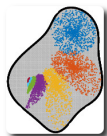


Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems

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Joint work with
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Didier Henrion (LAAS)
Xavier Thirioux (IRIT)

GT VS CPS Meeting
14 December 2017



The RS Problem

- Semialgebraic initial conditions

$$\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : g_1^0(\mathbf{x}) \geq 0, \dots, g_{m_0}^0(\mathbf{x}) \geq 0\}$$

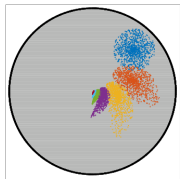
- Polynomial map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$,
 $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))$

- Reachable Set (RS) of admissible trajectories

$$\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t \in \mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$$

- $\mathbf{X}^\infty = \bigcup_{t \in \mathbb{N}} f^t(\mathbf{X}_0) \subseteq \mathbf{X} \subset \mathbb{R}^n$ (box or ball)

- Tractable approximations of RS \mathbf{X}^∞ ?



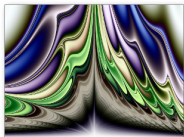
The RS Problem

- Occurs in several contexts :

- 1 program analysis: fixpoint computation

```
toyprogram (x1, x2)  
requires (0.25 ≤ x1 ≤ 0.75 && 0.25 ≤ x2 ≤ 0.75) ;  
while (x12 + x22 ≤ 1) {  
    x1 = x1 + 2x1x2 ;  
    x2 = 0.5(x2 - 2x13) ;  
}
```

- 2 hybrid systems, biology: Neuron Model, Growth Model



- 3 control: integrator, Hénon map

Related work: RS

- 1 Contractive methods based on LP relaxations and polyhedra projection [Bertsekas 72]
- 2 Extension to nonlinear systems [Harwood et al. 16]
- 3 Bernstein/Krivine-Handelman representations [Ben Sassi et al. 15, Ben Sassi et al. 12]

⊕ LP relaxations \implies scalability

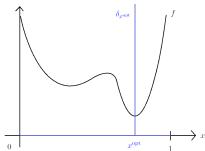
⊖ Convex approximations of nonconvex sets \implies coarse

⊖ No convergence guarantees (very often)

Related work: Lasserre hierarchy

💡 **Cast** a polynomial optimization problem as an *infinite-dimensional* LP over measures [Lasserre 2001]

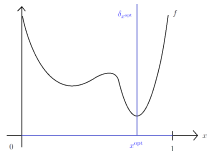
$$f^* := \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f(\mathbf{x}) d\mu$$



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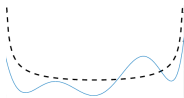
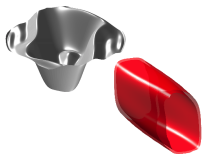
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↪ Regions of attraction [Henrion-Korda 14]

↪ Maximum invariants [Korda et al 13]

↪ Invariant 1D densities [Henrion 2012]



Related work: Lasserre hierarchy

- 5 SDP approximation of polynomial images of semialgebraic sets [M.-Henrion-Lasserre 15]
- $\mathbf{X}_1 := f(\mathbf{X}_0) \subseteq \mathbf{X}$, with $\mathbf{X} \subset \mathbb{R}^n$ a box or a ball
 \implies Discrete-time system with a single iteration
-  Approximation of image measure supports
 \implies certified SDP over approximations of \mathbf{X}_1
- $\mathbf{X}_t := f^t(\mathbf{X}_0)$
 - ⊖ $\deg f^t = d \times t \implies$ very expensive computation
 - ⊖ Would only approximate \mathbf{X}_t and not \mathbf{X}^∞

Contribution

- General framework to approximate X^∞
 - ⊕ **No discretization** is required

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Contribution

- General framework to approximate \mathbf{X}^∞
 - ⊕ **No discretization** is required
- Infinite-dimensional LP formulation
 - 💡 support of measures solving Liouville's Equation
- Finite-dimensional SDP relaxations
- $\mathbf{X}^\infty \subseteq \mathbf{X}^r := \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geq 1\}$
 - ⊕ Strong convergence guarantees $\lim_{r \rightarrow \infty} \text{vol}(\mathbf{X}^r \setminus \mathbf{X}^\infty) = 0$
 - ⊕ Compute w_r by solving one **semidefinite program**

The RS Problem

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for Reachable Sets (RS)

Application Examples

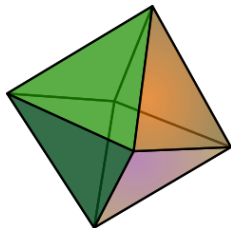
Conclusion

What is Semidefinite Programming?

- Linear Programming (LP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \mathbf{A} \mathbf{z} \geq \mathbf{d} . \end{aligned}$$

- Linear cost \mathbf{c}
- Linear inequalities “ $\sum_i A_{ij} z_j \geq d_i$ ”



Polyhedron

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 . \end{aligned}$$

- Linear cost \mathbf{c}
- Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities “ $\mathbf{F} \succcurlyeq 0$ ”
(\mathbf{F} has nonnegative eigenvalues)



Spectrahedron

What is Semidefinite Programming?

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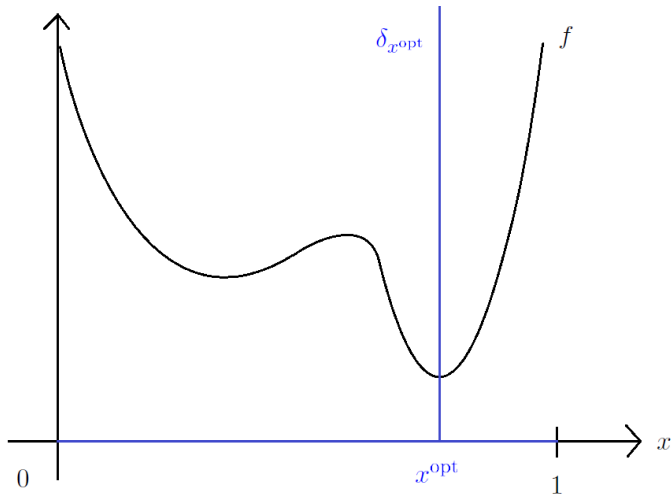
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Spectrahedron

Primal-dual Moment-SOS [Lasserre 01]

$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f d\mu$$



Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence \mathbf{z} has a representing measure on \mathbf{X} if there exists a finite measure μ supported on \mathbf{X} such that

$$\mathbf{z}_\alpha = \int_{\mathbf{X}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n.$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{X})$: space of **probability measures** supported on \mathbf{X}
- $\mathcal{Q}(\mathbf{X})$: combining **sums of squares** and polynomials g_j from \mathbf{X}

Polynomial Optimization Problems (POP)

$$\begin{array}{ll} \text{(Primal)} & \text{(Dual)} \\ \inf \int_{\mathbf{X}} f d\mu & = \sup m \\ \text{s.t. } \mu \in \mathcal{M}_+(\mathbf{X}) & \text{s.t. } m \in \mathbb{R}, \\ & f - m \in \mathcal{Q}(\mathbf{X}) \end{array}$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences \mathbf{z} of measures in $\mathcal{M}_+(\mathbf{X})$
- Truncated quadratic module $\mathcal{Q}_r(\mathbf{X})$

Polynomial Optimization Problems (POP)

(Moment)		(SOS)
$\inf \sum_{\alpha} f_{\alpha} \mathbf{z}_{\alpha}$	=	$\sup m$
s.t. $\mathbf{M}_{r-r_j}(\mathbf{g}_j \mathbf{z}) \succcurlyeq 0, \quad 0 \leq j \leq l,$		s.t. $m \in \mathbb{R},$
$\mathbf{z}_0 = 1$		$f - m \in \mathcal{Q}_r(\mathbf{X})$

The RS Problem

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Application Examples

Conclusion

Occupation Measures and Liouville's Equation

■ Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$

■ Pushforward $f_{\#} : \mathcal{M}_+(\mathbf{X}_0) \rightarrow \mathcal{M}_+(\mathbf{X})$:

$$f_{\#} \mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{X}_0 : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{X})$$

■ $f_{\#} \mu_0$ is the **image measure** of μ_0 under f

Occupation Measures and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ and

$$\mu_1 := f_{\#} \mu_0$$

...

$$\mu_t := f_{\#} \mu_{t-1}$$

$$v_t := \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} f_{\#}^i \mu_0$$

- The measures μ_t, v_t, μ_0 satisfy **Liouville's Equation**:

$$\mu_t + v_t = f_{\#} v_t + \mu_0$$

Occupation Measures and Liouville's Equation

- Let $\mu_t := \lambda_{\mathbf{X}_t}$: Lebesgue measure on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\mu_t = f_{\#}^t \mu_{0,t}$
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- Let $\lambda_{\mathbf{X}^T}$: Lebesgue measure restriction on $\mathbf{X}^T := \bigcup_{t=0}^T \mathbf{X}_t$
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$$\lambda_{\mathbf{X}^T} + \nu^T = f_{\#} \nu^T + \mu_0^T$$

ν^T average **occupation measure**: measures time spent in \mathbf{X}^T

Infinite Dimensional LP Formulation

Volume Assumption

Define $\mathbf{Y}^0 := \mathbf{X}^0$ and $\mathbf{Y}^t := \mathbf{X}_t \setminus \mathbf{X}^{t-1}$.

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T t \operatorname{vol} \mathbf{Y}^t \leq \infty.$$

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Lemma

Under **Volume Assumption**, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

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Lemma

Under **Volume Assumption**, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Proof

- $\lambda_{\mathbf{X}^T} = \sum_{t=0}^T \lambda_{\mathbf{Y}^t} \rightarrow \lambda_{\mathbf{X}^\infty}$ as $T \rightarrow \infty$
- $\mu_t + \nu_t = f_{\#} \nu_t + \mu_{0,t} \implies \nu^T := \sum_{t=0}^T \nu_t$ has mass $\leq \sum_{t=0}^T t \operatorname{vol} \mathbf{Y}^t$

Infinite Primal LP for RS Characterization

$$\begin{aligned} p^T &:= \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu \\ \text{s.t.} \quad & \int_{\mathbf{X}} \nu \leq T \text{vol} \mathbf{X}, \\ & \mu + \nu = f_{\#} \nu + \mu_0, \\ & \mu \leq \lambda_{\mathbf{X}}, \\ & \mu_0 \in \mathcal{M}_+(\mathbf{X}_0), \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X}). \end{aligned}$$

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Lemma

- 1 Optimal solution $\mu^* := \lambda_{\mathbf{S}^T}$ for some \mathbf{S}^T s.t. $\mathbf{X}^T \subseteq \mathbf{S}^T \subseteq \bar{\mathbf{X}}^\infty$
- 2 Under Volume Assumption, $\mathbf{S}^T = \bar{\mathbf{X}}^\infty$

LP Primal-dual Conic Formulation

The LP can be cast as follows:

$$\begin{aligned} p^T &= \sup_x \langle x, c \rangle_1 \\ \text{s.t. } & \mathcal{A}x = b, \\ & x \in E_1^+, \end{aligned}$$

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with

- $E_1 := \mathcal{M}(\mathbf{X}_0) \times \mathcal{M}(\mathbf{X})^3 \quad F_1 := \mathcal{C}(\mathbf{X}_0) \times \mathcal{C}(\mathbf{X})^3$
- $x = (\mu_0, \mu, \hat{\mu}, v, a) \quad c = (0, 1, 0, 0, 0) \quad b = (T \text{vol } \mathbf{X}, 0, \lambda_{\mathbf{X}})$
- the linear operator $\mathcal{A} : E_1 \rightarrow E_2$ given by

$$\mathcal{A}(\mu_0, \mu, \hat{\mu}, v, a) := \begin{bmatrix} \int_{\mathbf{X}} v + a \\ \mu + v - f_{\#} v - \mu_0 \\ \mu + \hat{\mu} \end{bmatrix}.$$

LP Primal-dual Conic Formulation

Primal LP

$$\begin{aligned} p^* &= \sup_x \langle x, c \rangle_1 \\ \text{s.t. } & Ax = b, \\ & x \in E_1^+. \end{aligned}$$

Dual LP

$$\begin{aligned} d^T &= \inf_y \langle b, y \rangle_2 \\ \text{s.t. } & A'y - c \in F_1^+. \end{aligned}$$

with

$$\blacksquare y := (u, v, w) \in \mathbb{R} \times \mathcal{C}(\mathbf{X})^2$$

$$\blacksquare A'(u, v, w) := \begin{bmatrix} -v \\ w + v \\ w \\ u + v - v \circ f \\ -v \end{bmatrix}.$$

LP Primal-dual Conic Formulation

Primal LP

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Dual LP

$$\begin{aligned} d^T &:= \inf_{u, \nu, w} \left(\int w(\mathbf{x}) + Tu \right) \lambda_{\mathbf{X}}(d\mathbf{x}) \\ \text{s.t.} \quad & \nu \in \mathcal{C}_+(\mathbf{X}_0), \\ & w - \nu - 1 \in \mathcal{C}_+(\mathbf{X}), \\ & w \in \mathcal{C}_+(\mathbf{X}), \\ & u + \nu \circ f - \nu \in \mathcal{C}_+(\mathbf{X}), \\ & u \geq 0, \\ & u \in \mathbb{R}, \nu, w \in \mathcal{C}(\mathbf{X}). \end{aligned}$$

Zero Duality Gap

Lemma

- 1 $p^T = d^T$ and \exists minimizing sequence (u_k, v_k, w_k) for dual LP.
- 2 $u_k = 0 \implies$ Volume Assumption $\implies p^T = d^T = \text{vol } \bar{X}^\infty$

Strong Convergence Properties

Strengthening of the dual LP:

$$\begin{aligned} d_r^T &:= \inf_{u,v,w} \sum_{\beta \in \mathbb{N}_{2r}^n} w_\beta z_\beta^X + u^T z_0^X \\ \text{s.t. } & v \in \mathcal{Q}_r(\mathbf{X}_0), \\ & w - v - 1 \in \mathcal{Q}_r(\mathbf{X}), \\ & u + v \circ f - v \in \mathcal{Q}_{rd}(\mathbf{X}), \\ & w \in \mathcal{Q}_r(\mathbf{X}), \\ & u \in \mathbb{R}^+. \end{aligned}$$

Strong Convergence Properties

Theorem

Assume that $\mathbf{x}^0, \mathbf{S}^T, \mathbf{X} \setminus \mathbf{S}^T$ have nonempty interior.

- 1 No duality gap between primal and dual SDP: $p_r^T = d_r^T$.

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- 2 Dual SDP has optimal solution $(\mathbf{u}_r, \mathbf{v}_r, \mathbf{w}_r)$:

$$\lim_{r \rightarrow \infty} \int_{\mathbf{X}} |\mathbf{w}_r + \mathbf{u}_r^T - \mathbf{1}_{\mathbf{S}^T}| = 0.$$

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- 4 $u_r = 0 \Rightarrow$ Volume Assumption $\Rightarrow \lim_{r \rightarrow \infty} \text{vol}(\mathbf{X}_r^\infty \setminus \mathbf{X}^\infty) = 0$.

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Application Examples

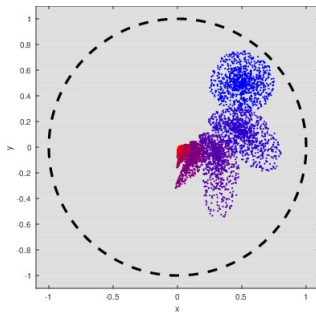
Conclusion

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leq \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2),$$

$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3),$$



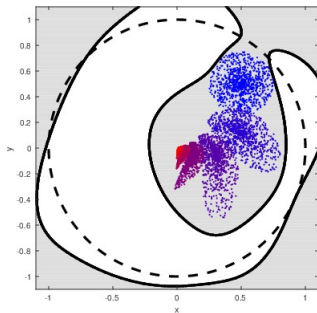
\mathbf{X}_2^∞

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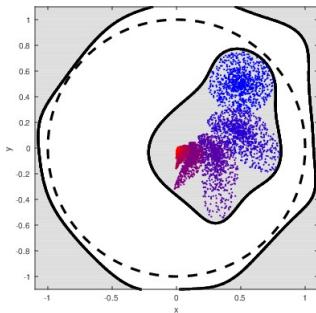
\mathbf{X}_3^∞

Toy Example

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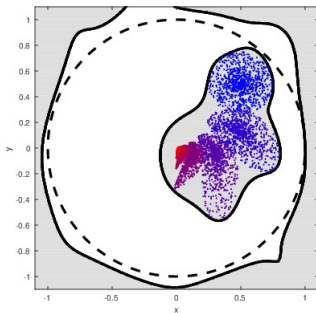
\mathbf{X}_4^∞

Toy Example

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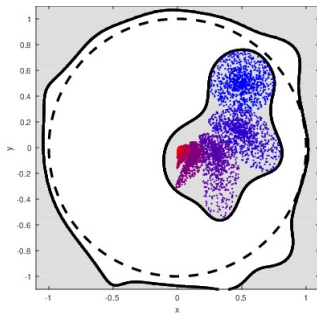
\mathbf{X}_5^∞

Toy Example

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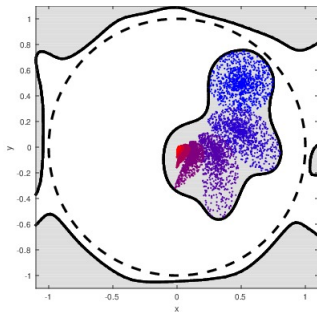
\mathbf{X}_6^∞

Toy Example

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$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2),$$

$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3),$$



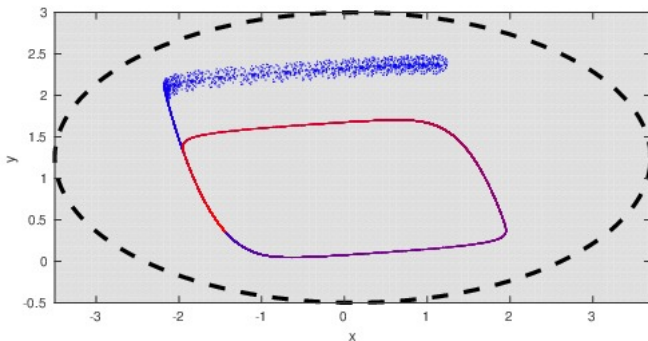
\mathbf{X}_7^∞

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{x}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$



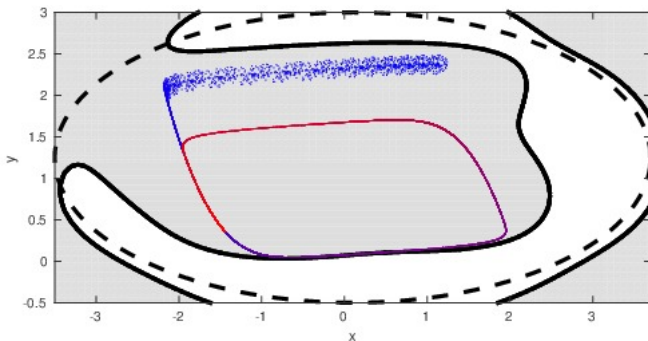
\mathbf{x}_2^∞

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$



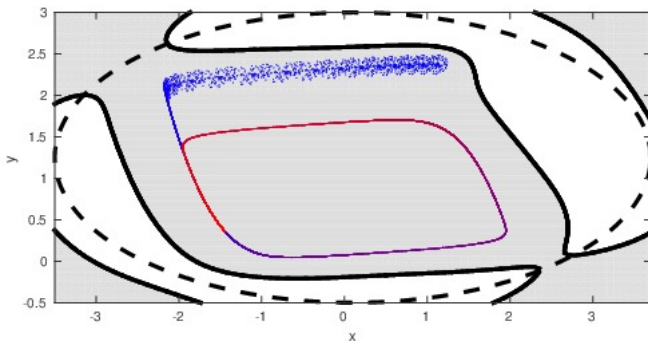
\mathbf{X}_3^∞

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$



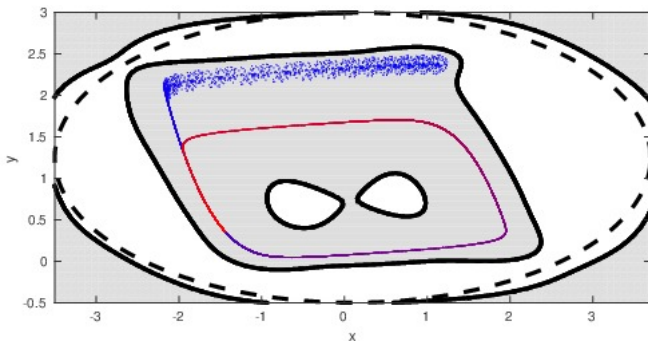
\mathbf{X}_4^∞

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$



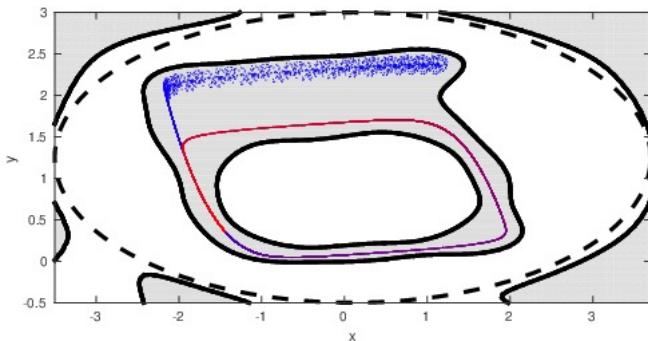
\mathbf{X}_5^∞

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$



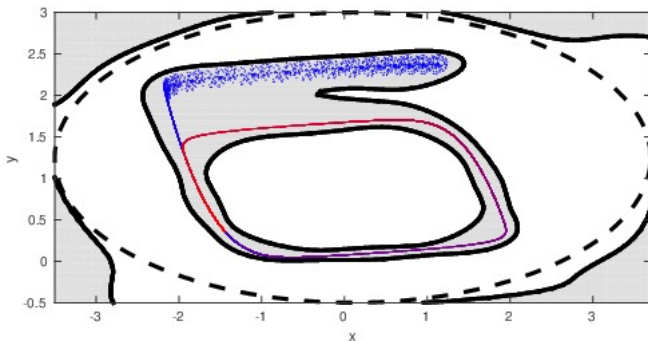
\mathbf{X}_6^∞

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875),$$

$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2)),$$

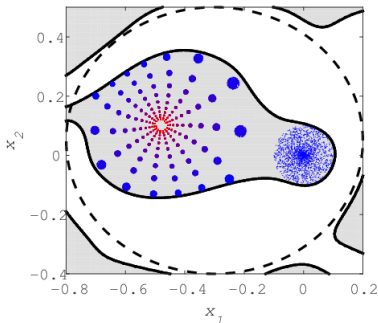


\mathbf{X}_7^∞

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1,$$

$$x_2^+ := 2x_1x_2 + c_2,$$



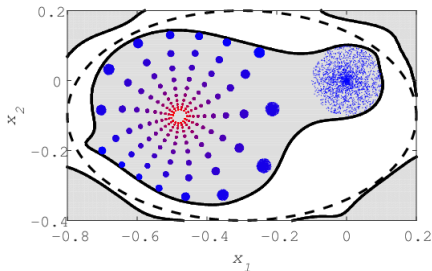
\mathbf{X}_5^∞ with $c_1 = -0.7$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1,$$

$$x_2^+ := 2x_1x_2 + c_2,$$



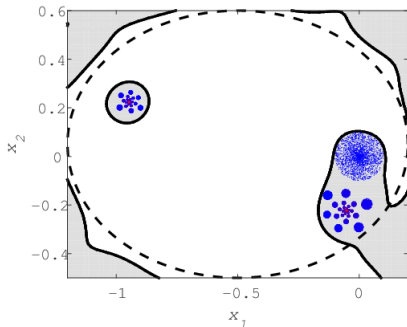
\mathbf{X}_5^∞ with $c_1 = -0.7$ and $c_2 = -0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1,$$

$$x_2^+ := 2x_1x_2 + c_2,$$



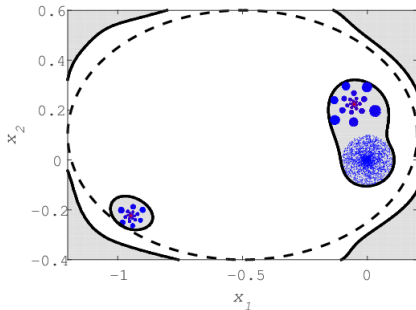
\mathbf{X}_5^∞ with $c_1 = -0.9$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1,$$

$$x_2^+ := 2x_1x_2 + c_2,$$



\mathbf{X}_5^∞ with $c_1 = -0.9$ and $c_2 = -0.2$

The RS Problem

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for Reachable Sets (RS)

Application Examples

Conclusion

Conclusion

- ⊕ Certified Approximation of the **whole reachable set** X^∞
- ⊖ Computational complexity: $\binom{n+2rd}{n}$ SDP variables
- ⊕ **Structure sparsity** can be exploited

Conclusion

Further research:

- **Volume Assumption:** $\lim_{T \rightarrow \infty} \sum_{t=0}^T t \text{ vol } \mathbf{Y}^t \leq \infty$
always true?
- finite time, continuous setting?
 - 💡 Use previous framework approximating:
 - 1 region of attraction
 - 2 maximum controlled invariant

Bibliography



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End

Thank you for your attention!

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