Certifying global optimality of AC-OPF Solutions via the CS-TSSOS hierarchy

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Joint work with Jean-Bernard Lasserre & Jie Wang

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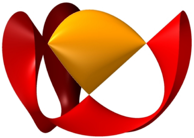
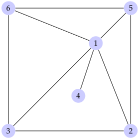
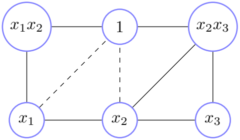
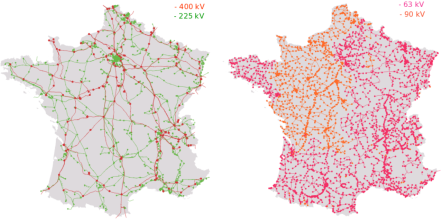






AC-OPF model

Minimize active power injections of an alternating current transmission network under physical + operational constraints



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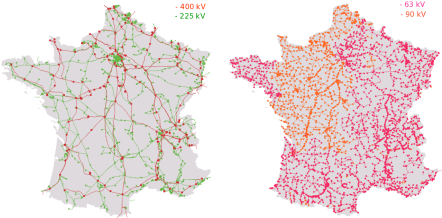

Artificial version of the control problem for electricity transmission network

AC-OPF model: polynomial optimization

Network = Graph with buses N, *from* edges E, *to* edges E^R

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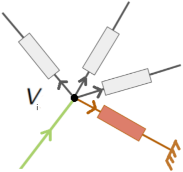

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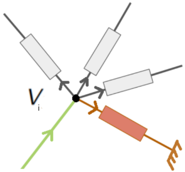

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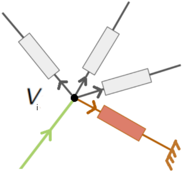

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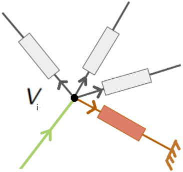

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Relation power-voltage-current: $\sum_{k \in G_i} S_k^g - \mathbf{S}_i^d = V_i I_i^\star$
$\rightsquigarrow$ leads to power-flow equations

AC-OPF model: polynomial optimization

$$
\begin{cases}
\inf\limits_{V_i, S_k^g \in \mathbb{C}} & \sum_{k \in G} \left(\mathbf{c}_{2k} (\Re(S_k^g))^2 + \mathbf{c}_{1k} \Re(S_k^g) + \mathbf{c}_{0k} \right) \\
\text{s.t.} & \angle V_r = 0 \\
& \mathbf{S}_k^{gl} \leq S_k^g \leq \mathbf{S}_k^{gu} \qquad \forall k \in G \\
& \boldsymbol{v}_i^l \leq |V_i| \leq \boldsymbol{v}_i^u \qquad \forall i \in N \\
& \sum_{k \in G_i} S_k^g - \mathbf{S}_i^d - \mathbf{Y}_i^k |V_i|^2 = \sum_{(i,j) \in E_i \cup E_i^R} S_{ij} \quad \forall i \in N \\
& S_{ij} = (\mathbf{Y}_{ij}^* - \mathbf{i} \frac{\mathbf{b}_{ij}^c}{2}) \frac{|V_i|^2}{|\mathbf{T}_{ij}|^2} - \mathbf{Y}_{ij}^* \frac{V_i V_j^*}{\mathbf{T}_{ij}} \quad \forall (i,j) \in E \\
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& |S_{ij}| \leq \mathbf{s}_{ij}^u \qquad \forall (i,j) \in E \cup E^R \\
& \boldsymbol{\theta}_{ij}^{\Delta l} \leq \angle (V_i V_j^*) \leq \boldsymbol{\theta}_{ij}^{\Delta u} \qquad \forall (i,j) \in E
\end{cases}
$$

AC-OPF model: existing methods

- **local solvers**: interior-point (IPOPT, KNITRO) for large-scale ($\simeq 10^5$ nodes)

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- **Global solutions** with branch & bound, Moment-SOS hierarchy ($\simeq 300$ nodes) [Kocuk et al. '14, Godard '19]

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💡 Exploiting sparsity
few terms [Reznick '78] or few correlations [Lasserre, Waki et al. '06]

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Correlative sparsity: few products between each variable and the others
$\rightsquigarrow x_1 x_2 + x_2 x_3 + \ldots x_{99} x_{100}$

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few terms [Reznick '78] or few correlations [Lasserre, Waki et al. '06]

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Term sparsity: few terms

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Correlative sparsity $n \simeq \mathbf{10^3}$ [Josz et al. '18]

Moment-SOS hierarchies

Correlative sparsity

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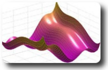
Moment-SOS hierarchies

NP-hard NON CONVEX Problem $f_{\min} = \inf f(\mathbf{x})$

Theory

(Primal)		(Dual)
$\inf \int f \, d\mu$		$\sup \lambda$
with μ proba $\Rightarrow$	**INFINITE LP**	$\Leftarrow$ with $f - \lambda \geqslant 0$

Moment-SOS hierarchies

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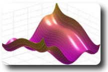
Practice

(Primal **Relaxation**)



(Dual **Strengthening**)

moments $\int \mathbf{x}^\alpha \, d\mu$

$f - \lambda =$ **sum of squares**

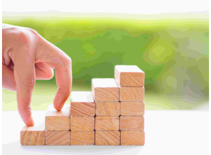
finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

[Lasserre '01] HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$

Based on representation of positive polynomials [Putinar '93]

✗ **Numerical** Solvers $\Longrightarrow$ **Approx** Certificate

✗ degree d, n vars $\Longrightarrow$ $\binom{n+d}{n}$ **SDP** VARIABLES



Moment-SOS hierarchies: an example

NP hard General Problem: $f_{\min} := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

Semialgebraic set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1-x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1-x_2)}^{g_2}$$

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Sums of squares (SOS) σ_j

Quadratic module: $\mathcal{M}(\mathbf{X})_d = \left\{ \sigma_0 + \sum_j \sigma_j g_j, \ \deg \sigma_j \, g_j \leqslant 2d \right\}$

Moment-SOS hierarchies

Hierarchy of SDP relaxations:

$$\lambda_d := \sup_{\lambda} \Big\{ \lambda : f - \lambda \in \mathcal{M}(\mathbf{X})_d \Big\}$$

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Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

"No Free Lunch" Rule: $\binom{n+2d}{n}$ SDP variables

Moment-SOS hierarchies

Correlative sparsity

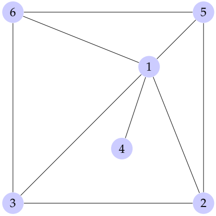
Term sparsity

Correlative sparsity

Exploit few links between **variables** [Lasserre, Waki et al. '06]

$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$

Chordal graph after adding edge $(3,5)$



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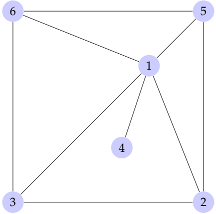


Chordal graph after adding edge $(3,5)$

maximal cliques I_k

Average size $\kappa \rightsquigarrow \kappa^{2d}$ vars

$I_1 = \{1,4\}$
$I_2 = \{1,2,3,5\}$
$I_3 = \{1,3,5,6\}$

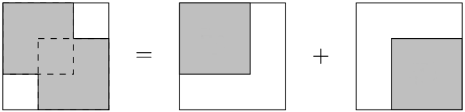
Dense SDP: 210 vars

Sparse SDP: 115 vars

Correlative sparsity

Theorem [Griewank Toint '84]

Chordal graph G with maximal cliques I_1, I_2
$Q_G \succcurlyeq 0$ with nonzero entries at edges of G
$\Longrightarrow Q_G = {P_1}^T Q_1 P_1 + {P_2}^T Q_2 P_2$ with $Q_k \succcurlyeq 0$ indexed by I_k



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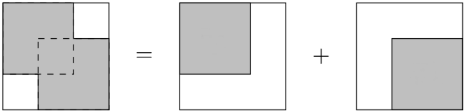

Sparse $f = f_1 + f_2$ where f_k involves **only** variables in I_k

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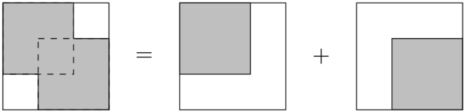

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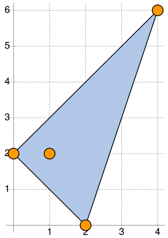
Theorem: Sparse Putinar's representation [Lasserre '06]

$f > 0$ on $\{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$
chordal graph G with cliques I_k $\Longrightarrow$
ball constraints for each $\mathbf{x}(I_k)$

$$\boxed{f = \sigma_{01} + \sigma_{02} + \sum_j \sigma_j g_j}$$

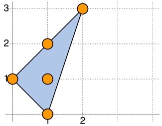
SOS σ_{0k} "sees" vars in I_k
σ_j "sees" vars from g_j

Term sparsity: unconstrained



$f = 4x_1^4 x_2^6 + x_1^2 - x_1 x_2^2 + x_2^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton polytope $\mathscr{B} = \operatorname{conv}(\text{spt}(f))$



Squares in SOS decomposition $\subseteq \frac{\mathscr{B}}{2} \cap \mathbb{N}^n$
[Reznick '78]

$$f = \begin{pmatrix} x_1 & x_2 & x_1x_2 & x_1x_2^2 & x_1^2x_2^3 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} x_1 \\ x_2 \\ x_1x_2 \\ x_1x_2^2 \\ x_1^2x_2^3 \end{pmatrix}$$

Term sparsity: unconstrained

[Postdoc Wang '19-21] ANR Tremplin-ERC

$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 9x_2^2x_3 - 45x_2x_3^2 + 142x_2^2x_3^2$$

[Reznick '78] → Newton polytope method

$$f = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_2x_1 & x_3x_2 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}$$

$\rightsquigarrow \frac{6\times 7}{2} = 21$ "unknown" entries in Q

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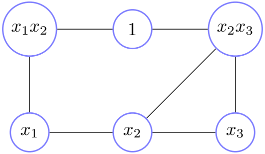
Term sparsity pattern graph G



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COPS

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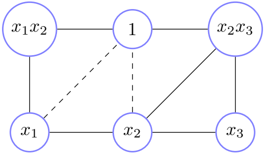
Term sparsity pattern graph G
+ chordal extension G'



Term sparsity: unconstrained

[Postdoc Wang '19-21] ANR Tremplin-ERC

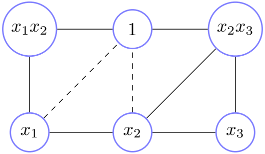
$$f = x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1^2x_2 + 2x_1^2x_2^2 - 2x_2x_3 + 6x_3^2 + 9x_2^2x_3 - 45x_2x_3^2 + 142x_2^2x_3^2$$

[Reznick '78] → Newton polytope method

$$f = \begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_2x_1 & x_3x_2 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1x_2 \\ x_2x_3 \end{pmatrix}$$

$\rightsquigarrow \frac{6 \times 7}{2} = 21$ "unknown" entries in Q

Term sparsity pattern graph G
+ chordal extension G'



Replace Q by $Q_{G'}$ with nonzero entries at edges of G'

$\rightsquigarrow$ 6 + 9 = 15 "unknown" entries in $Q_{G'}$

Term sparsity: constrained

At step d of the hierarchy, tsp graph G has

Nodes V = monomials of degree $\leqslant d$

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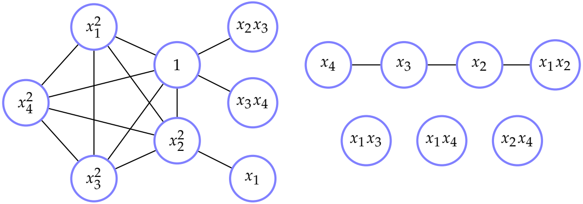
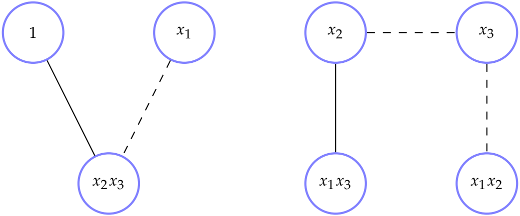
An example with $d = 2$
$f = x_1^4 + x_1 x_2^2 + x_2 x_3 + x_3^2 x_4^2$
$g_1 = 1 - x_1^2 - x_2^2 - x_3^2 \quad g_2 = 1 - x_3 x_4$



Term sparsity: support extension

$\alpha' + \beta' = \alpha + \beta$ and $(\alpha, \beta) \in E \Rightarrow (\alpha', \beta') \in E$



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By iteratively performing support extension & chordal extension

$$G^{(1)} = G' \subseteq \cdots \subseteq G^{(\ell)} \subseteq G^{(\ell+1)} \subseteq \cdots$$

Two-level hierarchy of lower bounds for $f_{\min}$, indexed by sparse order ℓ and relaxation order d

Term sparsity

CONVERGENCE GUARANTEES

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- choice of the CHORDAL EXTENSION: min / max

A few benchmarks on AC-OPF instances

mb: max block size

gap: the optimality gap w.r.t. local optimal solution

n	m	CS ($d = 2$) mb	CS ($d = 2$) opt	CS ($d = 2$) time	CS ($d = 2$) gap	CS+TS ($d = 2, \ell = 1$) mb	CS+TS ($d = 2, \ell = 1$) opt	CS+TS ($d = 2, \ell = 1$) time	CS+TS ($d = 2, \ell = 1$) gap
20	55	28	1.754e4	0.56	0.05%	22	1.754e4	0.30	0.05%
114	315	66	1.344e5	5.59	0.39%	31	1.339e5	2.01	0.73%
344	971	153	4.224e5	758	0.06%	44	4.207e5	96.0	0.48%
	1325	253	–	–	–	73	1.047e5	169	0.50%
1112	4613	231	4.241e4	3114	0.85%	39	4.240e4	46.6	0.86%
		496	–	–	–	31	7.239e4	410	0.25%
4356	18257	378	–	–	–	27	1.395e6	934	0.51%
6698	29283	1326	–	–	–	76	5.985e5	1886	0.47%

Conclusion and perspectives

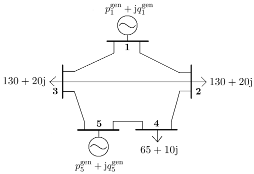
SPARSITY EXPLOITING CONVERGING HIERARCHIES to minimize large-scale polynomials

FAST IMPLEMENTATION IN JULIA: TSSOS

💡 Combine correlative & term sparsity $\rightsquigarrow$ solves problems with thousand variables

Conclusion and perspectives

Solving Alternative Current OPF to **global optimality**
$\rightarrow$ benchmarks [PGLIB '18] with up to **25 000 buses**!



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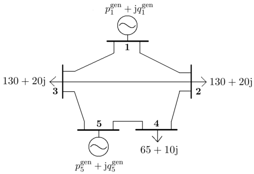

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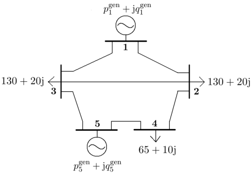

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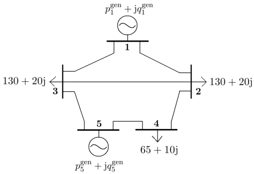

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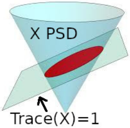

💡 Replace interior-point solvers by 1st-order methods
$\Rightarrow$ handle matrices of size up to 2000 with more than 1.5 million constraints… in 1 hour!

Take-away

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Why should you do polynomial optimization to solve AC-OPF?

- powerful & accurate MODELING tool
- CERTIFICATION cost $\simeq$ optimization cost
- EFFICIENCY guaranteed on structured applications

Thank you for your attention!

`https://homepages.laas.fr/vmagron`

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