succcurlyeq 0, \quad 0 \leqslant j \leqslant l, & \qquad \text{s.t.} \quad \lambda \in \mathbb{R}, \\[6pt]
\qquad\quad \mathbf{z}_1 = 1 & \qquad\qquad\quad f - \lambda \in \mathcal{Q}_k(\mathbf{S})
\end{array}
$$

Lasserre's Hierarchy of SDP relaxations

$$\ell_{\mathbf{z}}(q) : q \in \mathbb{R}[\mathbf{x}] \mapsto \sum_\alpha q_\alpha \mathbf{z}_\alpha$$

- Moment matrix
  $\mathbf{M}(\mathbf{z})_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := \ell_{\mathbf{z}}(\mathbf{x}^\alpha \mathbf{x}^\beta) = \mathbf{z}_{\alpha+\beta}$

- Localizing matrix $\mathbf{M}(g_j \mathbf{z})$ associated with g_j
  $\mathbf{M}(g_j \mathbf{z})_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := \ell_{\mathbf{z}}(g_j \mathbf{x}^\alpha \mathbf{x}^\beta) = \sum_\gamma g_{j,\gamma} \mathbf{z}_{\alpha+\beta+\gamma}$

Lasserre's Hierarchy of SDP relaxations

- $\mathbf{M}_k(\mathbf{z})$ contains $\binom{n+2k}{n}$ variables, has size $\binom{n+k}{n}$

- Truncated matrix of order $k = 2$ with variables x_1, x_2:

$$
\mathbf{M}_2(\mathbf{z}) =
\begin{array}{c}
\\
1 \\
- \\
x_1 \\
x_2 \\
- \\
x_1^2 \\
x_1 x_2 \\
x_2^2
\end{array}
\begin{array}{ccccccc}
1 & | & x_1 & x_2 & | & x_1^2 & x_1 x_2 & x_2^2 \\
\left(\begin{array}{c}1\end{array}\right. & | & z_{1,0} & z_{0,1} & | & z_{2,0} & z_{1,1} & z_{0,2} \\
- & - & - & - & - & - & - \\
z_{1,0} & | & z_{2,0} & z_{1,1} & | & z_{3,0} & z_{2,1} & z_{1,2} \\
z_{0,1} & | & z_{1,1} & z_{0,2} & | & z_{2,1} & z_{1,2} & z_{0,3} \\
- & - & - & - & - & - & - \\
z_{2,0} & | & z_{3,0} & z_{2,1} & | & z_{4,0} & z_{3,1} & z_{2,2} \\
z_{1,1} & | & z_{2,1} & z_{1,2} & | & z_{3,1} & z_{2,2} & z_{1,3} \\
z_{0,2} & | & z_{1,2} & z_{0,3} & | & z_{2,2} & z_{1,3} & z_{0,4} \left.\right)
\end{array}
$$

Lasserre's Hierarchy of SDP relaxations

- Consider $g_1(\mathbf{x}) := 2 - x_1^2 - x_2^2$. Then $v_1 = \lceil \deg g_1 / 2 \rceil = 1$.

$$
\mathbf{M}_1(g_1\,\mathbf{z}) = \begin{array}{c} 1 \\ x_1 \\ x_2 \end{array} \begin{pmatrix} 2 - z_{2,0} - z_{0,2} & 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{0,1} - z_{2,1} - z_{0,3} \\ 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{2,0} - z_{4,0} - z_{2,2} & 2z_{1,1} - z_{3,1} - z_{1,3} \\ 2z_{0,1} - z_{2,1} - z_{0,3} & 2z_{1,1} - z_{3,1} - z_{1,3} & 2z_{0,2} - z_{2,2} - z_{0,4} \end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{M}_1(g_1\,\mathbf{z})(3,3) &= \ell(g_1(\mathbf{x}) \cdot x_2 \cdot x_2) = \ell(2x_2^2 - x_1^2 x_2^2 - x_2^4) \\
&= 2z_{0,2} - z_{2,2} - z_{0,4}
\end{aligned}
$$

Lasserre's Hierarchy of SDP relaxations

- Truncation with moments of order at most $2k$

- $v_j := \lceil \deg g_j / 2 \rceil$

- Hierarchy of semidefinite relaxations:

$$\begin{cases}
\inf_{\mathbf{z}} \ell_{\mathbf{z}}(f) &= \sum_{\alpha} \int_{\mathbf{S}} f_{\alpha} \, \mathbf{x}^{\alpha} \, \mu(d\mathbf{x}) = \sum_{\alpha} f_{\alpha} \, \mathbf{z}_{\alpha} \\
\mathbf{M}_k(\mathbf{z}) &\succcurlyeq 0, \\
\mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) &\succcurlyeq 0, \quad 1 \leqslant j \leqslant l, \\
\mathbf{z}_1 &= 1.
\end{cases}$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$(SDP) \begin{cases} \mathcal{P}: & \inf_{\mathbf{z}} \quad \sum_\alpha c_\alpha \mathbf{z}_\alpha \\ & \text{s.t.} \quad \sum_\alpha F_\alpha \mathbf{z}_\alpha - F_0 \succcurlyeq 0 \\ \\ \mathcal{D}: & \sup_{\mathbf{Y}} \quad \text{Trace}\,(F_0 \mathbf{Y}) \\ & \text{s.t.} \quad \text{Trace}\,(F_\alpha \mathbf{Y}) = c_\alpha \;, \quad \mathbf{Y} \succcurlyeq 0 \;. \end{cases}$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{ \mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } f(\mathbf{x}) = \mathbf{y} \} \ ,$$

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } \|\mathbf{y} - f(\mathbf{x})\|_2^2 = 0\} \ ,$$

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } h_f(\mathbf{x}, \mathbf{y}) \geqslant 0\} \ ,$$

with

$$h_f(\mathbf{x}, \mathbf{y}) := -\|\mathbf{y} - f(\mathbf{x})\|_2^2 \ .$$

Approximation of sets defined with "∃"

Existential QE: approximate **F** as closely as desired [Lasserre 14]

$$\mathbf{F}_k^1 := \{ \mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0 \} \ ,$$

for some polynomials $q_k \in \mathbb{R}_{2k}[\mathbf{y}]$.

A hierarchy of outer approximations of F

- Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k-truncated quadratic module

- **REMEMBER**:
  $$q - h_f \in \mathcal{Q}_k(\mathbf{K}) \implies \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geqslant 0$$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

- Hierarchy of Semidefinite programs:

$$\inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K})) \right\} \ .$$

A hierarchy of outer approximations of F

Assuming the existence of solution q_k, the sublevel sets

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \supseteq \mathbf{F} \ ,$$

provide a sequence of certified outer approximations of $\mathbf{F}$.

A hierarchy of outer approximations of F

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provide a sequence of certified outer approximations of $\mathbf{F}$.

It comes from the following:

- q_k feasible solution, $q_k - h_f \in \mathcal{Q}_k(\mathbf{K})$

- $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q_k(\mathbf{y}) \geqslant h_f(\mathbf{x}, \mathbf{y}) \iff \forall \mathbf{y}, q_k(\mathbf{y}) \geqslant h(\mathbf{y}) \ .$

Strong convergence property

Theorem

Assuming that $\overset{\circ}{\mathbf{S}} \neq \varnothing$ and $\mathcal{Q}_k(\mathbf{K})$ is Archimedean,

1 The sequence of optimal solutions (q_k) converges to h w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0 \ , (q_k \to_{L_1} h)$$

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2.

$$\lim_{k \to \infty} \text{vol}(\mathbf{F}_k^1 \backslash \mathbf{F}) = 0 \;.$$

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Proof of existence

1. Existence of optimal q_k by **Slater's condition**

Strong convergence property

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   - Dual SDP:

$$\rho_k^* := \sup_{\mathbf{z}} \quad \ell_{\mathbf{z}}(h_f)$$

$$\text{s.t.} \quad \mathbf{M}_k(\mathbf{z}) \succcurlyeq 0,$$

$$\mathbf{M}_{k-v_j}(g_j\,\mathbf{z}) \succcurlyeq 0, \quad j = 1, \ldots, l,$$

$$\ell_{\mathbf{z}}(\mathbf{y}^\beta) = z_\beta^{\mathbf{B}}, \quad \forall \beta \in \mathbb{N}_{2k}^m.$$

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- Strictly feasible $\mathbf{z}$: moments of Lebesgue measure $\lambda_{\mathbf{K}}$

- $q = 0$ feasible for Primal SDP:

$$\rho_k := \inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K}) \right\} .$$

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   - By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \to_{L_1} h$

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   - By Monotone Convergence Theorem, $f_k \rightarrow_{L_1} h$.

   - By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \rightarrow_{L_1} h$

   - Apply Putinar's Positivstellensatz to $p_k - h_f + \epsilon / \operatorname{vol}(\mathbf{B})$:

   $$p_k - h_f + \epsilon / \operatorname{vol}(\mathbf{B}) = \sum_{j=0}^{l} \sigma_j \, g_j$$

Strong convergence property

Proof of volume convergence

   2 Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geqslant -1/r\}$

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Proof of volume convergence

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- $\text{vol}\,\mathbf{F}(r) \to \text{vol}\,\mathbf{F}$

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- $\mathrm{vol}\, \mathbf{F}(r) \to \mathrm{vol}\, \mathbf{F}$

- $\lim_{k \to \infty} \mathrm{vol}\, \mathbf{F}_k^1 \leqslant \mathrm{vol}\, \mathbf{F}(r)$

Strong convergence property

Proof of volume convergence

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- $\mathrm{vol}\, \mathbf{F}(r) \to \mathrm{vol}\, \mathbf{F}$

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- $\mathrm{vol}\, \mathbf{F} \leqslant \lim_{k \to \infty} \mathrm{vol}\, \mathbf{F}_k^1 \leqslant \mathrm{vol}\, \mathbf{F}(r)$

Infinite dimensional LP formulation

- **Pushforward** $f_\# : \mathcal{M}(\mathbf{S}) \to \mathcal{M}(\mathbf{B})$:

$$f_\# \mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{S} : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{B}), \forall \mu_0 \in \mathcal{M}(\mathbf{S})$$

- $f_\# \mu_0$ is the **image measure** of μ_0 under f

Infinite dimensional LP formulation

$$p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \quad \int_{\mathbf{B}} \mu_1$$

$$\text{s.t.} \quad \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}},$$

$$\mu_1 = f_{\#}\mu_0,$$

$$\mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}).$$

Lebesgue measure on $\mathbf{B}$ is $\lambda_{\mathbf{B}}(d\mathbf{y}) := \mathbf{1}_{\mathbf{B}}(\mathbf{y})\, d\mathbf{y}$

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$$\mu_1 = f_{\#} \mu_0,$$

$$\mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}).$$

Lemma

Let μ_1^* be an optimal solution of the above LP.

Then $\mu_1^* = \lambda_{\mathbf{F}}$ and $p^* = \text{vol } \mathbf{F}$.

LP Primal-dual conic formulation

The LP can be cast as follows:

$$p^* = \sup_x \quad \langle x, c \rangle_1$$
$$\text{s.t.} \quad \mathcal{A} x = b \,,$$
$$x \in E_1^+ \,,$$

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$$x \in E_1^+ \, ,$$

with

- $E_1 := \mathcal{M}(\mathbf{S}) \times \mathcal{M}(\mathbf{B})^2 \quad F_1 := \mathcal{C}(\mathbf{S}) \times \mathcal{C}(\mathbf{B})^2$

- $x := (\mu_0, \mu_1, \hat{\mu}_1) \quad c := (0, 1, 0) \in F_1 \quad b := (0, \lambda_{\mathbf{B}})$

- the linear operator $\mathcal{A} : E_1 \to E_2$ given by

$$\mathcal{A}(\mu_0, \mu_1, \hat{\mu}_1) := \begin{bmatrix} -f_{\#}\mu_0 + \mu_1 \\ \mu_1 + \hat{\mu}_1 \end{bmatrix} .$$

LP Primal-dual conic formulation

<div align="center">

Primal LP   Dual LP

</div>

$$p^* = \sup_x \quad \langle x, c \rangle_1 \qquad\qquad d^* = \inf_y \quad \langle b, y \rangle_2$$

$$\text{s.t.} \quad \mathcal{A}\, x = b\,, \qquad\qquad \text{s.t.} \quad \mathcal{A}' y - c \,\in\, \mathcal{C}_+(\mathbf{B})^2\,.$$

$$x \in E_1^+\,.$$

with

- $y := (v, w) \in \mathcal{M}(\mathbf{B})^2$

- $\mathcal{A}'(v, w) := \begin{bmatrix} -v \circ f \\ v + w \\ w \end{bmatrix}\,.$

LP Primal-dual conic formulation

Primal LP

$$p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int \mu_1$$

$$\text{s.t.} \quad \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}},$$

$$\mu_1 = f_{\#}\mu_0,$$

$$\mu_0 \in \mathcal{M}_+(\mathbf{S}),$$

$$\mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}).$$

Dual LP

$$d^* := \inf_{v, w} \int w(\mathbf{y}) \, \lambda_{\mathbf{B}}(d\mathbf{y})$$

$$\text{s.t.} \quad v(f(\mathbf{x})) \geqslant 0, \quad \forall \mathbf{x} \in \mathbf{S},$$

$$w(\mathbf{y}) \geqslant 1 + v(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbf{B},$$

$$w(\mathbf{y}) \geqslant 0, \quad \forall \mathbf{y} \in \mathbf{B},$$

$$v, w \in \mathcal{C}(\mathbf{B}).$$

Zero duality gap

Lemma

$p^* = d^*$

Strong convergence property

Strengthening of the dual LP:

$$d_k^* := \inf_{v,w} \quad \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}}$$

$$\text{s.t.} \quad v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}),$$

$$w - 1 - v \in \mathcal{Q}_k(\mathbf{B}),$$

$$w \in \mathcal{Q}_k(\mathbf{B}),$$

$$v, w \in \mathbb{R}_{2k}[\mathbf{y}].$$

Strong convergence property

Theorem

Assuming that $\overset{\circ}{\mathbf{F}} \neq \varnothing$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

1. The sequence (w_k) converges to $\mathbf{1_F}$ w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1_F}| d\mathbf{y} = 0 \ .$$

Strong convergence property

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1 The sequence (w_k) converges to $\mathbf{1_F}$ w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1_F}| d\mathbf{y} = 0 \ .$$

2 Let $\mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$. Then,

$$\lim_{k \to \infty} \mathrm{vol}(\mathbf{F}_k^2 \backslash \mathbf{F}) = 0 \ .$$

Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by
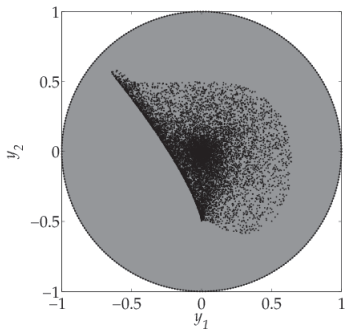
$$f(\mathbf{x}) := (x_1 + x_1 x_2, x_2 - x_1^3)/2$$
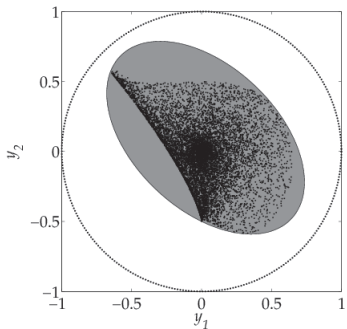
$\mathbf{F}_1^1$ $\mathbf{F}_1^2$

Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by
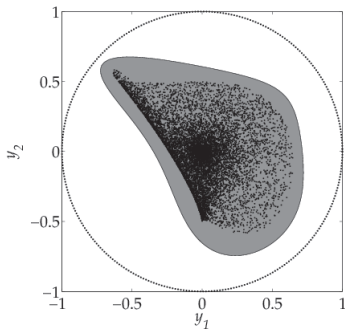
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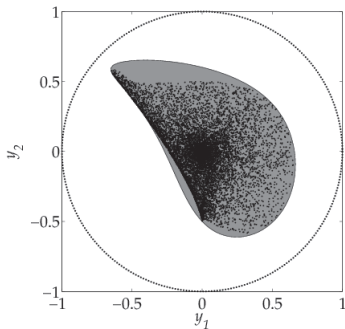
$\mathbf{F}_2^1$ $\mathbf{F}_2^2$

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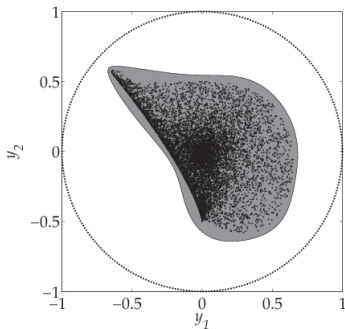
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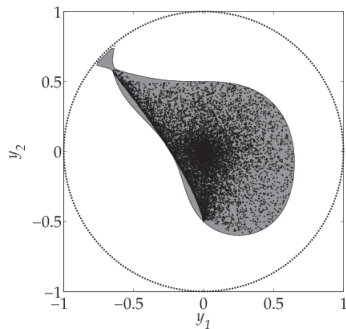
$\mathbf{F}_3^1$ $\mathbf{F}_3^2$

Polynomial image of the unit ball

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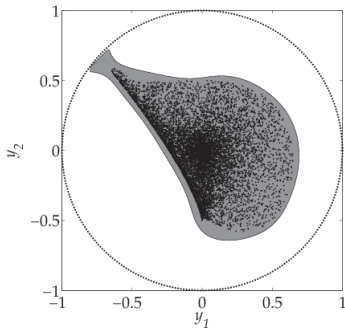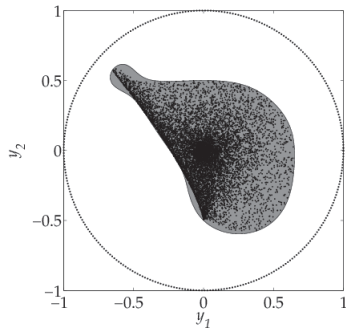
$\mathbf{F}_4^1$ $\mathbf{F}_4^2$

Semialgebraic set projections

Simpler formulation:

$$d_k^* := \inf_{v,w} \quad \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}} \qquad\qquad \inf_w \quad \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}}$$

$$\begin{aligned}
\text{s.t.} \quad & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), & \text{s.t.} \quad & w - 1 \in \mathcal{Q}_k(\mathbf{S}), \\
& w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), & & w \in \mathcal{Q}_k(\mathbf{B}), \\
& w \in \mathcal{Q}_k(\mathbf{B}), & & w \in \mathbb{R}_{2k}[x_1, \ldots, x_m]. \\
& v, w \in \mathbb{R}_{2k}[\mathbf{y}].
\end{aligned}$$

Semialgebraic set projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on $\mathbb{R}^2$ of the semialgebraic set

$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leqslant 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geqslant 0,$$
$$1/9 - (x_1 - 1/2)^4 - x_2^4 \geqslant 0\}$$
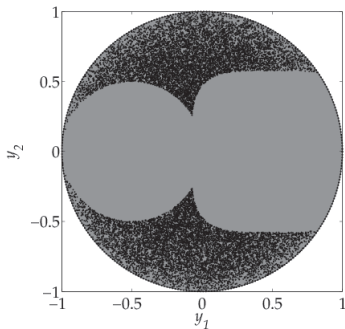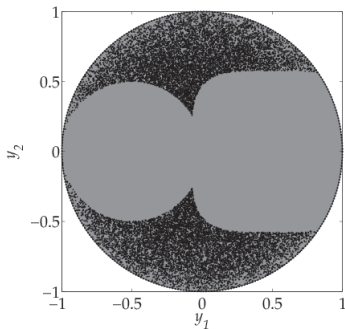
$\mathbf{F}_2^1$ $\mathbf{F}_2^2$

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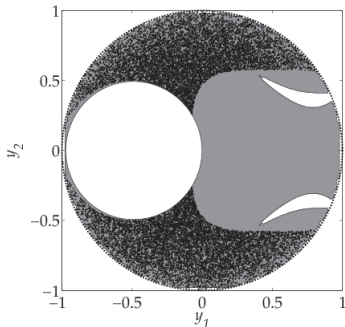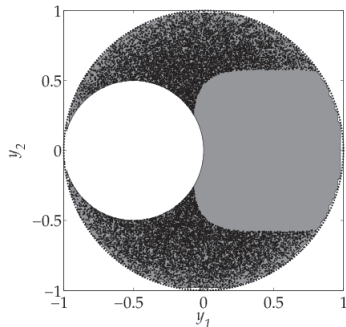
$\mathbf{F}_3^1$ $\mathbf{F}_3^2$

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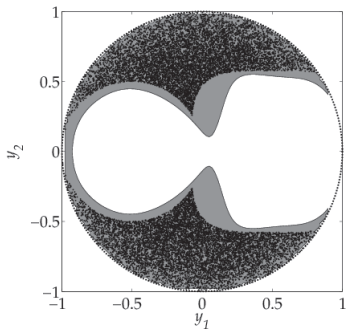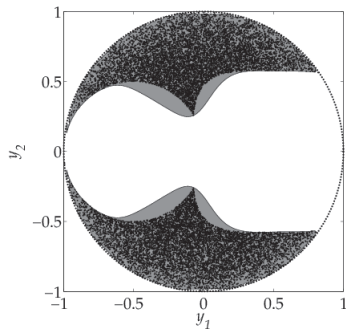
$\mathbf{F}_4^1$ $\mathbf{F}_4^2$

Bicriteria Optimization Problems

$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5$,

$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85$,

$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\}$.

$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2$,

$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4$.
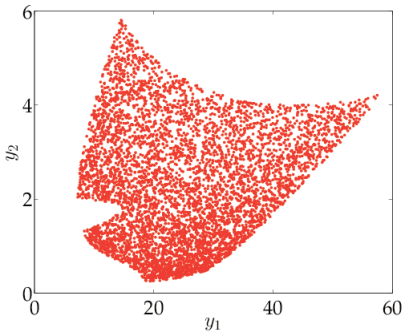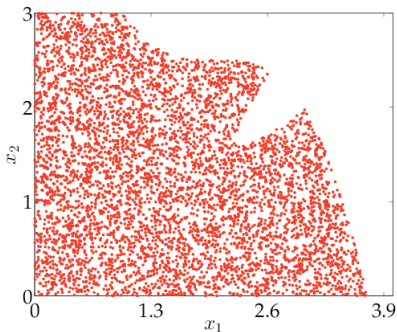
Previous Contributions

- Numerical schemes that **avoid computing finitely many points**.

- Pareto curve approximation with polynomials, **convergence guarantees** in L_1-norm

📄 V. Magron, D. Henrion, J.B. Lasserre. Approximating Pareto Curves using Semidefinite Relaxations. *Operations Research Letters*. arxiv:1404.4772, April 2014.
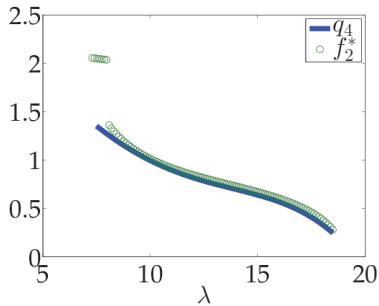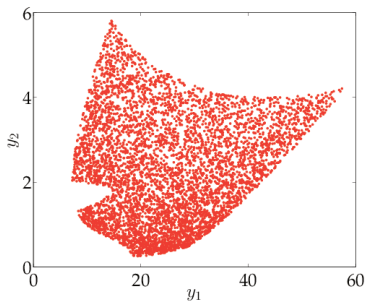
Degree 4
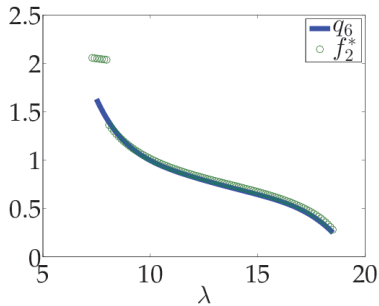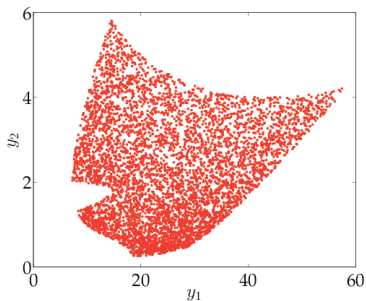
Previous Contributions

Degree 6
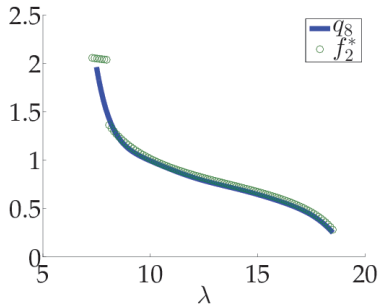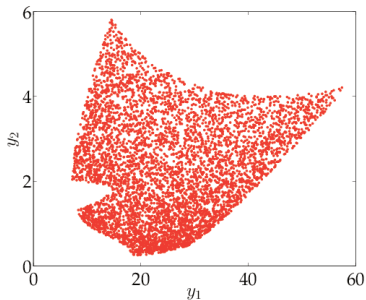
Degree 8

Approximating Pareto curves

Back on our previous nonconvex example:
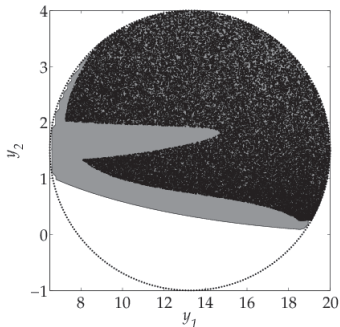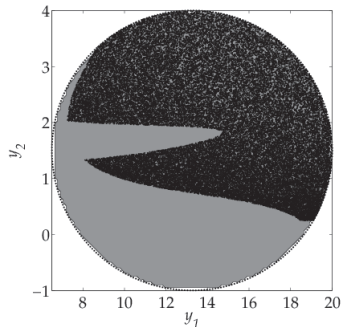
$\mathbf{F}_1^1$ $\mathbf{F}_1^2$

Approximating Pareto curves

Back on our previous nonconvex example:
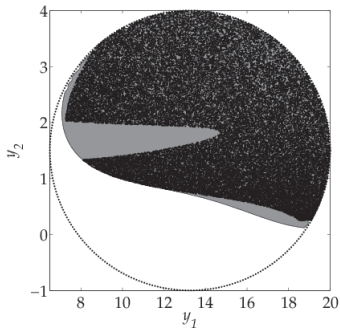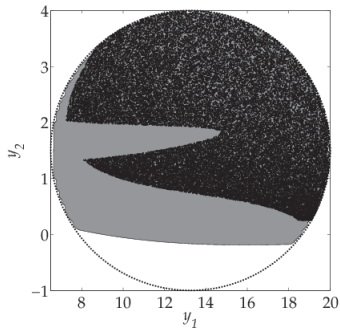
$\mathbf{F}_2^1$ $\qquad\qquad\qquad\qquad\qquad$ $\mathbf{F}_2^2$

Approximating Pareto curves

Back on our previous nonconvex example:
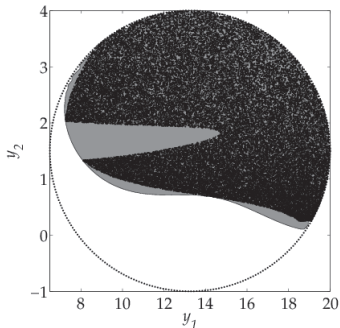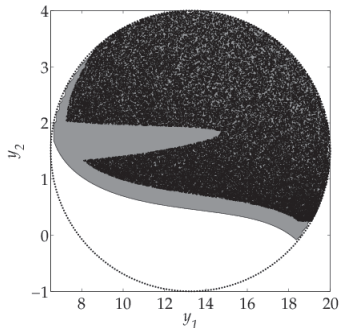
$$\mathbf{F}_3^1 \qquad\qquad \mathbf{F}_3^2$$

Approximating Pareto curves

"Zoom" on the region which is hard to approximate:
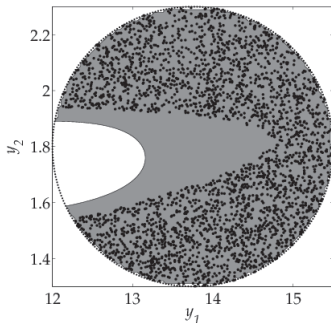
$\mathbf{F}_4^1$

Approximating Pareto curves

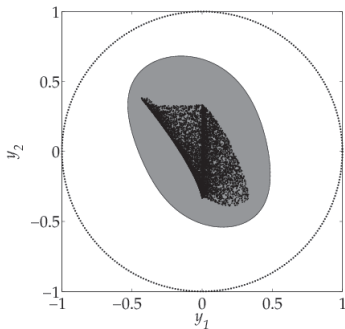"Zoom" on the region which is hard to approximate:
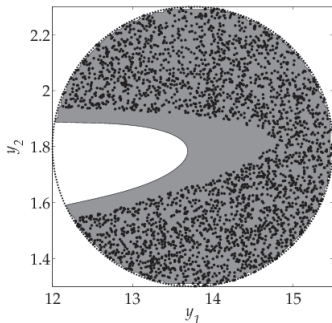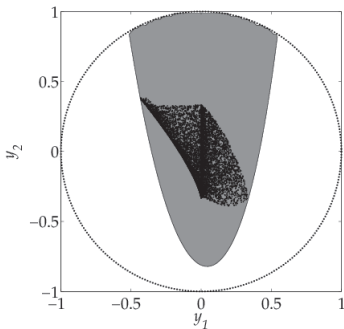

$\mathbf{F}_5^1$

Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1 x_2, x_1^2), x_2 - x_1^3)/3$$
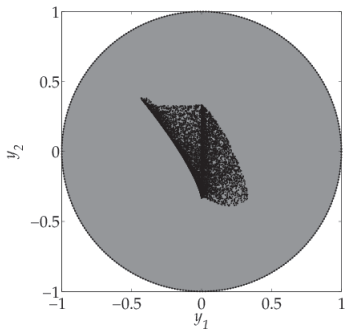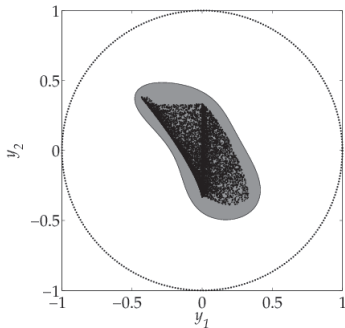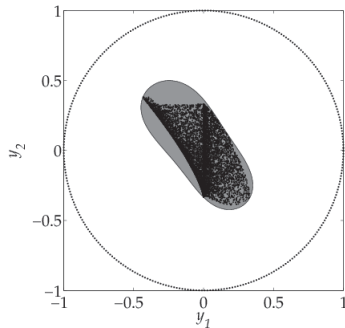

$\mathbf{F}_1^1$
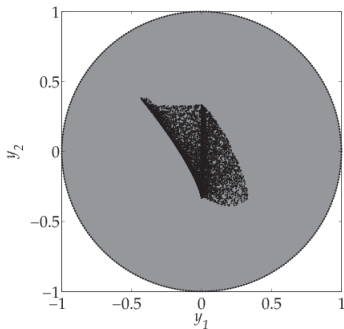
$\mathbf{F}_1^2$

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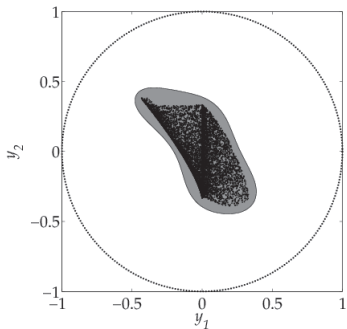$\mathbf{F}_2^1$ $\mathbf{F}_2^2$

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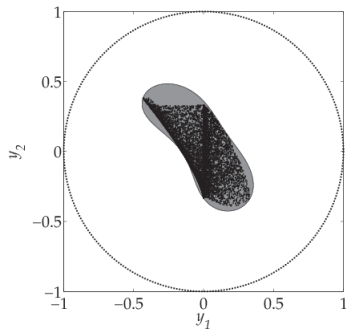$\mathbf{F}_3^1$ $\mathbf{F}_3^2$

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$\mathbf{F}_4^1$ $\mathbf{F}_4^2$

Conclusion

- **Unifying** framework: projections, Pareto curves

- Computational complexity

  1. Method 1: $\binom{n+m+2k}{2k}$ SDP variables

  2. Method 2: $\binom{n+2kd}{2kd}$ SDP variables

- **Structure sparsity** can be exploited

Conclusion

Further research:

- Alternative positivity certificates LP/SDP
  1. Less computationally demanding than SDP

  2. More efficient than LP (as generic convergence cannot occur)

End

📄 V. Magron, D. Henrion, J.B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. oo:2014.10.4606, October 2014.

Thank you for your attention!

cas.ee.ic.ac.uk/people/vmagron