Exact Polynomial Optimization via SOS, SONC and SAGE Decompositions

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Joint work with
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A^3 - Arctic Applied Algebra, Tromsø
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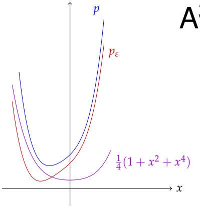

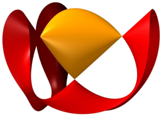

Deciding Nonnegativity & Exact Optimization

$X = (X_1, \dots, X_n)$ **co-NP hard problem: check** $f \geqslant 0$ **on** $\mathbf{K}$

$f \in \mathbb{Q}[X]$ **NP hard problem:** $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$

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1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

2. Constrained
   $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\} \quad g_j \in \mathbb{Q}[X]$

$\deg f, \deg g_j \leqslant d$

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[Collins 75] CAD **doubly exp. in** n **poly. in** d

[Grigoriev-Vorobjov 88, Basu-Pollack-Roy 98]
Critical points **singly exponential time** $(m+1)\,\tau\, d^{\mathcal{O}(n)}$

Deciding Nonnegativity & Exact Optimization

Sums of squares (SOS)

$\sigma = {h_1}^2 + \cdots + {h_p}^2$

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HILBERT 17TH PROBLEM: f SOS of rational functions?

[Artin 27] **YES**!

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$$\boxed{\simeq \quad \rightarrow \quad =}$$

The Question of Exact Certification

How to go from **approximate** to **exact** certification?

Decomposing Nonnegative Polynomials

1. **Reznick**'s representation $f = \frac{\sigma}{(X_1^2 + \cdots + X_n^2)^D}$
   positive definite form f
   [Reznick 95]

2. **Hilbert-Artin**'s representation $f = \frac{\sigma}{h^2}$
   $f \geqslant 0$
   [Artin 27]

3. **Putinar**'s representation
   $f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m$ $f > 0$ on compact K
   $\deg \sigma_i \leqslant 2D$
   [Putinar 93]

Decomposing Nonnegative Polynomials

- Deciding **polynomial nonnegativity**

$$f(a,b) = a^2 - 2ab + b^2 \geqslant 0$$

- $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$

- $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Decomposing Nonnegative Polynomials

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve **SDP**

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A} \, \mathbf{z} = \mathbf{d}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2$

- Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

Decomposing Nonnegative Polynomials

4. **Circuit** polynomial

$$f = b_{\boldsymbol{\alpha}(1)} X^{\boldsymbol{\alpha}(1)} + \cdots + b_{\boldsymbol{\alpha}(r)} X^{\boldsymbol{\alpha}(r)} + b_{\boldsymbol{\beta}} X^{\boldsymbol{\beta}}$$

$b_{\boldsymbol{\alpha}(j)} > 0 \quad \boldsymbol{\alpha}(j) \in (2\mathbb{N})^n$

$\boldsymbol{\beta} = \lambda_1 \boldsymbol{\alpha}(1) + \cdots + \lambda_r \boldsymbol{\alpha}(r) \quad \lambda_j > 0$ and $\lambda_1 + \cdots + \lambda_r = 1$

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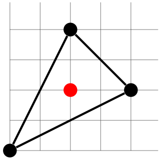
$$f = 1 + X_1^2 X_2^4 + X_1^4 X_2^2 - 3 X_1^2 X_2^2$$



Decomposing Nonnegative Polynomials

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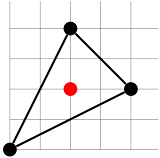

Circuit number $\Theta_f = \prod_{j=1}^{r} \left(\frac{b_{\alpha(j)}}{\lambda_j} \right)^{\lambda_j}$

Decomposing Nonnegative Polynomials

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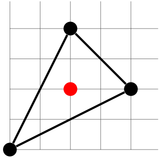

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Theorem (Illiman-de Wolff 16)

$f \geqslant 0 \Leftrightarrow |b_{\beta}| \leqslant \Theta_f$ or ($b_{\beta} \geqslant -\Theta_f$, β even)

SONC (SUMS OF NONNEGATIVE CIRCUITS)

Decomposing Nonnegative Polynomials

5 **arithmetic-geometric-mean-exponential (AGE)**

$f = c_1 \exp[X \cdot \boldsymbol{\alpha}(1)] + \cdots + c_t \exp[X \cdot \boldsymbol{\alpha}(t)] + \beta \exp[X \cdot \boldsymbol{\alpha}(0)]$

$c_j \in \mathbb{Q}_{>0} \quad \beta \in \mathbb{Q} \quad \boldsymbol{\alpha}(j) \in \mathbb{N}^n$

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Theorem (Chandrasekaran-Shah 16)

$f \geqslant 0 \Leftrightarrow \exists \boldsymbol{\nu} \mid D(\boldsymbol{\nu}, \mathbf{c}) \leqslant \beta$ and $\sum_j \boldsymbol{\alpha}(j) \nu_j = (\mathbf{1} \cdot \boldsymbol{\nu}) \boldsymbol{\alpha}(0)$

SAGE (SUMS OF AGE)

From Approximate to Exact Solutions

Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$
$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \to \quad =}$?

Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geqslant 0$ on $\mathbb{R}$ ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = {f_1}^2 + {f_2}^2$.

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Proof.

$f = h^2(q + ir)(q - ir)$ □

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

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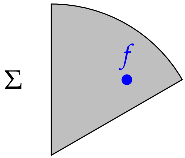
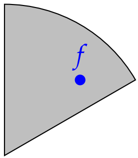
Round & Project Algorithm [Peyrl-Parrilo 08]



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Round & Project Algorithm [Peyrl-Parrilo 08]

Σ



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Find $\tilde{\mathbf{G}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{G}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{G}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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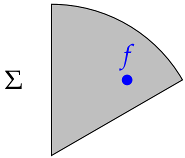

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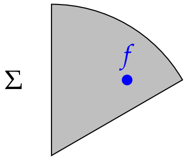

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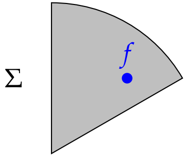

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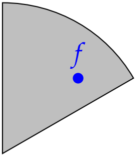

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$$G_{\alpha,\beta} \leftarrow \hat{G}_{\alpha,\beta} - \frac{1}{\eta(\alpha+\beta)} \left(\sum_{\alpha'+\beta'=\alpha+\beta} \hat{G}_{\alpha',\beta'} - f_{\alpha+\beta}\right)$$

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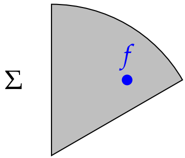

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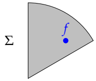
Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = \mathbf{v}_D{}^T(X)\, \mathbf{G}\, \mathbf{v}_D(X)$ and $\mathbf{G} \succcurlyeq 0$

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



sum of squares of f?

≃ Output!

From Approximate to Exact Solutions

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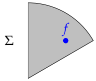


Hybrid Symbolic/Numeric Algorithms

sum of squares of $f + \varepsilon$?

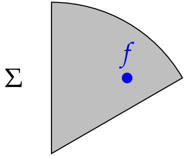
$\simeq$ Output!

Error Compensation

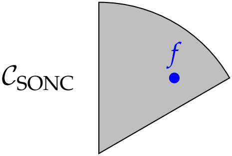
$\simeq \quad \to \quad =$

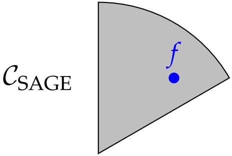
From Approximate to Exact Solutions

Exact SOS



Exact SONC/SAGE





Software: RealCertify and POEM

Exact optimization via SOS: RealCertify

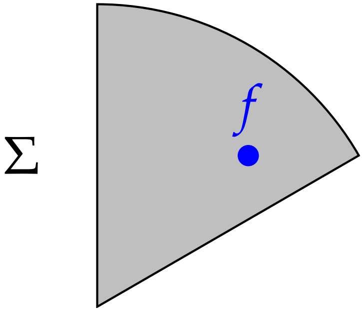
Maple & arbitrary precision SDP solver SDPA-GMP [Nakata 10]

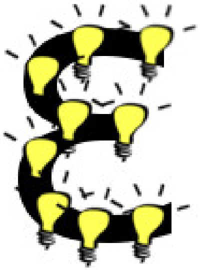
univsos	$n = 1$
multivsos	$n > 1$

Exact optimization via SONC/SAGE: POEM

Python (SymPy) & geometric programming/relative entropy ECOS [Domahidi-Chu-Boyd 13]

intsos with $n \geqslant 1$: Perturbation





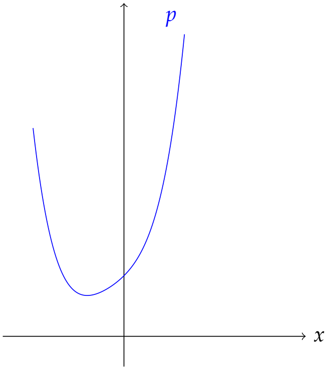
Perturbation idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$



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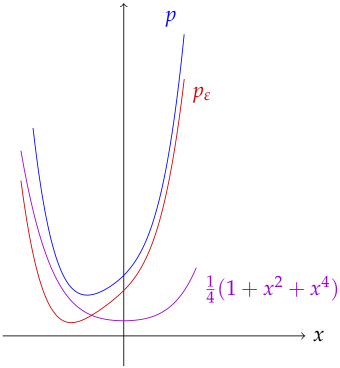
$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$

PERTURB: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$

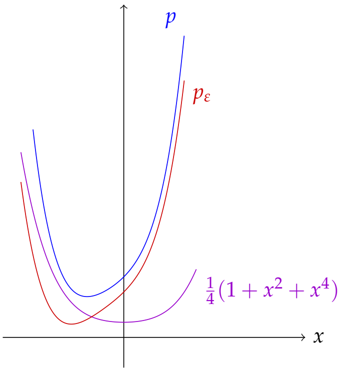
PERTURB: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$

SDP Approximation:

$$p - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

ABSORB: small enough u_i
$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS



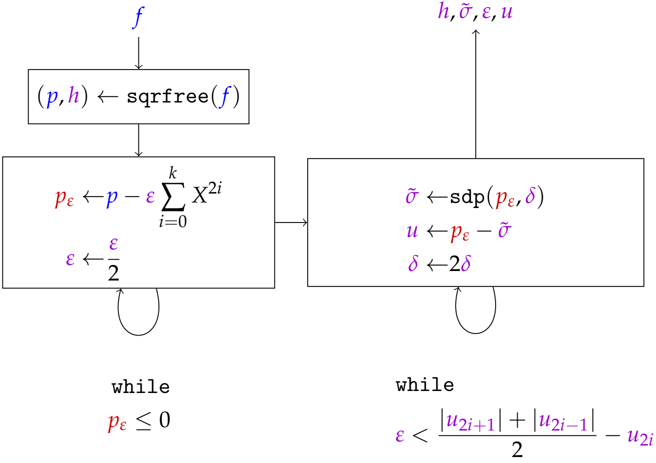
$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ and SDP Approximation

- **Input** $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



`intsos` with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

intsos with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

intsos with $n = 1$: Absorbtion

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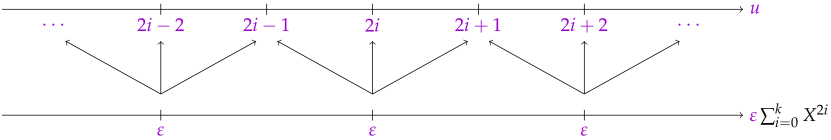

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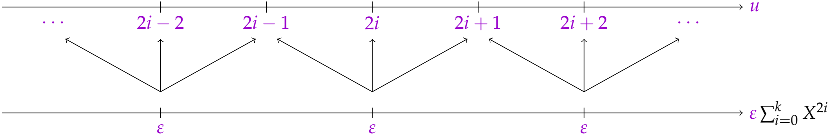
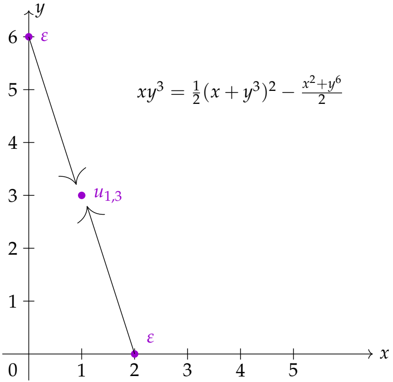


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

$\texttt{intsos}$ with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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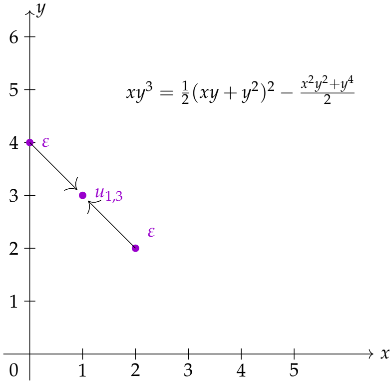

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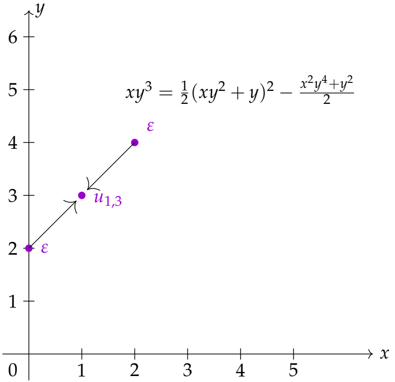
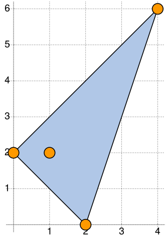


intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

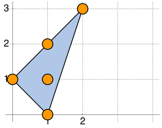
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$



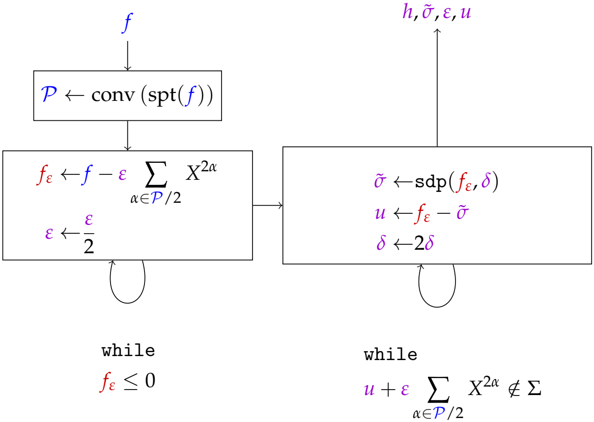
Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\text{spt}(f))$

Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm `intsos`

- **Input** $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



Algorithm `optsonc`: numerical steps

SONC (SUMS OF NONNEGATIVE CIRCUITS)

- **Input** $f = \sum_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}} X^{\boldsymbol{\alpha}}$ of degree d, $\hat{\delta} \in \mathbb{Q}^{>0}$, $\tilde{\delta} \in \mathbb{Q}^{>0}$
  Monomial squares = $\mathrm{MoSq}\,(f)$ Complement = $\mathrm{NoSq}\,(f)$

Algorithm optsonc: numerical steps

SONC (SUMS OF NONNEGATIVE CIRCUITS)

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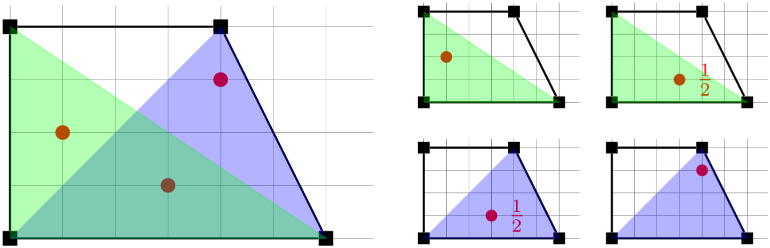
  Monomial squares = $\mathrm{MoSq}(f)$ Complement = $\mathrm{NoSq}(f)$

1. **Cover** each $\beta \in \mathrm{NoSq}(f)$ to get nonnegative circuit f_{β}
   $\implies \lambda^{\beta} \geqslant 0$ with $\sum_{\alpha \in \mathrm{MoSq}(f)} \lambda_{\alpha}^{\beta} \cdot \alpha = \beta$



Algorithm `optsonc`: numerical steps

- **Input** $f = \sum_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}} X^{\boldsymbol{\alpha}}$, $\hat{\delta}$, $\tilde{\delta}$

2. **Numerical** resolution of GEOMETRIC PROGRAM

$$
\begin{aligned}
f_{\text{SONC}} = \min_{\boldsymbol{G} > \mathbf{0}} \quad & \sum_{\boldsymbol{\beta} \in \mathrm{NoSq}(f)} G_{\boldsymbol{\beta},\mathbf{0}} \\
\text{s.t.} \quad & \sum_{\boldsymbol{\beta} \in \mathrm{NoSq}(f)} G_{\boldsymbol{\beta},\boldsymbol{\alpha}} \leqslant b_{\boldsymbol{\alpha}} \,, \quad \boldsymbol{\alpha} \in \mathrm{MoSq}(f)\,, \boldsymbol{\alpha} \neq \mathbf{0} \\
& \prod_{\boldsymbol{\alpha} \in \mathrm{Cov}^{\boldsymbol{\beta}}} \left(\frac{G_{\boldsymbol{\beta},\boldsymbol{\alpha}}}{\lambda_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}} \right)^{\lambda_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}} = -b_{\boldsymbol{\beta}} \,, \quad \boldsymbol{\beta} \in \mathrm{NoSq}(f)
\end{aligned}
$$

Algorithm `optsonc`: symbolic steps

- **Input** $f = \sum_{\alpha} b_{\alpha} X^{\alpha}$, $\hat{\delta}$, $\tilde{\delta}$

GEOMETRIC PROGRAM provides "IN THEORY"

$f_{\beta} = \sum_{\alpha \in \mathrm{Cov}^{\beta}} G_{\beta,\alpha} \cdot X^{\alpha} + b_{\beta} X^{\beta}$, $f + \sum_{\beta} G_{\beta,\mathbf{0}} - b_{\mathbf{0}} = \sum_{\beta} f_{\beta} \geqslant 0$

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Tolerance $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{G}$ violates the constraints

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Tolerance $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{G}$ violates the constraints

3. **Rounding step** $\hat{G} \leftarrow \texttt{round}\left(\tilde{G}, \hat{\delta}\right)$

Algorithm `optsonc`: symbolic steps

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GEOMETRIC PROGRAM provides "IN THEORY"

$$f_{\beta} = \sum_{\alpha \in \mathrm{Cov}^{\beta}} G_{\beta,\alpha} \cdot X^{\alpha} + b_{\beta} X^{\beta}, \; f + \sum_{\beta} G_{\beta,0} - b_0 = \sum_{\beta} f_{\beta} \geqslant 0$$

Tolerance $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{G}$ violates the constraints

3. **Rounding step** $\hat{G} \leftarrow \texttt{round}\left(\tilde{G}, \hat{\delta}\right)$
4. **Projection step**

$$G_{\beta,\alpha} \leftarrow b_{\alpha} \cdot \hat{G}_{\beta,\alpha} / \sum_{\beta' \in \mathrm{NoSq}(f)} \hat{G}_{\beta',\alpha}$$

$$\tilde{G}_{\beta,0} \leftarrow \lambda_0^{\beta} \left(-b_{\beta} \cdot \prod_{\alpha \in \mathrm{Cov}^{\beta}} \left(\frac{\lambda_{\alpha}^{\beta}}{G_{\beta,\alpha}} \right)^{\lambda_{\alpha}^{\beta}} \right)^{\frac{1}{\lambda_0^{\beta}}}$$

$$\hat{G}_{\beta,0} \leftarrow \texttt{round}\uparrow \left(\tilde{G}_{\beta,0}, \hat{\delta}\right)$$

Algorithm optsonc: symbolic steps

- **Input** $f = \sum_{\alpha} b_{\alpha} X^{\alpha}$, $\hat{\delta}$, $\tilde{\delta}$

GEOMETRIC PROGRAM provides "IN THEORY"

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$$\hat{G}_{\beta,0} \leftarrow \mathtt{round}\uparrow\left(\tilde{G}_{\beta,0}, \hat{\delta}\right)$$

$$\boxed{f \geqslant b_0 - \sum_{\beta} \hat{G}_{\beta,0}}$$

Algorithm `optsage`: numerical steps

SAGE (SUMS OF ARITHMETIC-GEOMETRIC-MEAN-EXPONENTIALS)

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

Algorithm `optsage`: numerical steps

SAGE (SUMS OF ARITHMETIC-GEOMETRIC-MEAN-EXPONENTIALS)

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

$f \in \mathcal{C}_{\text{SAGE}} \Leftrightarrow \exists \boldsymbol{\nu}^{(j)}, \boldsymbol{c}^{(j)}$ such that

$$\sum_j \boldsymbol{c}^{(j)} = \boldsymbol{b}\,, \quad \sum_i \boldsymbol{\alpha}(i) \boldsymbol{\nu}_i^{(j)} = \boldsymbol{0}\,, \quad -\boldsymbol{1} \cdot \boldsymbol{\nu}_{\backslash j}^{(j)} = \nu_j^{(j)}$$

$$\boldsymbol{c}_{\backslash j}^{(j)}, \boldsymbol{\nu}_{\backslash j}^{(j)} \geqslant \boldsymbol{0}\,, \quad D\left(\boldsymbol{\nu}_{\backslash j}^{(j)}, e\boldsymbol{c}_{\backslash j}^{(j)}\right) \leqslant c_j^{(j)}$$

Algorithm optsage: numerical steps

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1. **Numerical** resolution of RELATIVE ENTROPY PROGRAM

Precision $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{\boldsymbol{\nu}}, \tilde{\boldsymbol{c}}$ violate the constraints

Algorithm `optsage`: symbolic steps

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

Algorithm `optsage`: symbolic steps

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

Build the matrix $\boldsymbol{Q}$ with columns $(\boldsymbol{\alpha}(i), 1)$

Algorithm optsage: symbolic steps

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

Build the matrix $\boldsymbol{Q}$ with columns $(\boldsymbol{\alpha}(i), 1)$

2. **Rounding step** $\hat{\boldsymbol{v}} \leftarrow \texttt{round}\left(\tilde{\boldsymbol{v}}, \hat{\delta}\right)$, $\hat{\boldsymbol{c}} \leftarrow \texttt{round}\left(\tilde{\boldsymbol{c}}, \hat{\delta}\right)$

Algorithm `optsage`: symbolic steps

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

Build the matrix $\boldsymbol{Q}$ with columns $(\boldsymbol{\alpha}(i), 1)$

2. **Rounding step** $\hat{\boldsymbol{\nu}} \leftarrow \texttt{round}\left(\tilde{\boldsymbol{\nu}}, \hat{\delta}\right)$, $\hat{\boldsymbol{c}} \leftarrow \texttt{round}\left(\tilde{\boldsymbol{c}}, \hat{\delta}\right)$
3. **Projection step**
$\boldsymbol{\nu}^{(j)} \leftarrow (\boldsymbol{I} - \boldsymbol{Q}^{+}\boldsymbol{Q})\, \hat{\boldsymbol{\nu}}^{(j)}\,, \quad \boldsymbol{c}_{\backslash j}^{(j)} \leftarrow \hat{\boldsymbol{c}}_{\backslash j}^{(j)}\,, \quad c_j^{(j)} \leftarrow b_j - \mathbf{1} \cdot \boldsymbol{c}_{\backslash j}^{(j)}$

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$$\boldsymbol{\nu}^{(j)} \leftarrow (\boldsymbol{I} - \boldsymbol{Q}^{+}\boldsymbol{Q})\,\hat{\boldsymbol{\nu}}^{(j)}, \quad \boldsymbol{c}_{\setminus j}^{(j)} \leftarrow \hat{\boldsymbol{c}}_{\setminus j}^{(j)}, \quad c_j^{(j)} \leftarrow b_j - \boldsymbol{1} \cdot \boldsymbol{c}_{\setminus j}^{(j)}$$

Compute $c_j^{(1)}$ such that

$$c_j^{(j)} \geqslant D\left(\boldsymbol{\nu}_{\setminus j}^{(j)}, e\boldsymbol{c}_{\setminus j}^{(j)}\right) = \sum_{j \neq i > 1} \nu_i^{(j)} \log \frac{\nu_i^{(j)}}{ec_i^{(j)}} + \nu_1^{(j)} \log \frac{\nu_1^{(j)}}{ec_1^{(j)}}$$

Algorithm `optsage`: symbolic steps

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   $c^{(j)}_1 \leftarrow \texttt{round} \uparrow \left(\exp(\ldots), \hat{\delta}\right)$

Algorithm `optsage`: symbolic steps

- **Input** $f = \sum_i b_i \exp[X \cdot \boldsymbol{\alpha}(i)]$, $\hat{\delta}$, $\tilde{\delta}$

Build the matrix $\boldsymbol{Q}$ with columns $(\boldsymbol{\alpha}(i), 1)$

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$$c^{(j)}_1 \leftarrow \texttt{round}\uparrow\left(\exp(\ldots), \hat{\delta}\right)$$

$$\boxed{f \geqslant b_1 - \sum_j c^{(j)}_1}$$

SOS Benchmarks

- rounding-projection (SOS) [Peyrl-Parrilo]
- `RAGLib` (critical points) [Safey El Din]
- `SamplePoints` (CAD) [Moreno Maza-Alvandi et al.]

Id	n	d	`RealCertify` τ_1 (bits)	`RealCertify` t_1 (s)	`RoundProject` τ_2 (bits)	`RoundProject` t_2 (s)	`RAGLib` t_3 (s)	`CAD` t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	598.	–
f_1	10	4	344 347	2.45	8 374 082	4.59	–	–

SONC vs SAGE

terms	bit size		time	
	`optsonc`	`optsage`	`optsonc`	`optsage`
6	432	1005	0.06	0.26
9	806	2696	0.19	0.66
12	1261	5568	0.37	1.29
20	2592	19203	0.64	4.00
24	3826	32543	0.97	6.66
30	5029	53160	1.34	10.58
50	10622	167971	3.95	32.78

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"IN PRACTICE" `optsonc` faster and more concise than `optsage`

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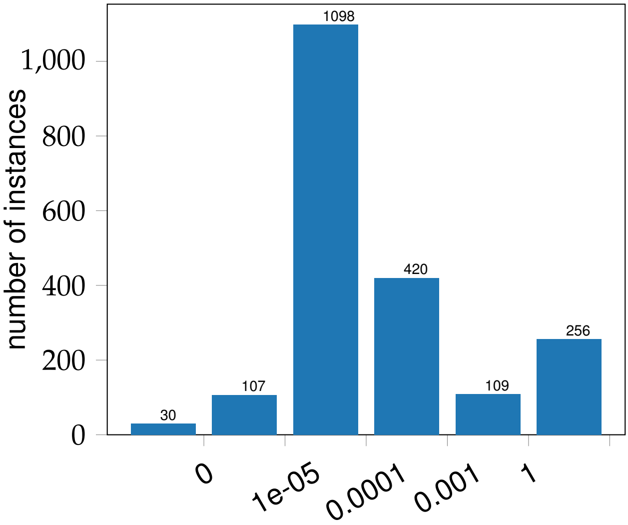
"IN PRACTICE" `optsonc` faster and more concise than `optsage`

"IN THEORY" `optsonc` less accurate than `optsage`

SONC: Gap between Numeric & Symbolic



Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	Output Bit Size
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau\, d^{d^{\mathcal{O}(n)}}$
`intsage`	$\mathring{\mathcal{C}}_{\text{SAGE}}$	$\mathbb{R}^n$	

Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	Output Bit Size
intsos	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau \, d^{d^{\mathcal{O}(n)}}$
intsage	$\mathring{\mathcal{C}}_{\text{SAGE}}$	$\mathbb{R}^n$	

- How to handle degenerate situations?
- Arbitrary precision SDP/GP/REP solvers
- Extension to other relaxations

Crucial need for polynomial systems certification
Available PhD/Postdoc Positions

Thank you for your attention!

RealCertify   POEM

https://homepages.laas.fr/vmagron

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