# Two-player games between polynomial optimizers and semidefinite solvers

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Joint work with

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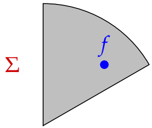
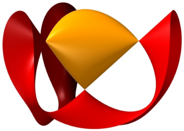
ICCOPT, Berlin, 07 August 2019

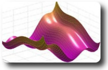


# SDP for Polynomial Optimization

**NP-hard NON CONVEX Problem** $f^\star = \inf f(\mathbf{x})$

## Theory

| (Primal) | | (Dual) |
|---|---|---|
| $\inf \int f \, d\mu$ |  | $\sup \lambda$ |
| with $\mu$ proba $\Rightarrow$ | INFINITE LP | $\Leftarrow$ with $p - \lambda \geqslant 0$ |

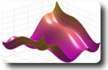
# SDP for Polynomial Optimization

**NP-hard NON CONVEX Problem** $f^\star = \inf f(\mathbf{x})$

**Practice**

(Primal **Relaxation**)

**moments** $\int \mathbf{x}^\alpha \, d\mu$



(Dual **Strengthening**)

$f - \lambda =$ **sum of squares**

**finite** number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $f_d^\star \uparrow f^*$ [Lasserre/Parrilo 01]

degree $d$
$n$ vars $\Longrightarrow \binom{n+d}{n}$ **SDP** VARIABLES

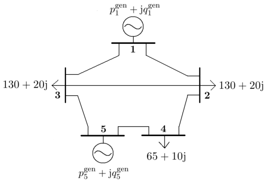
**Numeric** Solvers $\Longrightarrow$ **Approx** Certificate

# Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as $\infty$-*dimensional* LP over measures

STATIC **Polynomial Optimization**

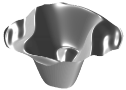
**Optimal Powerflow** $n \simeq \mathbf{10^3}$ [Josz et al 16]



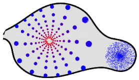

**Roundoff Error** $n \simeq \mathbf{10^2}$ [Magron et al 17]

DYNAMICAL **Polynomial Optimization**

**Regions of attraction** [Henrion et al 14]



**Reachable sets** [Magron et al 19]



**APPROXIMATE** OPTIMIZATION BOUNDS!

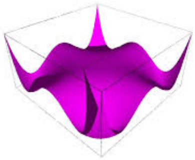
# Two-player Games: Optimizers vs Solvers

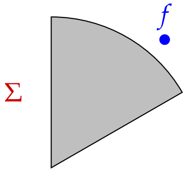
## Motzkin polynomial

sums of squares $= \Sigma$

$$f = \frac{1}{27} + x^2y^4 + x^4y^2 - x^2y^2$$

$f \geqslant 0$ but $f \notin \Sigma$





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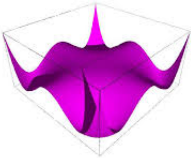

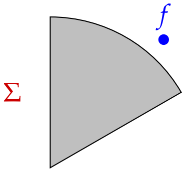


$$f^\star = \min_{(x,y)\in\mathbb{R}^2} f(x,y) = 0 \text{ for } |x^\star| = |y^\star| = \frac{\sqrt{3}}{3}$$

Lasserre's hierarchy:

- order 3 $\rightsquigarrow f_3^\star = -\infty$ unbounded SDP $\implies f \notin \Sigma$

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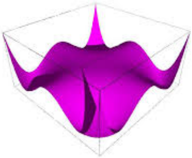

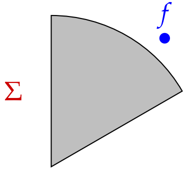


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- order 4 $\rightsquigarrow f_4^\star = -\infty$

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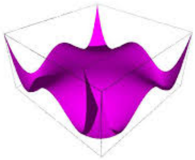

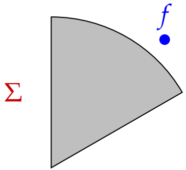


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- order 4 $\rightsquigarrow f_4^\star = -\infty$
- order 5 $\rightsquigarrow f_5^\star \simeq -0.4$

# Two-player Games: Optimizers vs Solvers

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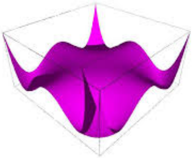

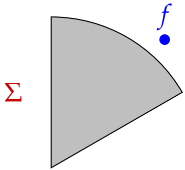


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- order 4 $\rightsquigarrow f_4^\star = -\infty$
- order 5 $\rightsquigarrow f_5^\star \simeq -0.4$
- order 8 $\rightsquigarrow f_8^\star \simeq -10^{-8} \oplus$ extraction of $x^\star, y^\star$ $\boxed{\textbf{Paradox}}$?!

# Two-player Games: Optimizers vs Solvers

## Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!



$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\boxed{\simeq \quad \rightarrow \quad =}\ ?$$

# Inaccurate SDP do Robust Optimization

$$f^\star = \inf \sum_\alpha f_\alpha \, \mathbf{x}^\alpha$$

**Moment matrix** $\mathbf{M}_d(\mathbf{y})_{\alpha,\beta} = y_{\alpha+\beta}$

## Accurate SDP Relaxations

| (Primal **Relaxation**) | (Dual **Strengthening**) |
|---|---|
| $\inf_{\mathbf{y}} \sum_\alpha f_\alpha \, y_\alpha$ | $\sup \ \lambda$ |
| s.t. $\mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$ | $f - \lambda = \sigma$ |
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$$\sup \ \lambda$$

$$| \, f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \, | \leqslant \varepsilon$$

$$\mathbf{Q} \succcurlyeq -\eta \mathbf{I}$$

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# Priority to Trace Equalities: $\varepsilon = 0$

$$\tilde{f} = f + \eta \sum_{\beta} \mathbf{x}^{2\beta}$$

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(Primal **Relaxation**) (Dual **Strengthening**)

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**Theorem [Lasserre-Magron 19]**

Inaccurate SDP relaxations of the **robust** problem

$$\max_{\tilde{f} \in \mathbf{B}_\infty(f, \eta)} \min_{\mathbf{x}} \tilde{f}(\mathbf{x})$$

# Priority to Trace Equalities: $\varepsilon = 0$

**Theorem [Lasserre 06]**

For fixed $n$, any $f \geqslant 0$ can be approximated arbitrarily closely by SOS polynomials.

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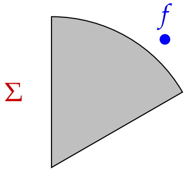
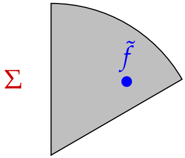



$$\tilde{f} = f + \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$$



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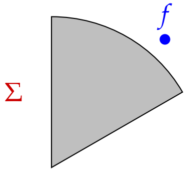


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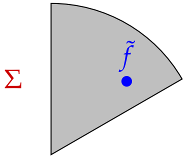
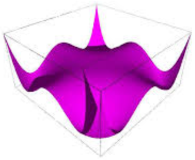



At fixed $\eta$, when $d \nearrow$, $\tilde{f} \in \Sigma$!



$$f + 10^{-7} \sum_{|\beta| \leqslant 4} \mathbf{x}^{2\beta} \in \Sigma$$

$\boxed{\textbf{Paradox Explanation}}$

# Priority to SDP Inequalities: $\eta = 0$

## Inaccurate SDP Relaxations

(Primal **Relaxation**)

$$\inf_{\mathbf{y}} \sum_{\alpha} f_\alpha y_\alpha + \varepsilon \|\mathbf{y}\|_1$$

$$\text{s.t. } \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0$$

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(Dual **Strengthening**)

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$$\left| f_\alpha - \lambda 1_{\alpha=0} - \langle \mathbf{B}_\alpha, \mathbf{Q} \rangle \right| \leqslant \varepsilon$$

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**Inaccurate SDP Relaxations**

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| $\inf_{\mathbf{y}} \sum_\alpha f_\alpha y_\alpha + \varepsilon \|\mathbf{y}\|_1$ | $\sup_{\lambda, \tilde{f}} \lambda$ |
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# Priority to SDP Inequalities: $\eta = 0$

**Theorem (Lasserre-Magron)**

Inaccurate SDP relaxations of the **robust** problem

$$\max_{\tilde{f} \in \mathbf{B}_\infty(f,\varepsilon)} \min_{\mathbf{x}} \tilde{f}(\mathbf{x})$$

# A Two-player Game Interpretation


max − min ROBUST OPTIMIZATION

Player 1 (solver) picks $\tilde{f} \in \mathbf{B}_\infty(f)$ $\rightsquigarrow$ **SDP leads**

Player 2 (optimizer) picks an SOS $\rightsquigarrow$ **User follows**

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**Convex** SDP relaxations $\implies \boxed{\max - \min = \min - \max}$

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$\min - \max$ ROBUST OPTIMIZATION

Player 1 (robust optimizer) picks an SOS $\rightsquigarrow$ **User leads**

Player 2 (solver) picks $\tilde{f} \in \mathbf{B}_\infty(f) \rightsquigarrow$ **SDP follows**

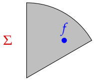
SDP for Polynomial Optimization

Optimization Game

Certification Game

# From Approximate to Exact Solutions

Win TWO-PLAYER GAME




sum of squares of $f$?

$\simeq$ Output!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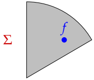


**Hybrid** Symbolic/Numeric Algorithms

sum of squares of $f - \varepsilon$?

$\simeq$ Output!


Error Compensation

$\simeq \quad \rightarrow \quad =$

# Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

**Existence Question**

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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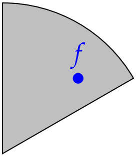
## Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 = ???$$

# Round & Project Algorithm [Peyrl-Parrilo 08]

$\Sigma$




$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

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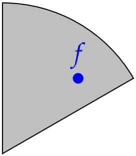


$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Find $\tilde{\mathbf{Q}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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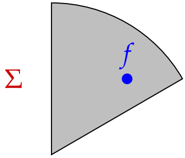


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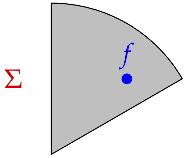
$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

Exact $Q \implies f_{\alpha+\beta} = \sum_{\alpha'+\beta'=\alpha+\beta} Q_{\alpha',\beta'}$

# Round & Project Algorithm [Peyrl-Parrilo 08]




$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Find $\tilde{\mathbf{Q}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq {\mathbf{v}_D}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

Exact $Q \implies f_{\alpha+\beta} = \sum_{\alpha'+\beta'=\alpha+\beta} Q_{\alpha',\beta'}$

1. **Rounding step** $\hat{Q} \leftarrow \texttt{round}\left(\tilde{Q}, \hat{\delta}\right)$

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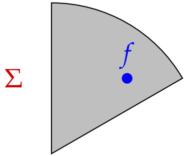


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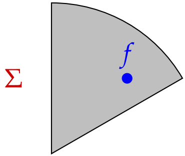


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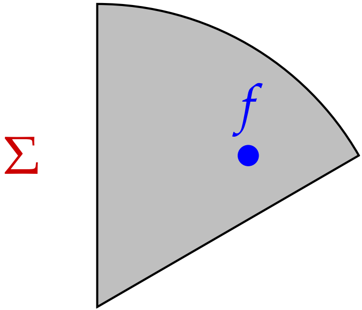
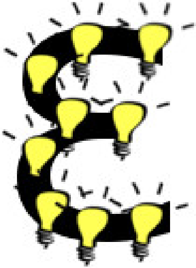
Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = \mathbf{v}_D{}^T(X)\, \mathbf{Q}\, \mathbf{v}_D(X)$ and $\mathbf{Q} \succcurlyeq 0$

# Our Alternative Approach



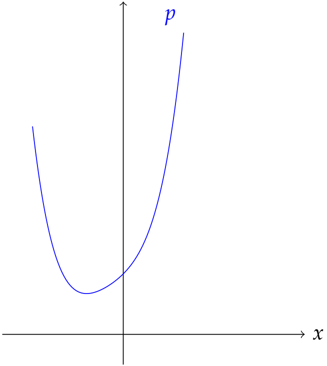
**PERTURBATION** idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

# `RealCertify` with $n = 1$ [Chevillard et. al 11]

$f \in \mathbb{Q}[X]$, $\deg f = d = 2k$, $f > 0$



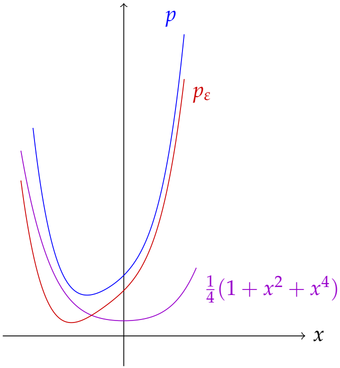

$f = 1 + X + X^2 + X^3 + X^4$

# RealCertify with $n = 1$ [Chevillard et. al 11]

$f \in \mathbb{Q}[X]$, $\deg f = d = 2k$, $f > 0$

**Perturb**: find $\varepsilon \in \mathbb{Q}$ s.t.

$$f_\varepsilon := f - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$




$$f = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$f > \frac{1}{4}(1 + X^2 + X^4)$$

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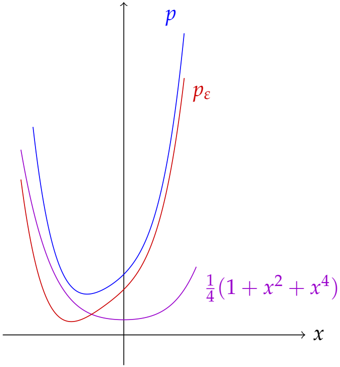
SDP Approximation:

$$f - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

**ABSORB**: small enough $u_i$

$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS




$$f = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$f > \frac{1}{4}(1 + X^2 + X^4)$$

# RealCertify with $n = 1$: SDP Approximation

$f$

$$f_\varepsilon \leftarrow f - \varepsilon \sum_{i=0}^{k} X^{2i}$$

$$\varepsilon \leftarrow \frac{\varepsilon}{2}$$

while

$f_\varepsilon \leq 0$

$$\tilde{\sigma} \leftarrow \texttt{sdp}(f_\varepsilon, \delta)$$

$$u \leftarrow f_\varepsilon - \tilde{\sigma}$$

$$\delta \leftarrow 2\delta$$

while

$$\varepsilon < \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i}$$

$\tilde{\sigma}, \varepsilon, u$

# RealCertify with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

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$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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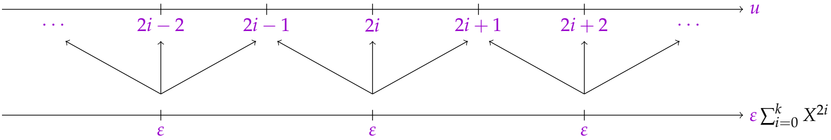

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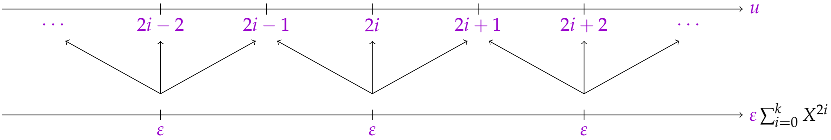
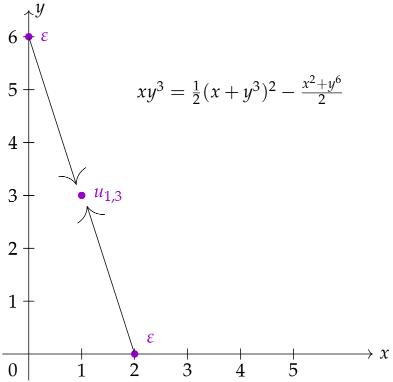



$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

## RealCertify with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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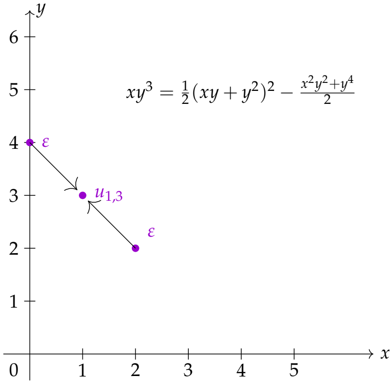

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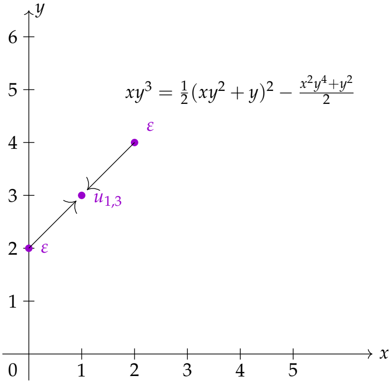

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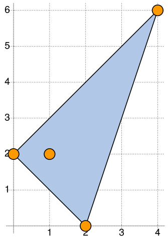
$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

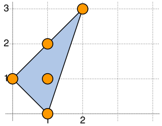
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt($f$) = $\{(4,6),(2,0),(1,2),(0,2)\}$




Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\mathrm{spt}(f))$

Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



# RealCertify: Benchmarks

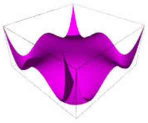
- RAGLib (critical points) [Safey El Din]
- SamplePoints (CAD) [Moreno Maza-Alvandi et al.]

| Id | $n$ | $d$ | RealCertify | | RoundProject | | RAGLib | CAD |
|---|---|---|---|---|---|---|---|---|
| | | | $\tau_1$ (bits) | $t_1$ (s) | $\tau_2$ (bits) | $t_2$ (s) | $t_3$ (s) | $t_4$ (s) |
| $f_{20}$ | 2 | 20 | 745 419 | 110. | 78 949 497 | 141. | 0.16 | 0.03 |
| $M$ | 3 | 8 | 17 232 | 0.35 | 18 831 | 0.29 | 0.15 | 0.03 |
| $f_2$ | 2 | 4 | 1 866 | 0.03 | 1 031 | 0.04 | 0.09 | 0.01 |
| $f_6$ | 6 | 4 | 56 890 | 0.34 | 475 359 | 0.54 | 598. | — |
| $f_{1}$ | 10 | 4 | 344 347 | 2.45 | 8 374 082 | 4.59 | — | — |

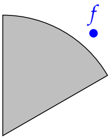
# Two-player Games: Perspectives

## OPTIMIZATION GAME

Solvers **OUTPUT** inaccurate certificates $\Rightarrow$ extract solutions

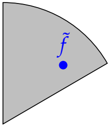


$\Sigma$




$$\tilde{f} = f + \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$$

$\Sigma$



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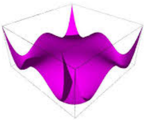

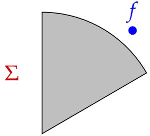


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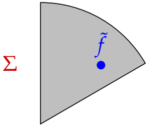



## CERTIFICATION GAME

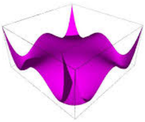
Optimizers **INPUT** inaccurate $\tilde{f} = f - \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$

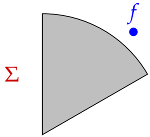
$\Longrightarrow$ exact certificates

# Two-player Games: Perspectives

### Optimization Game

Solvers **OUTPUT** inaccurate certificates $\Rightarrow$ extract solutions



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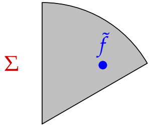



### Certification Game

Optimizers **INPUT** inaccurate $\tilde{f} = f - \eta \sum_{|\beta| \leqslant d} \mathbf{x}^{2\beta}$

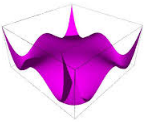
$\Longrightarrow$ exact certificates

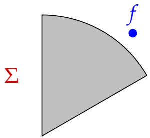
- Better arbitrary-precision SDP solvers
- Extension to other relaxations, sums of hermitian squares

# Two-player Games: Perspectives

## Optimization Game

Solvers **OUTPUT** inaccurate certificates $\Rightarrow$ extract solutions



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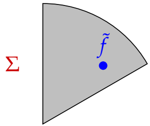


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**Crucial need for polynomial systems certification**

**Available PhD/Postdoc Positions**

# End

Thank you for your attention!

gricad-gitlab:RealCertify

https://homepages.laas.fr/vmagron


- Lasserre & Magron. In SDP relaxations, inaccurate solvers do robust optimization, *SIOPT*. arxiv:1811.02879

- Magron & Safey El Din. On Exact Polya and Putinar's Representations, *ISSAC'18*. arxiv:1802.10339

- Magron & Safey El Din. RealCertify: a Maple package for certifying non-negativity, *ISSAC'18*. arxiv:1805.02201