Semidefinite Approximations of Projections and Polynomial Images of Semialgebraic Sets

Victor Magron, CNRS VERIMAG

joint work with Didier Henrion and Jean-Bernard Lasserre (LAAS)

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The Problem

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$
- A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$,
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$
- $\deg f = d := \max\{\deg f_1, \ldots, \deg f_m\}$
- $\mathbf{F} := f(\mathbf{S}) \subseteq \mathbf{B}$, with $\mathbf{B} \subset \mathbb{R}^m$ a box or a ball
- Tractable approximations of $\mathbf{F}$?

The Problem

- Includes important special cases:

1. $m = 1$: **polynomial optimization**

$$\mathbf{F} \subseteq [\inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}), \sup_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})]$$

2. Approximate **projections** of $\mathbf{S}$ when $f(\mathbf{x}) := (x_1, \ldots, x_m)$

3. **Pareto curve** approximations

   For f_1,f_2 two conflicting criteria: $(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x})\, f_2(\mathbf{x}))^\top \right\}$

The Problem

3 **Pareto curve**: set of *weakly Edgeworth-Pareto optimal points*

$$(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x})\, f_2(\mathbf{x}))^\top \right\}$$

Definition

A point $\bar{\mathbf{x}} \in \mathbf{S}$ is called a *weakly Edgeworth-Pareto (EP) optimal point* of Problem $\mathbf{P}$, when there is no $\mathbf{x} \in \mathbf{S}$ such that $f_j(\mathbf{x}) < f_j(\bar{\mathbf{x}}),\ \ j = 1, 2.$

The Problem

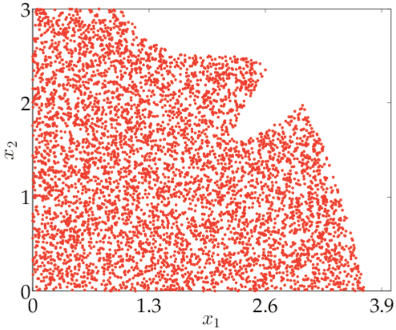
$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5$,

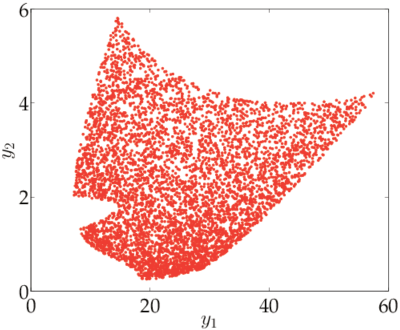
$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85$,

$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\}$.

$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2$,

$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4$.





Previous work

1. Exact description of projections with computer algebra

   - Real quantifier elimination (QE) [Tarski 51, Collins 74, Bochnak-Coste-Roy 98]
   - CAD: computational complexity $(sd)^{2^{O(n)}}$ for a finite set of s polynomials
   - Variant QE under radicality, equidimensionality [Hong-Safey 12]

Previous work

2 Scalarization methods for computing Pareto curve

- Numerical discretization schemes: modified Polak method [Pol 76]
- Iterative Eichfelder-Polak algorithm [Eich 09]
- Normal-boundary intersection method to find uniform spread of points [Das Dennis 98]

Contribution

- A unifying framework to handle projections, Pareto curve approximations and other applications

- **No discretization** is required

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- **No discretization** is required
- Two different methods:
  1. Existential QE: $\mathbf{F} \subseteq \mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\}$
  2. Image measure supports: $\mathbf{F} \subseteq \mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$

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- Strong convergence guarantees

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- Strong convergence guarantees

- Compute q_k or w_k with **Semidefinite programming** (SDP)

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Polynomial Optimization

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

- $p^* := \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})$: NP hard

- Sums of squares $\Sigma[\mathbf{x}]$
  e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$

- $\mathcal{Q}(\mathbf{S}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^{l} \sigma_j(\mathbf{x}) g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$

- **REMEMBER**: $f \in \mathcal{Q}(\mathbf{S}) \Longrightarrow \forall \mathbf{x} \in \mathbf{S}, f(\mathbf{x}) \geqslant 0$

Problem reformulation

- Borel σ-algebra $\mathcal{B}$ (generated by the open sets of $\mathbb{R}^n$)
- $\mathcal{M}_+(\mathbf{S})$: set of probability measures supported on $\mathbf{S}$.
  If $\mu \in \mathcal{M}_+(\mathbf{S})$ then
  1. $\mu : \mathcal{B} \to [0,1]$, $\mu(\emptyset) = 0$
  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any countable $(B_i) \subset \mathcal{B}$
  3. $\int_{\mathbf{S}} \mu(d\mathbf{x}) = 1$
- $\text{supp}(\mu)$ is the smallest set $\mathbf{S}$ such that $\mu(\mathbb{R}^n \backslash \mathbf{S}) = 0$

Problem reformulation

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \int_{\mathbf{S}} f \, d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{S}$ if there exists a finite measure μ supported on $\mathbf{S}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{S}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n .$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{S})$: space of probability measures supported on $\mathbf{S}$
- $\mathcal{Q}(\mathbf{S})$: quadratic module

Polynomial Optimization Problems (POP)

$$
\begin{array}{lll}
 & \text{(Primal)} & \\
\inf & \int_{\mathbf{S}} f \, d\mu & \\
\text{s.t.} & \mu \in \mathcal{M}_+(\mathbf{S}) &
\end{array}
\quad = \quad
\begin{array}{ll}
 & \text{(Dual)} \\
\sup & \lambda \\
\text{s.t.} & \lambda \in \mathbb{R} \, , \\
 & f - \lambda \in \mathcal{Q}(\mathbf{S})
\end{array}
$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{S})$
- Truncated quadratic module $\mathcal{Q}_k(\mathbf{S}) := \mathcal{Q}(\mathbf{S}) \cap \mathbb{R}_{2k}[\mathbf{x}]$

Polynomial Optimization Problems (POP)

$$\begin{aligned}
&\text{(Moment)} && && \text{(SOS)} \\
&\inf \quad \sum_\alpha f_\alpha \, \mathbf{z}_\alpha && = && \sup \quad \lambda \\
&\text{s.t.} \quad \mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) \succcurlyeq 0 \,, \quad 0 \leqslant j \leqslant l, && && \text{s.t.} \quad \lambda \in \mathbb{R} \;, \\
&\qquad \mathbf{z}_1 = 1 && && \qquad f - \lambda \in \mathcal{Q}_k(\mathbf{S})
\end{aligned}$$

Lasserre's Hierarchy of SDP relaxations

$$\ell_{\mathbf{z}}(q) : q \in \mathbb{R}[\mathbf{x}] \mapsto \sum_{\alpha} q_\alpha \mathbf{z}_\alpha$$

- Moment matrix
  $\mathbf{M}(\mathbf{z})_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := \ell_{\mathbf{z}}(\mathbf{x}^\alpha \, \mathbf{x}^\beta) = \mathbf{z}_{\alpha+\beta}$
- Localizing matrix $\mathbf{M}(g_j \, \mathbf{z})$ associated with g_j
  $\mathbf{M}(g_j \, \mathbf{z})_{\mathbf{x}^\alpha, \mathbf{x}^\beta} := \ell_{\mathbf{z}}(g_j \, \mathbf{x}^\alpha \, \mathbf{x}^\beta) = \sum_\gamma g_{j,\gamma} \, \mathbf{z}_{\alpha+\beta+\gamma}$

Lasserre's Hierarchy of SDP relaxations

- $\mathbf{M}_k(\mathbf{z})$ contains $\binom{n+2k}{n}$ variables, has size $\binom{n+k}{n}$
- Truncated matrix of order $k = 2$ with variables x_1, x_2:

$$
\begin{array}{c c}
 & \begin{array}{ccccccccc} 1 & | & x_1 & x_2 & | & x_1^2 & x_1x_2 & x_2^2 \end{array} \\
\mathbf{M}_2(\mathbf{z}) = \begin{array}{c} 1 \\ - \\ x_1 \\ x_2 \\ - \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} &
\left(\begin{array}{cccccccccc}
1 & | & z_{1,0} & z_{0,1} & | & z_{2,0} & z_{1,1} & z_{0,2} \\
- & - & - & - & - & - & - & \\
z_{1,0} & | & z_{2,0} & z_{1,1} & | & z_{3,0} & z_{2,1} & z_{1,2} \\
z_{0,1} & | & z_{1,1} & z_{0,2} & | & z_{2,1} & z_{1,2} & z_{0,3} \\
- & - & - & - & - & - & - & - \\
z_{2,0} & | & z_{3,0} & z_{2,1} & | & z_{4,0} & z_{3,1} & z_{2,2} \\
z_{1,1} & | & z_{2,1} & z_{1,2} & | & z_{3,1} & z_{2,2} & z_{1,3} \\
z_{0,2} & | & z_{1,2} & z_{0,3} & | & z_{2,2} & z_{1,3} & z_{0,4}
\end{array}\right)
\end{array}
$$

Lasserre's Hierarchy of SDP relaxations

- Consider $g_1(\mathbf{x}) := 2 - x_1^2 - x_2^2$. Then $v_1 = \lceil \deg g_1 / 2 \rceil = 1$.

$$\mathbf{M}_1(g_1\, \mathbf{z}) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x_1 & x_2 \end{array} \\ \begin{pmatrix} 2 - z_{2,0} - z_{0,2} & 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{0,1} - z_{2,1} - z_{0,3} \\ 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{2,0} - z_{4,0} - z_{2,2} & 2z_{1,1} - z_{3,1} - z_{1,3} \\ 2z_{0,1} - z_{2,1} - z_{0,3} & 2z_{1,1} - z_{3,1} - z_{1,3} & 2z_{0,2} - z_{2,2} - z_{0,4} \end{pmatrix} \end{array}$$

$$\begin{aligned} \mathbf{M}_1(g_1\, \mathbf{z})(3,3) = \ell(g_1(\mathbf{x}) \cdot x_2 \cdot x_2) &= \ell(2x_2^2 - x_1^2 x_2^2 - x_2^4) \\ &= 2z_{0,2} - z_{2,2} - z_{0,4} \end{aligned}$$

Lasserre's Hierarchy of SDP relaxations

- Truncation with moments of order at most $2k$
- $v_j := \lceil \deg g_j / 2 \rceil$
- Hierarchy of semidefinite relaxations:

$$\begin{cases} \inf_{\mathbf{z}} \ell_{\mathbf{z}}(f) & = & \sum_\alpha \int_{\mathbf{S}} f_\alpha \, \mathbf{x}^\alpha \, \mu(d\mathbf{x}) = \sum_\alpha f_\alpha \, \mathbf{z}_\alpha \\ \mathbf{M}_k(\mathbf{z}) & \succcurlyeq & 0 \,, \\ \mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) & \succcurlyeq & 0 \,, \quad 1 \leqslant j \leqslant l, \\ \mathbf{z}_1 & = & 1 \,. \end{cases}$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$(SDP) \begin{cases} \mathcal{P}: & \inf_{\mathbf{z}} & \sum_\alpha c_\alpha \mathbf{z}_\alpha \\ & \text{s.t.} & \sum_\alpha F_\alpha \mathbf{z}_\alpha - F_0 \succcurlyeq 0 \\ & & \\ \mathcal{D}: & \sup_{\mathbf{Y}} & \text{Trace}\,(F_0\, \mathbf{Y}) \\ & \text{s.t.} & \text{Trace}\,(F_\alpha\, \mathbf{Y}) = c_\alpha \ , \quad \mathbf{Y} \succcurlyeq 0 \ . \end{cases}$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

The Problem

$m = 1$: Polynomial Optimization

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Method 2: support of image measures

Application examples

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Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } f(\mathbf{x}) = \mathbf{y}\} \ ,$$

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } \|\mathbf{y} - f(\mathbf{x})\|_2^2 = 0\} \ ,$$

Approximation of sets defined with "∃"

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } h_f(\mathbf{x}, \mathbf{y}) \geqslant 0\} \ ,$$

with

$$h_f(\mathbf{x}, \mathbf{y}) := -\|\mathbf{y} - f(\mathbf{x})\|_2^2 \ .$$

Approximation of sets defined with “∃”

Existential QE: approximate $\mathbf{F}$ as closely as desired [Lasserre 14]

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \ ,$$

for some polynomials $q_k \in \mathbb{R}_{2k}[\mathbf{y}]$.

A hierarchy of outer approximations of F

- Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k-truncated quadratic module
- **REMEMBER**:
  $q - h_f \in \mathcal{Q}_k(\mathbf{K}) \implies \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geqslant 0$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$
- Hierarchy of Semidefinite programs:

$$\inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K}) \right\} .$$

A hierarchy of outer approximations of F

Assuming the existence of solution q_k, the sublevel sets

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \supseteq \mathbf{F} \ ,$$

provide a sequence of certified outer approximations of $\mathbf{F}$.

A hierarchy of outer approximations of F

Assuming the existence of solution q_k, the sublevel sets

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provide a sequence of certified outer approximations of $\mathbf{F}$.

It comes from the following:

- q_k feasible solution, $q_k - h_f \in \mathcal{Q}_k(\mathbf{K})$
- $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q_k(\mathbf{y}) \geqslant h_f(\mathbf{x}, \mathbf{y}) \Longleftrightarrow \forall \mathbf{y}, q_k(\mathbf{y}) \geqslant h(\mathbf{y}) \ .$

Strong convergence property

Theorem

Assuming that $\mathring{\mathbf{S}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{K})$ is Archimedean,

1. The sequence of optimal solutions (q_k) converges to h w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k\to\infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0 \ , (q_k \to_{L_1} h)$$

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2.

$$\lim_{k \to \infty} \mathrm{vol}(\mathbf{F}_k^1 \backslash \mathbf{F}) = 0 \ .$$

Strong convergence property

Proof of existence

1. Existence of optimal q_k by **Slater's condition**

Strong convergence property

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   - Dual SDP:

$$\begin{aligned} \rho_k^* := \sup_{\mathbf{z}} \quad & \ell_{\mathbf{z}}(h_f) \\ \text{s.t.} \quad & \mathbf{M}_k(\mathbf{z}) \succcurlyeq 0, \\ & \mathbf{M}_{k-v_j}(g_j\,\mathbf{z}) \succcurlyeq 0, \quad j = 1, \ldots, l, \\ & \ell_{\mathbf{z}}(\mathbf{y}^\beta) = z_\beta^{\mathbf{B}}, \quad \forall \beta \in \mathbb{N}_{2k}^m. \end{aligned}$$

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\end{aligned}
$$

   - Strictly feasible $\mathbf{z}$: moments of Lebesgue measure $\lambda_{\mathbf{K}}$
   - $q = 0$ feasible for Primal SDP:

$$
\rho_k := \inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K})) \right\} .
$$

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   - By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \to_{L_1} h$

Strong convergence property

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   - By Monotone Convergence Theorem, $f_k \to_{L_1} h$.
   - By Stone-Weierstrass Theorem existence of p_k s.t. $p_k \to_{L_1} h$
   - Apply Putinar's Positivstellensatz to $p_k - h_f + \epsilon / \operatorname{vol}(\mathbf{B})$:

$$p_k - h_f + \epsilon / \operatorname{vol}(\mathbf{B}) = \sum_{j=0}^{l} \sigma_j \, g_j$$

Strong convergence property

Proof of volume convergence

2. Define $\mathbf{F}(r) := \{\mathbf{y} \in \mathbf{B} : h(\mathbf{y}) \geqslant -1/r\}$

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Strong convergence property

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   - $\operatorname{vol} \mathbf{F}(r) \to \operatorname{vol} \mathbf{F}$
   - $\lim_{k \to \infty} \operatorname{vol} \mathbf{F}_k^1 \leqslant \operatorname{vol} \mathbf{F}(r)$

Strong convergence property

Proof of volume convergence

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   - $\operatorname{vol} \mathbf{F}(r) \to \operatorname{vol} \mathbf{F}$

   - $\lim_{k\to\infty} \operatorname{vol} \mathbf{F}_k^1 \leqslant \operatorname{vol} \mathbf{F}(r)$

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The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Infinite dimensional LP formulation

- **Pushforward** $f_{\#} : \mathcal{M}(\mathbf{S}) \to \mathcal{M}(\mathbf{B})$:

$$f_{\#}\mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{S} : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{B}), \forall \mu_0 \in \mathcal{M}(\mathbf{S})$$

- $f_{\#}\mu_0$ is the **image measure** of μ_0 under f

Infinite dimensional LP formulation

$$
\begin{aligned}
p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \quad & \int_{\mathbf{B}} \mu_1 \\
\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}} \,, \\
& \mu_1 = f_{\#} \mu_0 \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}) \,.
\end{aligned}
$$

Lebesgue measure on $\mathbf{B}$ is $\lambda_{\mathbf{B}}(d\mathbf{y}) := \mathbf{1}_{\mathbf{B}}(\mathbf{y})\, d\mathbf{y}$

Infinite dimensional LP formulation

$$
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\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}} \,, \\
& \mu_1 = f_{\#} \mu_0 \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}) \,.
\end{aligned}
$$

Lemma

Let μ_1^* be an optimal solution of the above LP.
Then $\mu_1^* = \lambda_{\mathbf{F}}$ and $p^* = \text{vol}\, \mathbf{F}$.

LP Primal-dual conic formulation

The LP can be cast as follows:

$$\begin{aligned} p^* = \sup_{x} \quad & \langle x, c \rangle_1 \\ \text{s.t.} \quad & \mathcal{A}\, x = b\,, \\ & x \in E_1^+\,, \end{aligned}$$

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$$

with

- $E_1 := \mathcal{M}(\mathbf{S}) \times \mathcal{M}(\mathbf{B})^2 \quad F_1 := \mathcal{C}(\mathbf{S}) \times \mathcal{C}(\mathbf{B})^2$
- $x := (\mu_0, \mu_1, \hat{\mu}_1) \quad c := (0, 1, 0) \in F_1 \quad b := (0, \lambda_{\mathbf{B}})$
- the linear operator $\mathcal{A} : E_1 \to E_2$ given by

$$
\mathcal{A}\,(\mu_0, \mu_1, \hat{\mu}_1) := \begin{bmatrix} -f_{\#}\mu_0 + \mu_1 \\ \mu_1 + \hat{\mu}_1 \end{bmatrix}.
$$

LP Primal-dual conic formulation

Primal LP

$$
\begin{aligned}
p^* = \sup_{x} \quad & \langle x, c \rangle_1 \\
\text{s.t.} \quad & \mathcal{A}\, x = b \,, \\
& x \in E_1^+ \,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^* = \inf_{y} \quad & \langle b, y \rangle_2 \\
\text{s.t.} \quad & \mathcal{A}'\, y - c \ \in \mathcal{C}_+(\mathbf{B})^2 \,.
\end{aligned}
$$

with

- $y := (v, w) \in \mathcal{M}(\mathbf{B})^2$
- $\mathcal{A}'\,(v, w) := \begin{bmatrix} -v \circ f \\ v + w \\ w \end{bmatrix}.$

LP Primal-dual conic formulation

Primal LP

$$
\begin{aligned}
p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \quad & \int \mu_1 \\
\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}}\,, \\
& \mu_1 = f_{\#}\mu_0\,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S})\,, \\
& \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B})\,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^* := \inf_{v,w} \quad & \int w(\mathbf{y})\, \lambda_{\mathbf{B}}(d\mathbf{y}) \\
\text{s.t.} \quad & v(f(\mathbf{x})) \geqslant 0, \quad \forall \mathbf{x} \in \mathbf{S}\,, \\
& w(\mathbf{y}) \geqslant 1 + v(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbf{B}\,, \\
& w(\mathbf{y}) \geqslant 0, \quad \forall \mathbf{y} \in \mathbf{B}\,, \\
& v, w \in \mathcal{C}(\mathbf{B})\,.
\end{aligned}
$$

Zero duality gap

Lemma

$p^* = d^*$

Strong convergence property

Strengthening of the dual LP:

$$
\begin{aligned}
d_k^* := \inf_{v,w} \quad & \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}} \\
\text{s.t.} \quad & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\
& w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), \\
& w \in \mathcal{Q}_k(\mathbf{B}), \\
& v, w \in \mathbb{R}_{2k}[\mathbf{y}].
\end{aligned}
$$

Strong convergence property

Theorem

Assuming that $\mathring{\mathbf{F}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

1. The sequence (w_k) converges to $\mathbf{1}_{\mathbf{F}}$ w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 \enspace .$$

Strong convergence property

Theorem

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$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 \ .$$

2. Let $\mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$. Then,

$$\lim_{k \to \infty} \mathrm{vol}(\mathbf{F}_k^2 \backslash \mathbf{F}) = 0 \ .$$

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

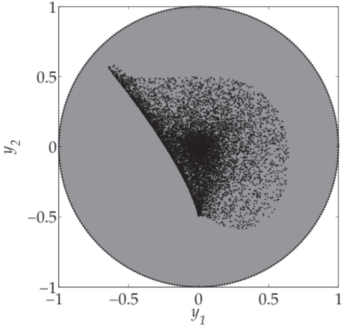
Application examples

Conclusion

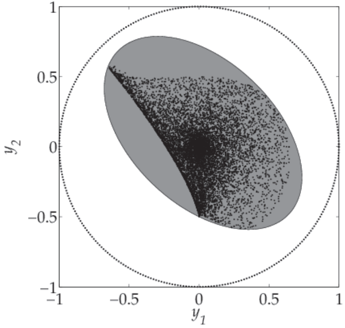
Polynomial image of the unit ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1 x_2, x_2 - x_1^3)/2$$



$\mathbf{F}_1^1$



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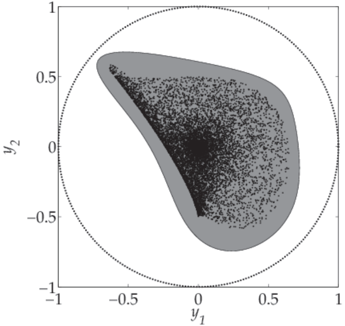
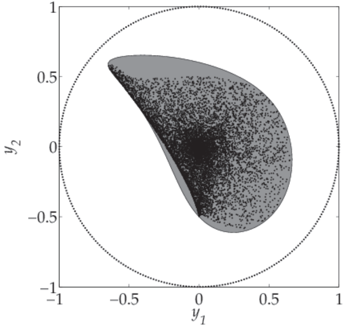


$\mathbf{F}_2^1$



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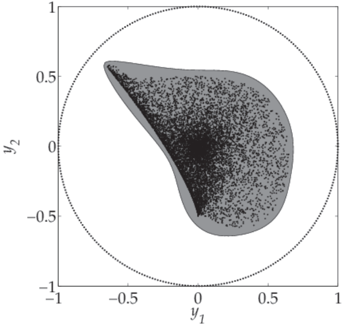
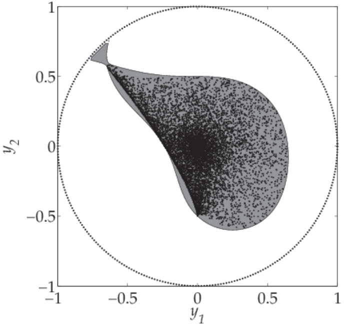


$\mathbf{F}_3^1$



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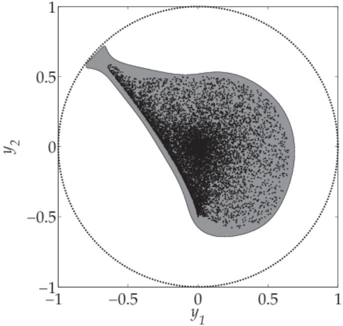
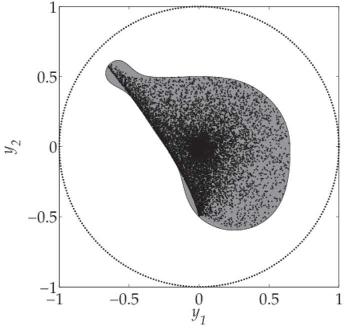


$\mathbf{F}_4^1$



$\mathbf{F}_4^2$

Semialgebraic set projections

Simpler formulation:

$$
\begin{aligned}
d_k^* := \inf_{v,w} \quad & \sum_{\beta \in \mathbb{N}^m_{2k}} w_\beta z_\beta^{\mathbf{B}} \\
\text{s.t.} \quad & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\
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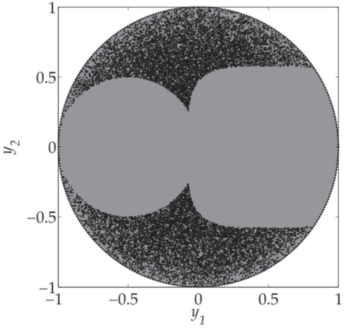
& w \in \mathcal{Q}_k(\mathbf{B}), \\
& v, w \in \mathbb{R}_{2k}[\mathbf{y}].
\end{aligned}
\qquad
\begin{aligned}
\inf_{w} \quad & \sum_{\beta \in \mathbb{N}^m_{2k}} w_\beta z_\beta^{\mathbf{B}} \\
\text{s.t.} \quad & w - 1 \in \mathcal{Q}_k(\mathbf{S}), \\
& w \in \mathcal{Q}_k(\mathbf{B}), \\
& w \in \mathbb{R}_{2k}[x_1, \ldots, x_m].
\end{aligned}
$$

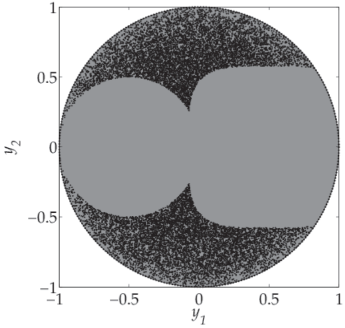
Semialgebraic set projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on $\mathbb{R}^2$ of the semialgebraic set

$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leqslant 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geqslant 0,$$
$$1/9 - (x_1 - 1/2)^4 - x_2^4 \geqslant 0\}$$



$\mathbf{F}_2^1$



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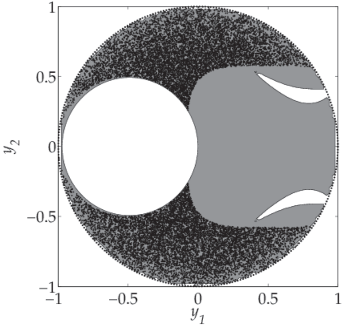
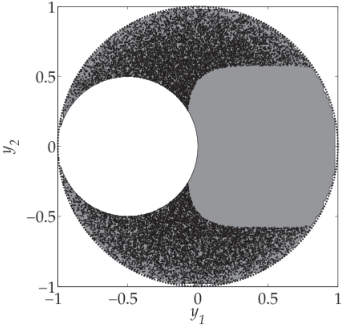


$\mathbf{F}_3^1$



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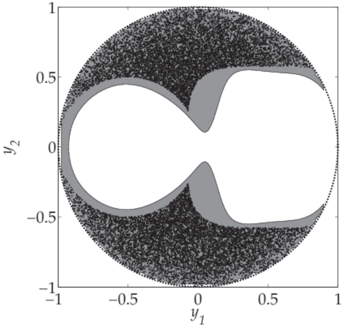
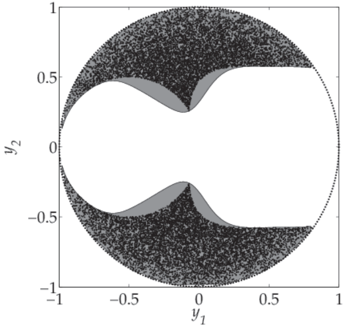


$\mathbf{F}_4^1$

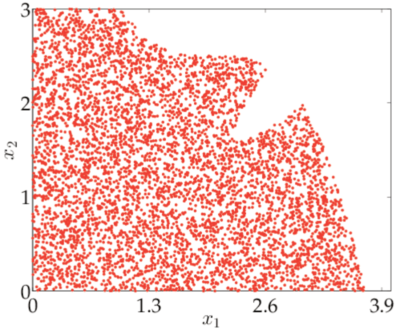


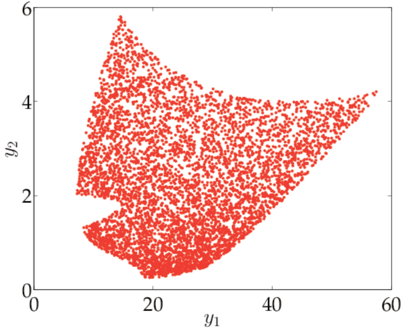
$\mathbf{F}_4^2$

Bicriteria Optimization Problems

$$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 \ ,$$
$$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 \ ,$$
$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\} \ .$$

$$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 \ ,$$
$$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 \ .$$





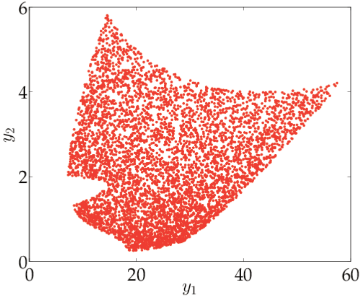
Previous Contributions

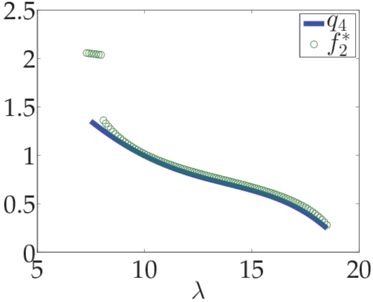
- Numerical schemes that **avoid computing finitely many points**.

- Pareto curve approximation with polynomials, **convergence guarantees** in L_1-norm

V. Magron, D. Henrion, J.B. Lasserre. Approximating Pareto Curves using Semidefinite Relaxations. *Operations Research Letters*. arxiv:1404.4772, April 2014.

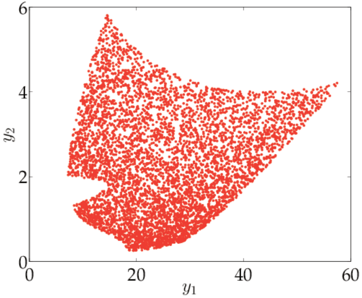
Previous Contributions

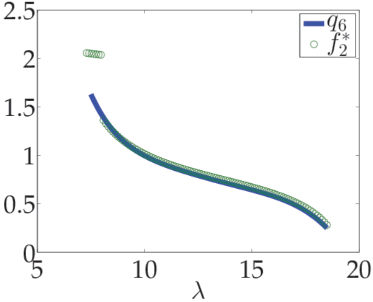




Degree 4

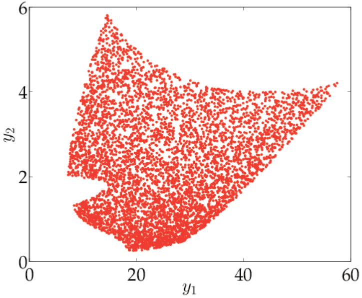
Previous Contributions

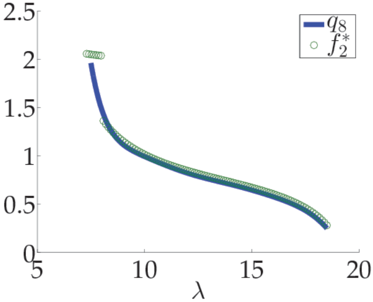




Degree 6

Previous Contributions

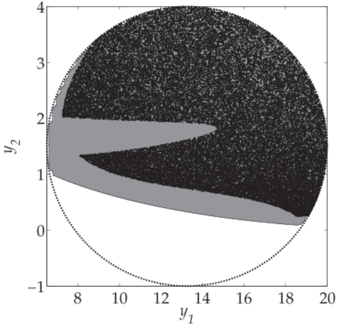




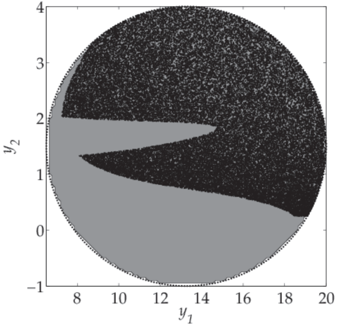
Degree 8

Approximating Pareto curves

Back on our previous nonconvex example:



$\mathbf{F}_1^1$



$\mathbf{F}_1^2$

Approximating Pareto curves

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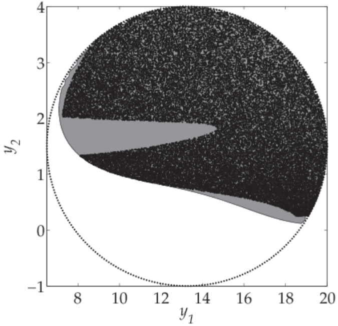
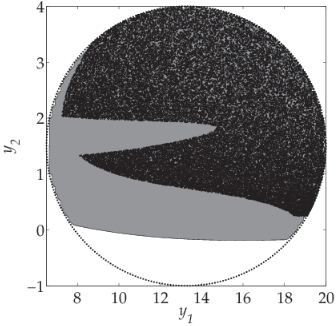


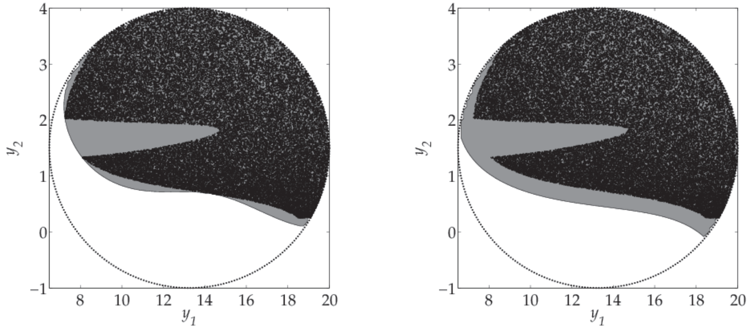
$\mathbf{F}_2^1$



$\mathbf{F}_2^2$

Approximating Pareto curves

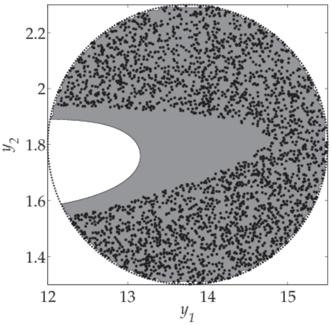
Back on our previous nonconvex example:



$\mathbf{F}_3^1$ $\mathbf{F}_3^2$

Approximating Pareto curves

"Zoom" on the region which is hard to approximate:



$\mathbf{F}_4^1$

Approximating Pareto curves

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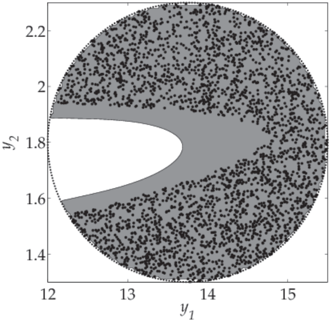
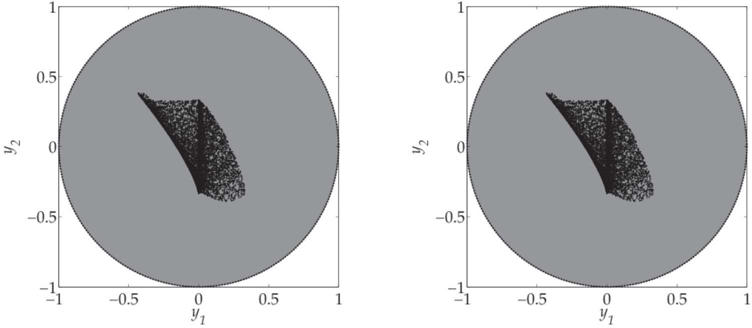


$\mathbf{F}_5^1$

Semialgebraic image of semialgebraic sets

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by

$$f(\mathbf{x}) := (\min(x_1 + x_1 x_2, x_1^2), x_2 - x_1^3)/3$$



$\mathbf{F}_1^1$ $\mathbf{F}_1^2$

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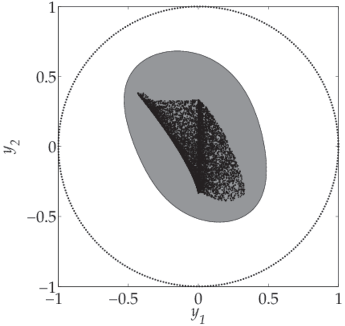
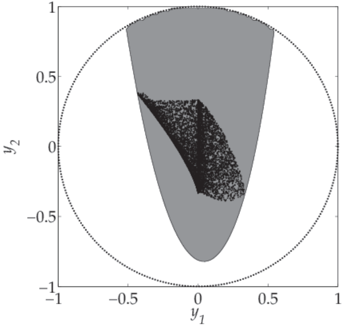


$\mathbf{F}_2^1$



$\mathbf{F}_2^2$

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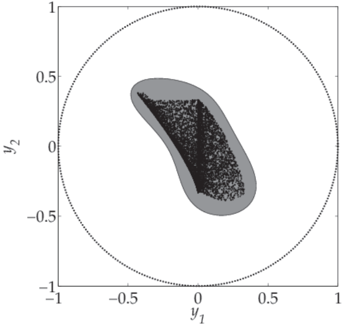
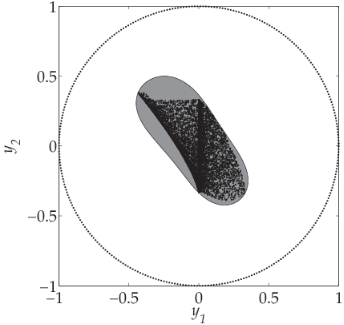


$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

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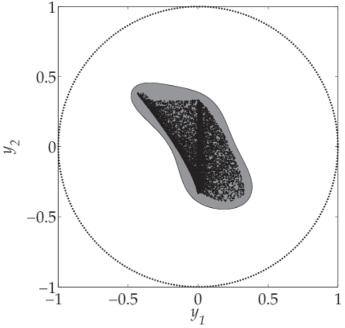
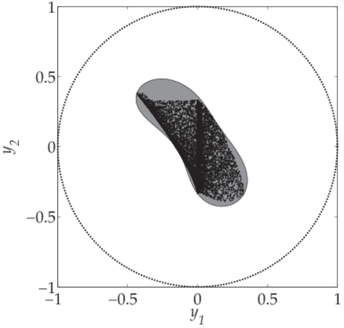


$\mathbf{F}_4^1$



$\mathbf{F}_4^2$

The Problem

$m = 1$: Polynomial Optimization

Method 1: existential quantifier elimination

Method 2: support of image measures

Application examples

Conclusion

Conclusion

- **Unifying** framework: projections, Pareto curves
- Computational complexity
  1. Method 1: $\binom{n+m+2k}{2k}$ SDP variables
  2. Method 2: $\binom{n+2kd}{2kd}$ SDP variables
- **Structure sparsity** can be exploited

Conclusion

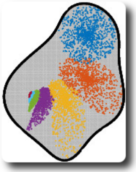
Further research:

- Alternative positivity certificates LP/SDP
  1. Less computationally demanding than SDP
  2. More efficient than LP (as generic convergence cannot occur)

Conclusion

Further research:

Discrete-time polynomial systems: $\mathbf{x}_{k+1} = f(\mathbf{x}_k)$



certified/convergent SDP hierarchies

image measure supports conservation eqs.

End

V. Magron, D. Henrion, J.B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. *SIAM J. Optimization*, 2015.

Thank you for your attention!

`http://www-verimag.imag.fr/~magron`