Semidefinite Approximations of Invariant Measure Supports and Reachable Sets for Discrete-time Polynomial Systems

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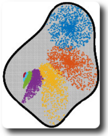


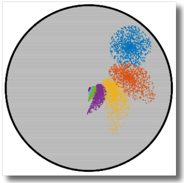
The Problem

- Semialgebraic initial conditions
  $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : g_1^0(\mathbf{x}) \geqslant 0, \ldots, g_{m_0}^0(\mathbf{x}) \geqslant 0\}$

- Polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$,
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

- $\deg f = d := \max\{\deg f_1, \ldots, \deg f_n\}$

- Set of admissible trajectories
  $\mathbf{X}^{(\infty)} := \{(\mathbf{x}_t)_{t\in\mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$



- $\mathbf{X}^{(\infty)} = \bigcup_{t\in\mathbb{N}} f^t(\mathbf{X}_0) \subseteq \mathbf{X}$, with $\mathbf{X} \subset \mathbb{R}^n$ a box or a ball

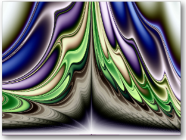
- Tractable approximations of $\mathbf{X}^{(\infty)}$?

The Problem

- Occurs in several contexts :
  1. program analysis: fixpoint computation

     toyprogram (x_1, x_2)
     requires $(0.25 \leqslant x_1 \leqslant 0.75$ && $0.25 \leqslant x_2 \leqslant 0.75)$;
     while $(x_1^2 + x_2^2 \leqslant 1)$ {
     $x_1 = x_1 + 2x_1x_2$;
     $x_2 = 0.5(x_2 - 2x_1^3)$;
     }

  2. hybrid systems, biology: Neuron Model, Growth Model
  3. control: integrator, Hénon map



Related work: LP relaxations

1. Contractive methods based on LP relaxations and polyhedra projection [Bertsekas 72]

2. Extension to nonlinear systems [Harwood et al. 16]

3. Bernstein/Krivine-Handelman representations [Ben Sassi-et al. 15, Ben Sassi et al. 12]

$\oplus$ LP relaxations $\implies$ scalability

$\ominus$ Convex approximations of nonconvex sets $\implies$ coarse

$\ominus$ No convergence guarantees (very often)

Related work: SDP relaxations

1. Upper bounds of the volume of a semialgebraic set [Henrion et al. 09]
2. Tractable approximations of sets defined with quantifiers $\exists, \forall$ [Lasserre 15]
3. Semidefinite characterization of region of attraction [Henrion-Korda 14]
4. Convex computation of maximum controlled invariant [Korda-Henrion-Jones 13]

Related work: SDP relaxations

5. SDP approximation of polynomial images of semialgebraic sets [Magron-Henrion-Lasserre 15]

- $\mathbf{X}_1 := f(\mathbf{X}_0) \subseteq \mathbf{X}$, with $\mathbf{X} \subset \mathbb{R}^n$ a box or a ball
  $\Longrightarrow$ Discrete-time system with a single iteration

- 💡 Approximation of image measure supports
  $\Longrightarrow$ certified SDP over approximations of $\mathbf{X}_1$

- $\mathbf{X}_t := f^t(\mathbf{X}_0)$

  $\ominus$ $\deg f^t = d \times t \implies$ very expensive computation

  $\ominus$ Would only approximate $\mathbf{X}_t$ and not $\mathbf{X}^{(\infty)}$

Contribution

- General framework to approximate $\mathbf{X}^{(\infty)}$

  $\oplus$ **No discretization** is required

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- Infinite-dimensional LP formulation
  💡 support of measures solving Liouville's Equation

Contribution

- General framework to approximate $\mathbf{X}^{(\infty)}$
  $\oplus$ **No discretization** is required
- Infinite-dimensional LP formulation
  💡 support of measures solving Liouville's Equation
- Finite-dimensional SDP relaxations
- $\mathbf{X}^{(\infty)} \subseteq \mathbf{X}^r := \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geqslant 1\}$
  $\oplus$ Strong convergence guarantees
  $\lim_{r\to\infty} \text{vol}(\mathbf{X}^r \backslash \mathbf{X}^{(\infty)}) = 0$
  $\oplus$ Compute w_r by solving one **semidefinite program**

The Problem

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for Reachability

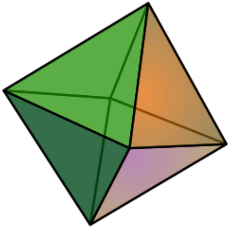
Application examples

Conclusion

What is Semidefinite Programming?

- Linear Programming (LP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d}\ . \end{aligned}$$



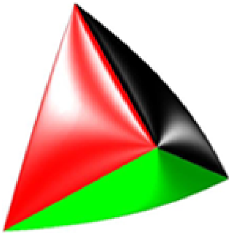
Polyhedron

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij}\, z_j \geqslant d_i$"

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \, .$$



Spectrahedron

- Linear cost $\mathbf{c}$
- Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \ .$$

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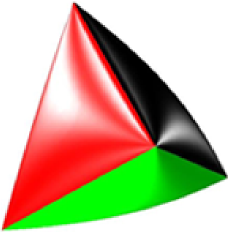

Spectrahedron

Applications of SDP

- Combinatorial optimization
- Control theory
- Matrix completion
- Unique Games Conjecture (Khot '02) :
  "A *single concrete algorithm* provides **optimal guarantees** among all efficient algorithms for a large class of computational problems."
  (Barak and Steurer survey at ICM'14)
- Solving polynomial optimization (Lasserre '01)

Polynomial Optimization

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

- $p^* := \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$: NP hard

- Sums of squares $\Sigma[\mathbf{x}]$
  e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$

- $\mathcal{Q}(\mathbf{X}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^{l} \sigma_j(\mathbf{x}) g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$

- **REMEMBER**: $f \in \mathcal{Q}(\mathbf{X}) \implies \forall \mathbf{x} \in \mathbf{X}, f(\mathbf{x}) \geqslant 0$

Infinite LP Reformulation

- Borel σ-algebra $\mathcal{B}(\mathbf{X})$ (generated by the open sets of $\mathbf{X}$)
- $\mathcal{M}_+(\mathbf{X})$: set of probability measures supported on $\mathbf{X}$.
  If $\mu \in \mathcal{M}_+(\mathbf{X})$ then
  1. $\mu : \mathcal{B} \to [0, \infty)$, $\mu(\emptyset) = 0$
  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any disjoint countable $(B_i) \subset \mathcal{B}(\mathbf{X})$
  3. Lebesgue **Volume** of $B \in \mathcal{B}(\mathbf{X})$

$$\operatorname{vol} B := \int_{\mathbf{X}} \lambda_B \,, \text{ with } \lambda_B(d\mathbf{x}) := \mathbf{1}_B(\mathbf{x})\, d\mathbf{x}$$

- $\operatorname{supp} \mu$ is the smallest set $\mathbf{X}$ such that $\mu(\mathbb{R}^n \backslash \mathbf{X}) = 0$

Infinite LP Reformulation

$$p^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{X}$ if there exists a finite measure μ supported on $\mathbf{X}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{X}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n .$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{X})$: space of probability measures supported on $\mathbf{X}$
- $\mathcal{Q}(\mathbf{X})$: quadratic module

Polynomial Optimization Problems (POP)

(Primal) (Dual)

$$\begin{array}{llcll} \inf & \int_{\mathbf{X}} f \, d\mu & = & \sup & m \\ \text{s.t.} & \mu \in \mathcal{M}_+(\mathbf{X}) & & \text{s.t.} & m \in \mathbb{R} \, , \\ & & & & f - m \in \mathcal{Q}(\mathbf{X}) \end{array}$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{X})$
- Truncated quadratic module $\mathcal{Q}_r(\mathbf{X}) := \mathcal{Q}(\mathbf{X}) \cap \mathbb{R}_{2r}[\mathbf{x}]$

Polynomial Optimization Problems (POP)

$$\begin{array}{llcll}
 & \text{(Moment)} & & & \text{(SOS)} \\
\inf & \sum_\alpha f_\alpha \mathbf{z}_\alpha & = & \sup & m \\
\text{s.t.} & \mathbf{M}_{r-v_j}(g_j \mathbf{z}) \succcurlyeq 0\,, \quad 0 \leqslant j \leqslant l, & & \text{s.t.} & m \in \mathbb{R}\ , \\
 & \mathbf{z}_1 = 1 & & & f - m \in \mathcal{Q}_r(\mathbf{X})
\end{array}$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$
(SDP) \begin{cases} \mathcal{P}: & \inf_{\mathbf{z}} & \sum_\alpha c_\alpha \mathbf{z}_\alpha \\ & \text{s.t.} & \sum_\alpha F_\alpha \, \mathbf{z}_\alpha - F_0 \succcurlyeq 0 \\ \\ \mathcal{D}: & \sup_{\mathbf{Y}} & \mathrm{Trace}\,(F_0 \, \mathbf{Y}) \\ & \text{s.t.} & \mathrm{Trace}\,(F_\alpha \, \mathbf{Y}) = c_\alpha \ , \quad \mathbf{Y} \succcurlyeq 0 \ . \end{cases}
$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

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Pushforward and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$

- **Pushforward** $f_\# : \mathcal{M}_+(\mathbf{X}_0) \to \mathcal{M}_+(\mathbf{X})$:

$$f_\# \mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{X}_0 : f(\mathbf{x}) \in \mathbf{A}\}) , \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{X})$$

- $f_\# \mu_0$ is the **image measure** of μ_0 under f

Pushforward and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$, $\alpha > 1$ and define

$$\mu_1 := \alpha f_\# \mu_0$$
$$\dots$$
$$\mu_t := \alpha f_\# \mu_{t-1}$$
$$\mu := \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} \alpha^i f_\#^i \mu_0$$

- The measures μ_t, μ, μ_0 satisfy **Liouville's Equation**:

$$\mu_t + \mu = \alpha f_\# \mu + \mu_0$$

Pushforward and Liouville's Equation

- Let $\mu_t := \lambda_{\mathbf{X}_t}$: Lebesgue measure restriction on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\mu_t = \alpha^t f^t_{\#} \mu_0$
  $\implies$ μ_t satisfies **Liouville's Equation**!

Pushforward and Liouville's Equation

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Proof

Define $\mu := \sum_{i=0}^{t-1} \alpha^i f^i_\# \mu_0$. Then, $\mu_t + \mu = \alpha f_\# \mu + \mu_0$.

Pushforward and Liouville's Equation

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Proof

Define $\mu := \sum_{i=0}^{t-1} \alpha^i f^i_{\#}\, \mu_0$. Then, $\mu_t + \mu = \alpha f_{\#}\, \mu + \mu_0$.

- Let $\lambda_{\mathbf{X}^{(T)}}$: Lebesgue measure restriction on $\bigcup_{t=0}^{T} \mathbf{X}_t$
  $\implies$ $\lambda_{\mathbf{X}^{(T)}}$ satisfies **Liouville's Equation** by superposition

Pushforward and Liouville's Equation

- Let $\mu_t := \lambda_{\mathbf{X}_t}$: Lebesgue measure restriction on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
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  $\implies \mu_t$ satisfies **Liouville's Equation**!

Proof

Define $\mu := \sum_{i=0}^{t-1} \alpha^i f^i_{\#}\, \mu_0$. Then, $\mu_t + \mu = \alpha f_{\#}\, \mu + \mu_0$.

- Let $\lambda_{\mathbf{X}(T)}$: Lebesgue measure restriction on $\bigcup_{t=0}^{T} \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}(T)}$ satisfies **Liouville's Equation** by superposition

Lemma

$\alpha > 1 \implies \lambda_{\mathbf{X}(\infty)}$ satisfies **Liouville's Equation**

Infinite Primal LP

$$
\begin{aligned}
p^* := \sup_{\mu_\infty, \mu, \mu_0} \quad & \int_{\mathbf{X}} \mu_\infty \\
\text{s.t.} \quad & \mu_\infty + \mu = \alpha f_\# \mu + \mu_0 \,, \\
& \mu_\infty \leqslant \lambda_{\mathbf{X}} \,, \\
& \mu_\infty, \mu \in \mathcal{M}_+(\mathbf{X}) \,, \quad \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,.
\end{aligned}
$$

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- **Question**: $\lambda_{\mathbf{X}^{(\infty)}}$ optimal for this infinite primal LP ?

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- **Answer** (in general): No!

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\end{aligned}
$$

- **Question**: $\lambda_{\mathbf{X}^{(\infty)}}$ optimal for this infinite primal LP ?
- **Answer** (in general): No!

Proof

Let μ be any invariant measure w.r.t. f on $\mathbf{X} \backslash \mathbf{X}^{(\infty)}$:

- $\mu = f_{\#} \mu$, $\mu_{\text{inv}} := (\alpha - 1) \mu$ satisfies Liouville's Equation
- $\lambda_{\mathbf{X}^{(\infty)}} + \mu_{\text{inv}}$ satisfies Liouville's Equation

$\text{vol}(\text{supp}\, \mu) > 0 \implies \text{vol}\, \mathbf{X}^{(\infty)} < \int_{\mathbf{X}} (\lambda_{\mathbf{X}^{(\infty)}} + \mu_{\text{inv}}) \leqslant \text{vol}\, \mathbf{X}$.

Infinite Primal LP

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& \mu_\infty \leqslant \lambda_{\mathbf{X}} \,, \\
& \mu_\infty, \mu \in \mathcal{M}_+(\mathbf{X}) \,, \quad \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,.
\end{aligned}
$$

Lemma

Let μ_{inv} be invariant w.r.t. f with maximal support $\mathbf{X}^{\text{inv}}$.
Then the above LP has optimal solution $\lambda_{\mathbf{X}^{(\infty)} \cup \mathbf{X}^{\text{inv}}}$ and $p^* = \text{vol}(\mathbf{X}^{(\infty)} \cup \mathbf{X}^{\text{inv}})$.

Infinite Primal LP

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p^* := \sup_{\mu_\infty, \mu, \mu_0} \quad & \int_{\mathbf{X}} \mu_\infty \\
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Lemma

Let μ_{inv} be invariant w.r.t. f with maximal support $\mathbf{X}^{\text{inv}}$.
Then the above LP has optimal solution $\lambda_{\mathbf{X}^{(\infty)} \cup \mathbf{X}^{\text{inv}}}$ and
$p^* = \text{vol}(\mathbf{X}^{(\infty)} \cup \mathbf{X}^{\text{inv}})$.

$\ominus$ Assuming that $\text{vol}\, \mathbf{X}^{\text{inv}} = 0$ is a **strong hypothesis**!
$f(x) = x, \mathbf{X}_0 := [0, 1/2], \mathbf{X} = [0, 1]$
$\implies \mathbf{X}^{(\infty)} = \mathbf{X}_0$ and $p^* = \text{vol}\, \mathbf{X}$.

LP Primal-dual conic formulation

The LP can be cast as follows:

$$
\begin{aligned}
p^* = \sup_{x} \quad & \langle x, c \rangle_1 \\
\text{s.t.} \quad & \mathcal{A}\, x = b \,, \\
& x \in E_1^+ \,,
\end{aligned}
$$

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with

- $E_1 := \mathcal{M}(\mathbf{X})^3 \times \mathcal{M}(\mathbf{X}_0) \quad F_1 := \mathcal{C}(\mathbf{X})^3 \times \mathcal{C}(\mathbf{X}_0)$
- $x := (\mu_\infty, \hat{\mu}_\infty, \mu, \mu_0) \quad c := (1, 0, 0, 0) \in F_1 \quad b := (0, \lambda_{\mathbf{X}})$
- the linear operator $\mathcal{A} : E_1 \to E_2$ given by

$$\mathcal{A}\,(\mu_\infty, \hat{\mu}_\infty, \mu, \mu_0) := \begin{bmatrix} \mu_\infty + \mu - \alpha f_{\#}\, \mu - \mu_0 \\ \mu_\infty + \hat{\mu}_\infty \end{bmatrix}.$$

LP Primal-dual conic formulation

Primal LP

$$\begin{aligned} p^* = \sup_x \quad & \langle x, c \rangle_1 \\ \text{s.t.} \quad & \mathcal{A}\, x = b\,, \\ & x \in E_1^+\,. \end{aligned}$$

Dual LP

$$\begin{aligned} d^* = \inf_y \quad & \langle b, y \rangle_2 \\ \text{s.t.} \quad & \mathcal{A}'\, y - c \in F_1^+\,. \end{aligned}$$

with

- $y := (v, w) \in \mathcal{M}(\mathbf{X})^2$
- $\mathcal{A}'\,(v, w) := \begin{bmatrix} v + w \\ v - \alpha\, v \circ f \\ w \\ -v \end{bmatrix}.$

LP Primal-dual conic formulation

Primal LP

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& \mu_\infty, \mu \in \mathcal{M}_+(\mathbf{X}) \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^* := \inf_{v, w} \quad & \int w(\mathbf{x}) \, \lambda_{\mathbf{X}}(d\mathbf{x}) \\
\text{s.t.} \quad & w - v - 1 \in \mathcal{C}_+(\mathbf{X}) \,, \\
& \alpha \, v \circ f - v \in \mathcal{C}_+(\mathbf{X}) \,, \\
& w \in \mathcal{C}_+(\mathbf{X}) \,, \\
& v \in \mathcal{C}_+(\mathbf{X}_0) \,.
\end{aligned}
$$

Zero duality gap

Lemma

$p^* = d^*$

Strong convergence property

Strengthening of the dual LP:

$$\begin{aligned} d_r^* := \inf_{v,w} \quad & \sum_{\beta \in \mathbb{N}_{2r}^n} w_\beta z_\beta^{\mathbf{X}} \\ \text{s.t.} \quad & w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \,, \\ & \alpha\, v \circ f - v \in \mathcal{Q}_{rd}(\mathbf{X}) \,, \\ & w \in \mathcal{Q}_r(\mathbf{X}) \,, \\ & v \in \mathcal{Q}_r(\mathbf{X}_0) \,. \end{aligned}$$

Strong convergence property

Theorem

Assume that $\mathbf{X}^* := \mathbf{X}^{(\infty)} \cup \mathbf{X}^{\text{inv}}$ has nonempty interior and $\mathcal{Q}_r(\mathbf{X}_0)$ (resp. $\mathcal{Q}_r(\mathbf{X})$) is Archimedean.

1. The sequence (w_r) converges to $\mathbf{1}_{\mathbf{X}^*}$ w.r.t the $L_1(\mathbf{X})$-norm:

$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r - \mathbf{1}_{\mathbf{X}^*}| = 0 \ .$$

Strong convergence property

Theorem

Assume that $\mathbf{X}^* := \mathbf{X}^{(\infty)} \cup \mathbf{X}^{\text{inv}}$ has nonempty interior and $\mathcal{Q}_r(\mathbf{X}_0)$ (resp. $\mathcal{Q}_r(\mathbf{X})$) is Archimedean.

1. The sequence (w_r) converges to $\mathbf{1}_{\mathbf{X}^*}$ w.r.t the $L_1(\mathbf{X})$-norm:

$$\lim_{r\to\infty} \int_{\mathbf{X}} |w_r - \mathbf{1}_{\mathbf{X}^*}| = 0 \ .$$

2. Let $\mathbf{X}^r := \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geqslant 1\}$. Then,

$$\lim_{r\to\infty} \text{vol}(\mathbf{X}^r \backslash \mathbf{X}^*) = 0 \ .$$

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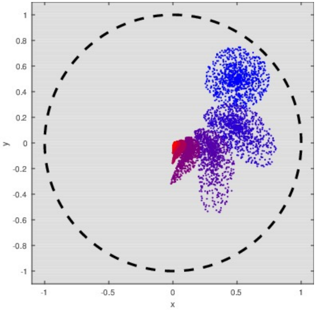
Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$



$\mathbf{X}^4$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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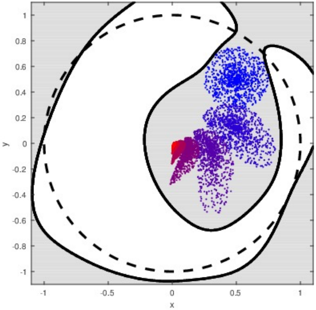

$\mathbf{X}^6$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
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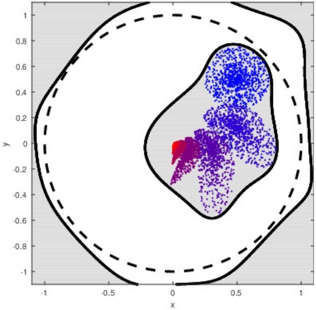

$\mathbf{X}^8$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1 x_2)\,,$$
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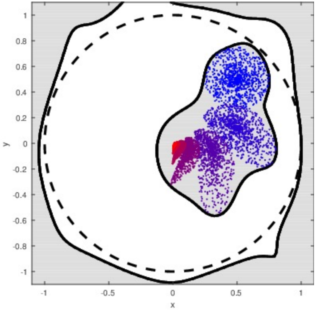

$\mathbf{X}^{10}$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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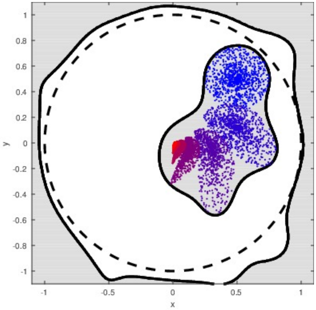
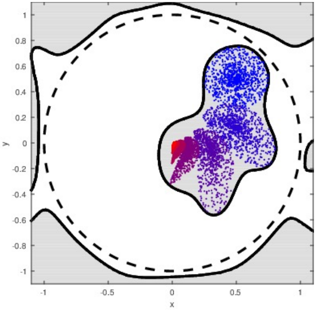


$\mathbf{X}^{12}$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2),$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3),$$

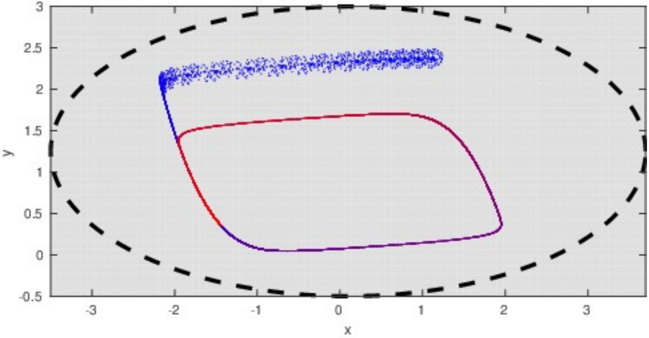


$\mathbf{X}^{14}$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

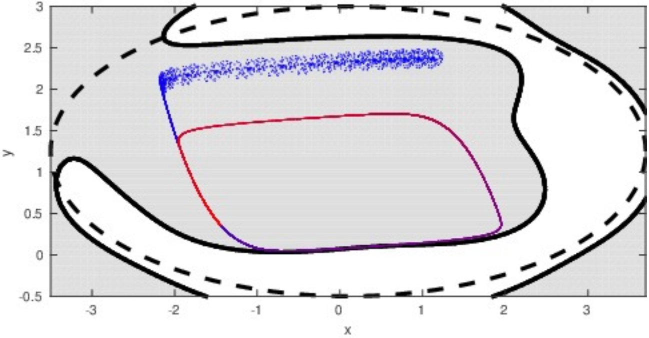


$\mathbf{X}^4$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

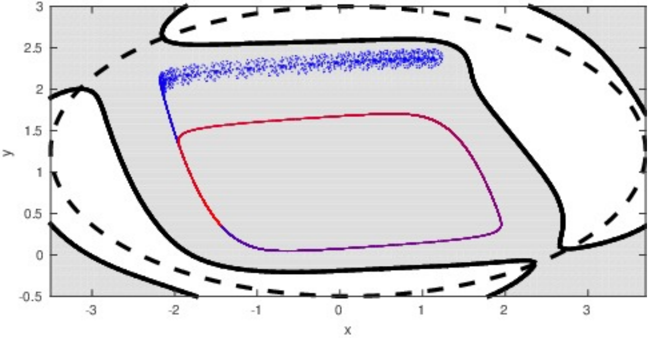


$\mathbf{X}^6$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

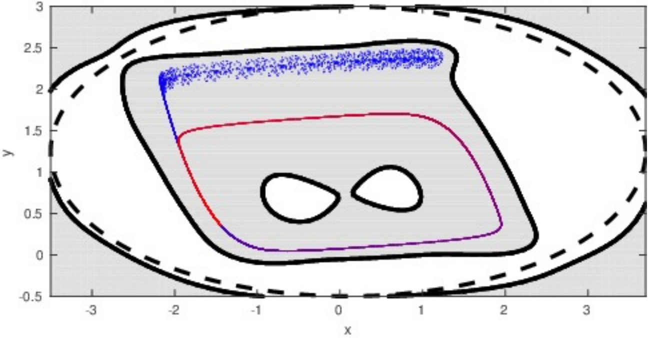


$\mathbf{X}^8$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

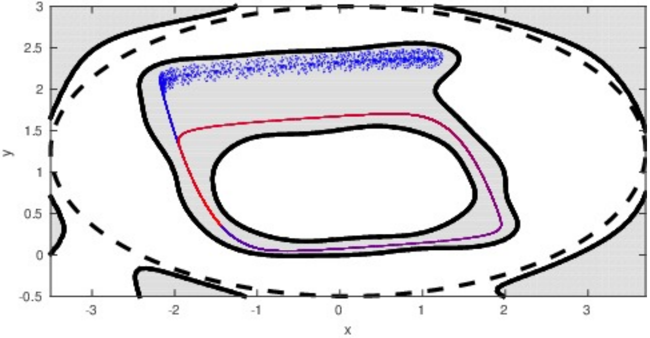


$\mathbf{X}^{10}$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

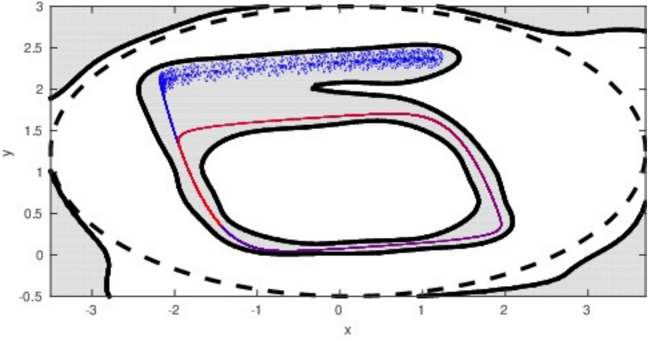


$\mathbf{X}^{12}$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

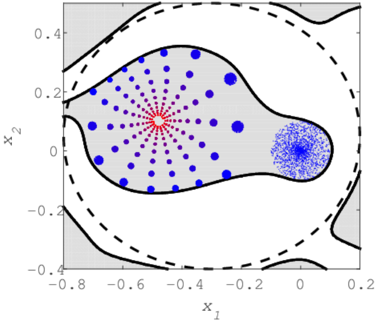


$\mathbf{X}^{14}$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1\,,$$
$$x_2^+ := 2x_1x_2 + c_2\,,$$

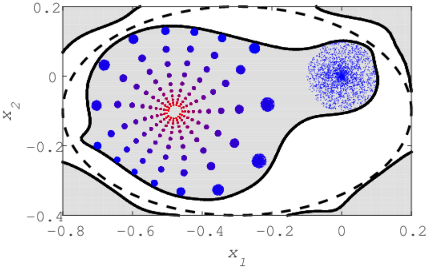


$\mathbf{X}^{12}, c_1 = -0.7, c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 ,$$
$$x_2^+ := 2x_1x_2 + c_2 ,$$



$\mathbf{X}^{12}, c_1 = -0.7, c_2 = -0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

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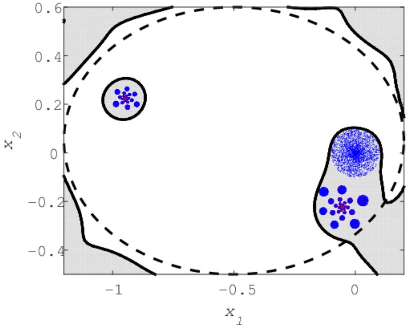
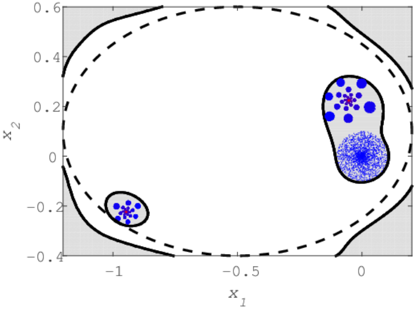


$\mathbf{X}^{12}, c_1 = -0.9, c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
$$x_2^+ := 2x_1 x_2 + c_2 \,,$$

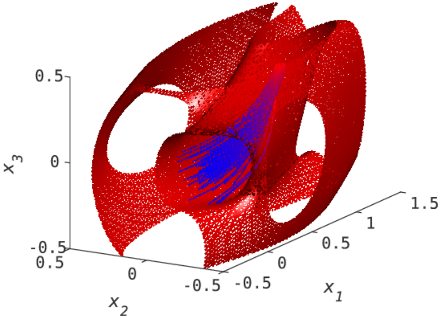


$\mathbf{X}^{12}, c_1 = -0.9, c_2 = -0.2$

Phytoplankton Growth Model

Trajectories from $\mathbf{X}_0 := [-0.3, -0.2]^2 \times [-0.05, 0.05]$ under

$$x_1^+ := x_1 + 0.01(1 - x_1 - 0.25x_1x_2)\,,$$
$$x_2^+ := x_2 + 0.01(2x_3 - 1)x_2\,,$$
$$x_3^+ := x_3 + 0.01(0.25x_1 - 2x_3^2)\,,$$

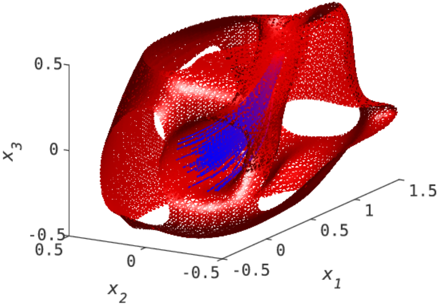


$\mathbf{X}^4$

Phytoplankton Growth Model

Trajectories from $\mathbf{X}_0 := [-0.3, -0.2]^2 \times [-0.05, 0.05]$ under

$$x_1^+ := x_1 + 0.01(1 - x_1 - 0.25 x_1 x_2)\,,$$
$$x_2^+ := x_2 + 0.01(2x_3 - 1)x_2\,,$$
$$x_3^+ := x_3 + 0.01(0.25 x_1 - 2x_3^2)\,,$$

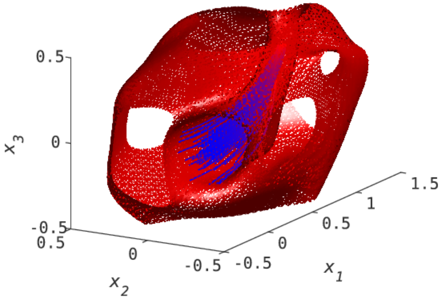


$\mathbf{X}^6$

Phytoplankton Growth Model

Trajectories from $\mathbf{X}_0 := [-0.3, -0.2]^2 \times [-0.05, 0.05]$ under

$$x_1^+ := x_1 + 0.01(1 - x_1 - 0.25 x_1 x_2)\,,$$
$$x_2^+ := x_2 + 0.01(2x_3 - 1)x_2\,,$$
$$x_3^+ := x_3 + 0.01(0.25 x_1 - 2x_3^2)\,,$$

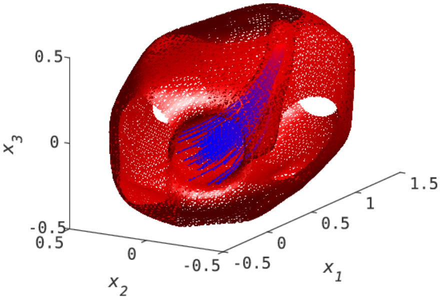


$\mathbf{X}^8$

Phytoplankton Growth Model

Trajectories from $\mathbf{X}_0 := [-0.3, -0.2]^2 \times [-0.05, 0.05]$ under

$$x_1^+ := x_1 + 0.01(1 - x_1 - 0.25 x_1 x_2)\,,$$
$$x_2^+ := x_2 + 0.01(2x_3 - 1)x_2\,,$$
$$x_3^+ := x_3 + 0.01(0.25 x_1 - 2x_3^2)\,,$$

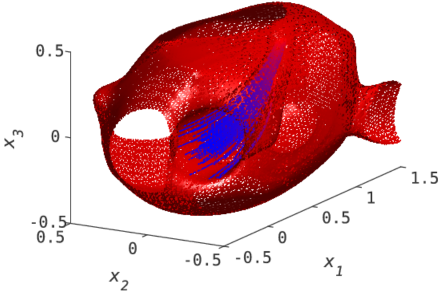


$\mathbf{X}^{10}$

Phytoplankton Growth Model

Trajectories from $\mathbf{X}_0 := [-0.3, -0.2]^2 \times [-0.05, 0.05]$ under

$$x_1^+ := x_1 + 0.01(1 - x_1 - 0.25 x_1 x_2)\,,$$
$$x_2^+ := x_2 + 0.01(2x_3 - 1)x_2\,,$$
$$x_3^+ := x_3 + 0.01(0.25 x_1 - 2x_3^2)\,,$$

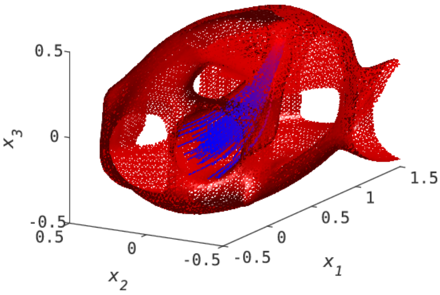


$\mathbf{X}^{12}$

Phytoplankton Growth Model

Trajectories from $\mathbf{X}_0 := [-0.3, -0.2]^2 \times [-0.05, 0.05]$ under

$$x_1^+ := x_1 + 0.01(1 - x_1 - 0.25 x_1 x_2)\,,$$
$$x_2^+ := x_2 + 0.01(2x_3 - 1)x_2\,,$$
$$x_3^+ := x_3 + 0.01(0.25 x_1 - 2x_3^2)\,,$$



$\mathbf{X}^{14}$

The Problem

Infinite LP Formulation for Polynomial Optimization

Infinite LP Formulation for Reachability

Application examples

Conclusion

Conclusion

$\oplus$ Certified Approximation of the **whole reachable set** $\mathbf{X}^{(\infty)}$

$\ominus$ Cannot avoid to approximate $\mathbf{X}^{\mathrm{inv}}$

$\ominus$ Computational complexity: $\binom{n+2rd}{n}$ SDP variables

$\oplus$ **Structure sparsity** can be exploited

Conclusion

Further research:

- Infinite Primal LP characterization of $\mathbf{X}^{(\infty)}$ only ?
- Discrete finite-time, continuous finite/infinite horizon ?
  💡Use previous framework approximating:

  1. region of attraction
  2. maximum controlled invariant

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End

Thank you for your attention!

http://www-verimag.imag.fr/~magron