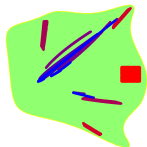
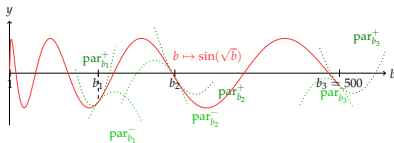
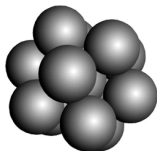


# New Applications of Moment-SOS Hierarchies

Victor Magron, RA Imperial College

17 November 2014

CAS Seminar  
IC



# Personal Background

---

- 2008 – 2010: Master at Tokyo University  
**HIERARCHICAL DOMAIN DECOMPOSITION METHODS**  
(S. Yoshimura)
- 2010 – 2013: PhD at Inria Saclay LIX/CMAP  
**FORMAL PROOFS FOR NONLINEAR OPTIMIZATION**  
(S. Gaubert and B. Werner)
- 2014 Jan-Sept: Postdoc at LAAS-CNRS  
**MOMENT-SOS APPLICATIONS**  
(D. Henrion and J.B. Lasserre)

# Errors and Proofs

---

- Mathematicians want to eliminate all the uncertainties on their results. Why?



M. Lecat, *Erreurs des Mathématiciens des origines à nos jours*, 1935.

130 pages of errors! (Euler, Fermat, Sylvester, ...)

# Errors and Proofs

---

- Possible workaround: proof assistants

COQ (Coquand, Huet 1984) 🐣

HOL-LIGHT (Harrison, Gordon 1980)



Built in top of OCAML 🐪

- Tool: Formal Bounds for Global Optimization

- Collaboration with:



Benjamin Werner (LIX Polytechnique)



Stéphane Gaubert (Maxplus Team CMAP/INRIA Polytechnique)




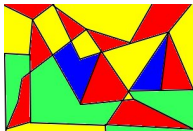
Xavier Allamigeon (Maxplus Team)

# Complex Proofs

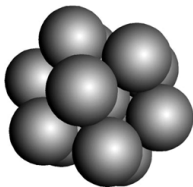
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- Complex mathematical proofs / mandatory computation

 K. Appel and W. Haken , Every Planar Map is Four-Colorable, 1989.



 T. Hales, A Proof of the Kepler Conjecture, 1994.

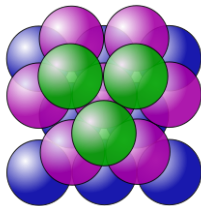


# From Oranges Stack...

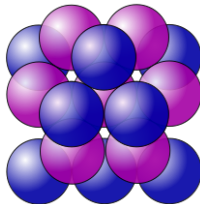
---

Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is  $\frac{\pi}{\sqrt{18}}$



Face-centered cubic Packing



Hexagonal Compact Packing

## ...to Flyspeck Nonlinear Inequalities

---

- The proof of T. Hales (1998) contains mathematical and computational parts
- Computation: check thousands of nonlinear inequalities
- Robert MacPherson, editor of The Annals of Mathematics: “[...] the mathematical community will have to get used to this state of affairs.”
- **Flyspeck** [Hales 06]: **Formal Proof of Kepler Conjecture**



## ...to Flyspeck Nonlinear Inequalities

---

- The proof of T. Hales (1998) contains mathematical and computational parts
- Computation: check thousands of nonlinear inequalities
- Robert MacPherson, editor of The Annals of Mathematics: “[...] the mathematical community will have to get used to this state of affairs.”
- **Flyspeck [Hales 06]: Formal Proof of Kepler Conjecture**
- **Project Completion on 10 August by the Flyspeck team!!**

# ...to Floyespeck Nonlinear Inequalities

---

- Nonlinear inequalities: quantified reasoning with “ $\forall$ ”

$$\forall \mathbf{x} \in \mathbf{K}, f(\mathbf{x}) \geq 0$$

- NP-hard optimization problem

# A “Simple” Example

---

## In the computational part:

- Multivariate Polynomials:

$$\Delta \mathbf{x} := x_1 x_4 (-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2 x_5 (x_1 - x_2 + x_3 + x_4 - x_5 + x_6) + x_3 x_6 (x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2 (x_3 x_4 + x_1 x_6) - x_5 (x_1 x_3 + x_4 x_6)$$

# A “Simple” Example

---

## In the computational part:

- **Semialgebraic** functions: composition of polynomials with  $|\cdot|, \sqrt{\cdot}, +, -, \times, /, \sup, \inf, \dots$

$$p(\mathbf{x}) := \partial_4 \Delta \mathbf{x} \quad q(\mathbf{x}) := 4x_1 \Delta \mathbf{x}$$

$$r(\mathbf{x}) := p(\mathbf{x}) / \sqrt{q(\mathbf{x})}$$

$$l(\mathbf{x}) := -\frac{\pi}{2} + 1.6294 - 0.2213 (\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913 (\sqrt{x_4} - 2.52) + 0.728 (\sqrt{x_1} - 2.0)$$

# A “Simple” Example

---

## In the computational part:

- **Transcendental** functions  $\mathcal{T}$ : composition of semialgebraic functions with  $\arctan, \exp, \sin, +, -, \times, \dots$

# A “Simple” Example

---

## In the computational part:

- Feasible set  $\mathbf{K} := [4, 6.3504]^3 \times [6.3504, 8] \times [4, 6.3504]^2$

Lemma<sub>9922699028</sub> from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{p(\mathbf{x})}{\sqrt{q(\mathbf{x})}}\right) + l(\mathbf{x}) \geq 0$$

# Existing Formal Frameworks

---

## Formal proofs for Global Optimization:

- Bernstein polynomial methods [Zumkeller's PhD 08]
- SMT methods [Gao et al. 12]
- Interval analysis and Sums of squares

# Existing Formal Frameworks

---

## Interval analysis

- Certified interval arithmetic in COQ [Melquiond 12]
- Taylor methods in HOL Light [Solovyev thesis 13]
  - Formal verification of floating-point operations
- robust but subject to the **Curse of Dimensionality**

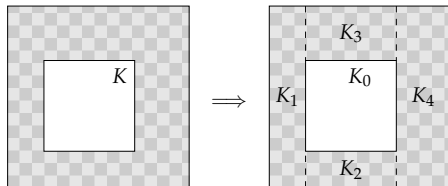


# Existing Formal Frameworks

Lemma<sub>9922699028</sub> from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{\partial_4 \Delta \mathbf{x}}{\sqrt{4x_1 \Delta \mathbf{x}}}\right) + l(\mathbf{x}) \geq 0$$

- Dependency issue using Interval Calculus:
  - One can bound  $\partial_4 \Delta \mathbf{x} / \sqrt{4x_1 \Delta \mathbf{x}}$  and  $l(\mathbf{x})$  separately
  - Too coarse lower bound:  $-0.87$
  - Subdivide  $\mathbf{K}$  to prove the inequality



# Existing Formal Frameworks

---

## Sums of squares (SOS) techniques

- Formalized in HOL-LIGHT [Harrison 07] COQ [Besson 07]
  - Precise methods but scalability and robustness issues (numerical)
  - powerful: global optimality certificates without branching
- but
- not so robust: handles moderate size problems
  - Restricted to polynomials

# Existing Formal Frameworks

---

- *Caprasse* Problem:

$$\forall \mathbf{x} \in [-0.5, 0.5]^4, -x_1x_3^3 + 4x_2x_3^2x_4 + 4x_1x_3x_4^2 + 2x_2x_4^3 + 4x_1x_3 + 4x_3^2 - 10x_2x_4 - 10x_4^2 + 5.1801 \geq 0.$$

- Decompose the polynomial as SOS of degree at most 4
- Gives a nonnegative bound!

# Existing Formal Frameworks

---

## The “No Free Lunch” Rule:

- Exponential dependency in
  - 1 Relaxation order  $k$  (SOS degree)
  - 2 number of variables  $n$  of the polynomial
- Computing  $k$ -th bound involves  $\binom{n+2k}{n}$  variables
- At fixed  $k$ ,  $O(n^{2k})$  variables

# Existing Formal Frameworks

---

Approximation theory: Chebyshev/Taylor models

- mandatory for non-polynomial problems
- hard to combine with SOS techniques (degree of approximation)

# Existing Formal Frameworks

---

Can we develop a new approach with both keeping the respective strength of interval and precision of SOS?

Proving Flyspeck Inequalities is challenging: medium-size and tight

# New Framework (in my PhD thesis)

---

- Certificates for lower bounds of Nonlinear optimization using:
  - Moment-SOS hierarchies
  - Maxplus approximation (Optimal Control)
- Verification of these certificates inside COQ

# New Framework (in my PhD thesis)

---

## Software Implementation NLCertify:

- <https://forge.ocamlcore.org/projects/nl-certify/>



15 000 lines of OCAML code



4000 lines of COQ code



# Moment-SOS Hierarchies for Polynomial Optimization

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Conclusion

# Polynomial Optimization

---

- Semialgebraic set  $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_l(\mathbf{x}) \geq 0\}$
- $p^* := \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})$ : NP hard
- Sums of squares  $\Sigma[\mathbf{x}]$   
e.g.  $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$
- $\mathcal{Q}(\mathbf{S}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^l \sigma_j(\mathbf{x})g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$
- **REMEMBER:**  $f \in \mathcal{Q}(\mathbf{S}) \implies \forall \mathbf{x} \in \mathbf{S}, f(\mathbf{x}) \geq 0$

# Problem reformulation [Lasserre 01]

---

- Borel  $\sigma$ -algebra  $\mathcal{B}$  (generated by the open sets of  $\mathbb{R}^n$ )
- $\mathcal{M}_+(\mathbf{S})$ : set of probability measures supported on  $\mathbf{S}$ .  
If  $\mu \in \mathcal{M}_+(\mathbf{S})$  then
  - 1  $\mu : \mathcal{B} \rightarrow [0, 1], \mu(\emptyset) = 0$
  - 2  $\mu(\cup_i B_i) = \sum_i \mu(B_i)$ , for any countable  $(B_i) \subset \mathcal{B}$
  - 3  $\int_{\mathbf{S}} \mu(dx) = 1$
- $\text{supp}(\mu)$  is the smallest set  $\mathbf{S}$  such that  $\mu(\mathbb{R}^n \setminus \mathbf{S}) = 0$

# Problem reformulation [Lasserre 01]

---

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \int_{\mathbf{S}} f \mu(d\mathbf{x})$$

# Problem reformulation [Lasserre 01]

---

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \sum_{\alpha} f_{\alpha} \int_{\mathbf{S}} \mathbf{x}^{\alpha} \mu(d\mathbf{x})$$

# Primal-dual Moment-SOS [Lasserre 01]

---

- Let  $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$  be the monomial basis

## Definition

A sequence  $\mathbf{z}$  has a representing measure on  $\mathbf{S}$  if there exists a finite measure  $\mu$  supported on  $\mathbf{S}$  such that

$$\mathbf{z}_\alpha = \int_{\mathbf{S}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n.$$

# Primal-dual Moment-SOS [Lasserre 01]

---

- $\mathcal{M}_+(\mathbf{S})$ : space of probability measures supported on  $\mathbf{S}$
- $\mathcal{Q}(\mathbf{S})$ : quadratic module

## Polynomial Optimization Problems (POP)

$$\begin{array}{ll} \text{(Primal)} & \text{(Dual)} \\ \inf \int_{\mathbf{S}} f d\mu & = \sup \lambda \\ \text{s.t. } \mu \in \mathcal{M}_+(\mathbf{S}) & \text{s.t. } \lambda \in \mathbb{R}, \\ & f - \lambda \in \mathcal{Q}(\mathbf{S}) \end{array}$$

# Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences  $\mathbf{z}$  of measures in  $\mathcal{M}_+(\mathbf{S})$
- Truncated quadratic module  $\mathcal{Q}_k(\mathbf{S}) := \mathcal{Q}(\mathbf{S}) \cap \mathbb{R}_{2k}[\mathbf{x}]$

## Polynomial Optimization Problems (POP)

(Moment)		(SOS)
$\inf \sum_{\alpha} f_{\alpha} \mathbf{z}_{\alpha}$	=	$\sup \lambda$
s.t. $\mathbf{M}_{k-v_j}(g_j \mathbf{z}) \succcurlyeq 0, \quad 0 \leq j \leq l,$		s.t. $\lambda \in \mathbb{R},$
$\mathbf{z}_1 = 1$		$f - \lambda \in \mathcal{Q}_k(\mathbf{S})$



# Semidefinite Optimization

---

- $F_0, F_\alpha$  symmetric real matrices, cost vector  $c$

Primal-dual pair of semidefinite programs:

$$(SDP) \begin{cases} \mathcal{P} : & \inf_{\mathbf{z}} \quad \sum_{\alpha} c_{\alpha} \mathbf{z}_{\alpha} \\ & \text{s.t.} \quad \sum_{\alpha} F_{\alpha} \mathbf{z}_{\alpha} - F_0 \succcurlyeq 0 \\ \\ \mathcal{D} : & \sup_{\mathbf{Y}} \quad \text{Trace} (F_0 \mathbf{Y}) \\ & \text{s.t.} \quad \text{Trace} (F_{\alpha} \mathbf{Y}) = c_{\alpha} , \quad \mathbf{Y} \succcurlyeq 0 . \end{cases}$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

Moment-SOS Hierarchies for Polynomial Optimization

New Applications of Moment-SOS Hierarchies

Conclusion

## Moment-SOS Hierarchies for Polynomial Optimization

### New Applications of Moment-SOS Hierarchies

Semialgebraic Maxplus Optimization

Formal Nonlinear Optimization

Pareto Curves

Polynomial Images of Semialgebraic Sets

Program Analysis with Polynomial Templates

### Conclusion

# General informal Framework

---

Given  $\mathbf{K}$  a compact set and  $f$  a **transcendental** function, bound  $f^* = \inf_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$  and prove  $f^* \geq 0$

- $f$  is under-approximated by a **semialgebraic** function  $f_{\text{sa}}$
- Reduce the problem  $f_{\text{sa}}^* := \inf_{\mathbf{x} \in \mathbf{K}} f_{\text{sa}}(\mathbf{x})$  to a **polynomial optimization problem (POP)**

# General informal Framework

---

Approximation theory: Chebyshev/Taylor models

- mandatory for non-polynomial problems
- hard to combine with Sum-of-Squares techniques (degree of approximation)

# Maxplus Approximation

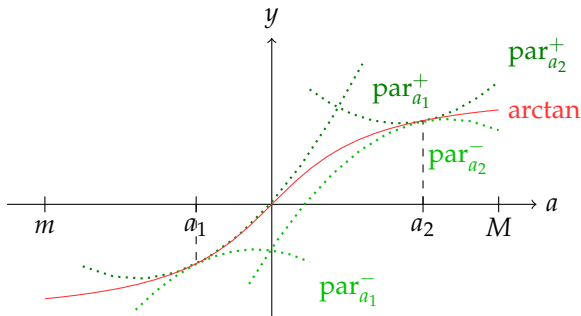
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- Initially introduced to solve Optimal Control Problems [Fleming-McEneaney 00]
- **Curse of dimensionality** reduction [McEneaney Kluberg, Gaubert-McEneaney-Qu 11, Qu 13].  
Allowed to solve instances of dim up to 15 (inaccessible by grid methods)
- In our context: approximate **transcendental** functions

# Maxplus Approximation

## Definition

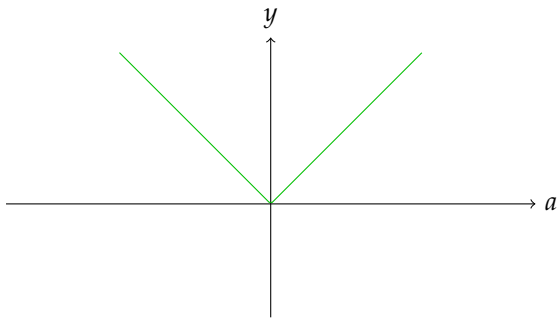
Let  $\gamma \geq 0$ . A function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $\gamma$ -semiconvex if the function  $\mathbf{x} \mapsto \phi(\mathbf{x}) + \frac{\gamma}{2} \|\mathbf{x}\|_2^2$  is convex.



# Nonlinear Function Representation

---

Exact parsimonious maxplus representations

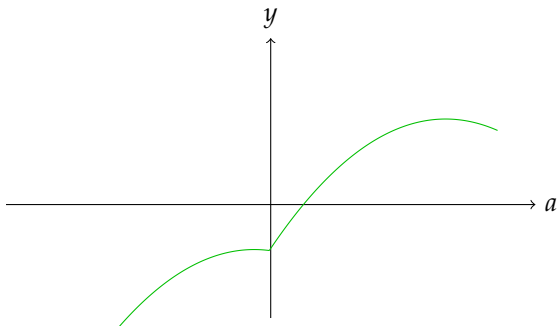




# Nonlinear Function Representation

---

Exact parsimonious maxplus representations



# Nonlinear Function Representation

---

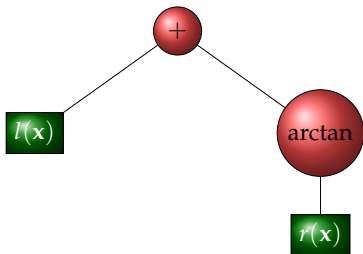
Abstract syntax tree representations of multivariate transcendental functions:

- leaves are **semialgebraic** functions of  $\mathcal{A}$
- nodes are univariate functions of  $\mathcal{D}$  or binary operations

# Nonlinear Function Representation

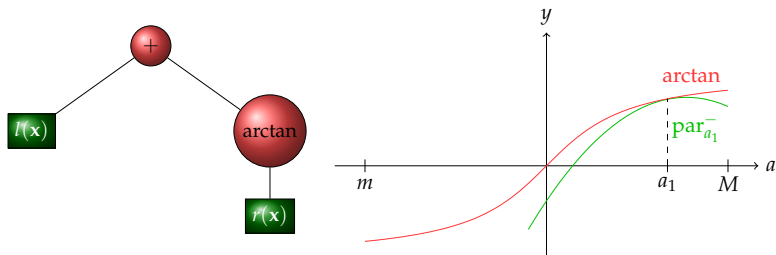
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- For the “Simple” Example from Flyspeck:



# Maxplus Optimization Algorithm

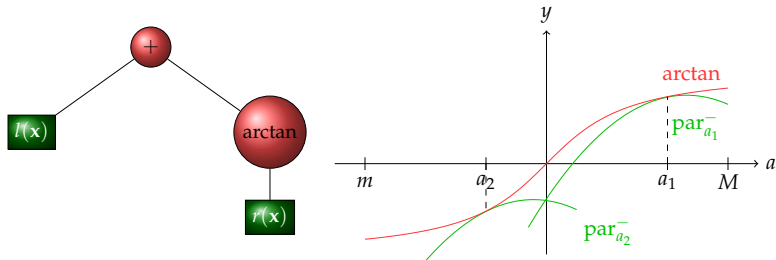
First iteration:



- 1 control point  $\{a_1\}$ :  $m_1 = -4.7 \times 10^{-3} < 0$

# Maxplus Optimization Algorithm

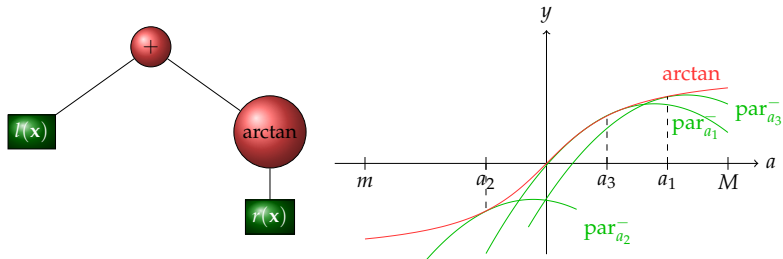
Second iteration:



2 control points  $\{a_1, a_2\}$ :  $m_2 = -6.1 \times 10^{-5} < 0$

# Maxplus Optimization Algorithm

Third iteration:



3 control points  $\{a_1, a_2, a_3\}$ :  $m_3 = 4.1 \times 10^{-6} > 0$

OK!

# Contributions

---



V. Magron, X. Allamigeon, S. Gaubert, and B. Werner.  
Certification of Real Inequalities – Templates and Sums of  
Squares, arxiv:1403.5899, 2014. Accepted for publication in  
*Mathematical Programming SERIES B, volume on Polynomial  
Optimization.*

## Moment-SOS Hierarchies for Polynomial Optimization

### New Applications of Moment-SOS Hierarchies

Semialgebraic Maxplus Optimization

**Formal Nonlinear Optimization**

Pareto Curves

Polynomial Images of Semialgebraic Sets




Program Analysis with Polynomial Templates

### Conclusion



# The General “Formal Framework”

---

-  We check the correctness of SOS certificates for **POP**
-  We build certificates to prove interval bounds for **semialgebraic** functions
-  We bound formally **transcendental** functions with semialgebraic approximations

# Formal SOS bounds

---

When  $q \in \mathcal{Q}(\mathbf{K})$ ,  $\sigma_0, \dots, \sigma_m$  is a positivity certificate for  $q$

Check **symbolic polynomial equalities**  $q = q'$  in COQ



Existing tactic `ring` [Grégoire-Mahboubi 05]



Polynomials coefficients: arbitrary-size rationals `bigQ`  
[Grégoire-Théry 06]





Much simpler to verify certificates using *sceptical approach*



Extends also to **semialgebraic** functions

# Formal Nonlinear Optimization

---

Inequality	#boxes	 Time	 Time
9922699028	39	190 s	2218 s
3318775219	338	1560 s	19136 s

- Comparable with Taylor interval methods in HOL-LIGHT [Hales-Solovyev 13]



Bottleneck of informal optimizer is SOS solver



22 times slower!  $\implies$  Current bottleneck is to check polynomial equalities

# Contribution

---

**For more details on the formal side:**



X. Allamigeon, S. Gaubert, V. Magron and B. Werner. Formal Proofs for Nonlinear Optimization. Under revision, arxiv:1404.7282

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# Bicriteria Optimization Problems

---

- Let  $f_1, f_2 \in \mathbb{R}[\mathbf{x}]$  two conflicting criteria
- Let  $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_m(\mathbf{x}) \geq 0\}$  a semialgebraic set

$$(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) \ f_2(\mathbf{x}))^\top \right\}$$

## Assumption

The image space  $\mathbb{R}^2$  is partially ordered in a natural way ( $\mathbb{R}_+^2$  is the ordering cone).

# Bicriteria Optimization Problems

---

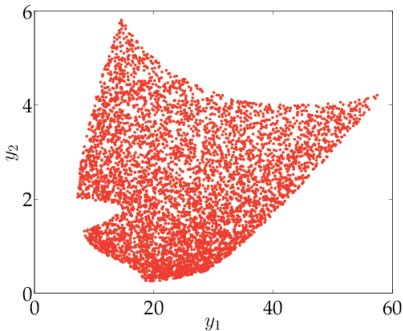
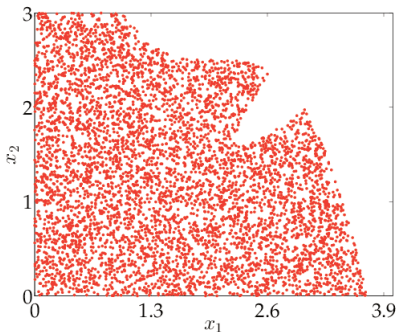
$$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 ,$$

$$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 ,$$

$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geq 0, g_2(\mathbf{x}) \geq 0\} .$$

$$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 ,$$

$$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 .$$



# Parametric Sublevel Set Approximations

---

- Inspired by previous research on multiobjective linear optimization [Gorissen-den Hertog 12]
- Workaround: reduce  $\mathbf{P}$  to a **parametric POP**

$$(\mathbf{P}_\lambda) : f^*(\lambda) := \min_{\mathbf{x} \in \mathbf{S}} \{ f_2(\mathbf{x}) : f_1(\mathbf{x}) \leq \lambda \} ,$$

- variable  $(\mathbf{x}, \lambda) \in \mathbf{K} = \mathbf{S} \times [0, 1]$



# A Hierarchy of Polynomial Approximations

---

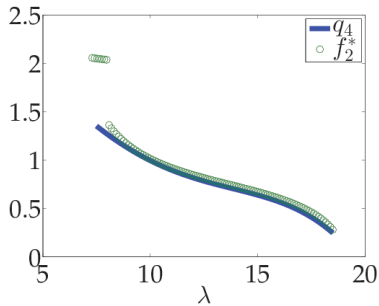
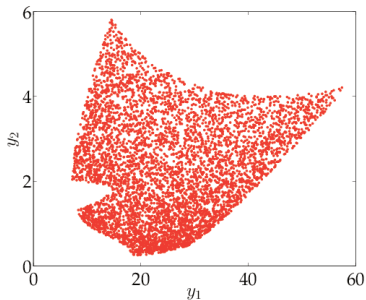
Moment-SOS approach [Lasserre 10]:

$$(D_k) \left\{ \begin{array}{l} \max_{q \in \mathbb{R}_{2k}[\lambda]} \sum_{i=0}^{2k} q_i / (1+i) \\ \text{s.t. } f_2(\mathbf{x}) - q(\lambda) \in \mathcal{Q}_{2k}(\mathbf{K}) . \end{array} \right.$$

- The hierarchy  $(D_k)$  provides a sequence  $(q_k)$  of **polynomial under-approximations** of  $f^*(\lambda)$ .
- $\lim_{d \rightarrow \infty} \int_0^1 (f^*(\lambda) - q_k(\lambda)) d\lambda = 0$

# A Hierarchy of Polynomial Approximations

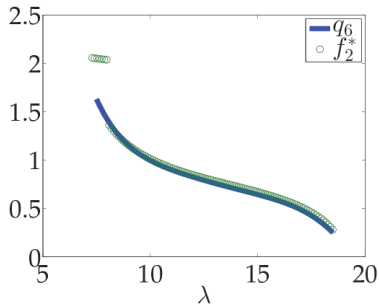
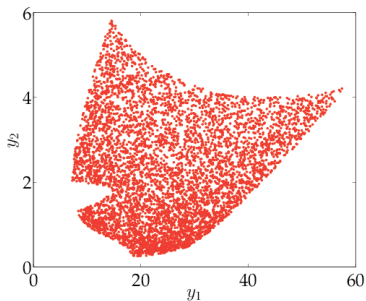
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Degree 4

# A Hierarchy of Polynomial Approximations

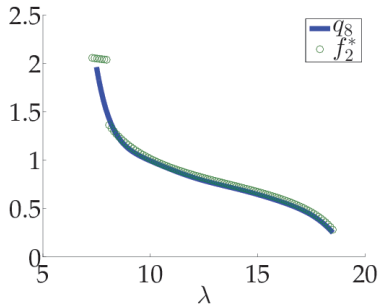
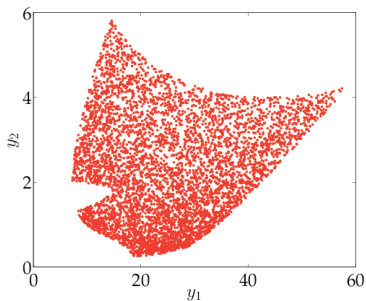
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Degree 6

# A Hierarchy of Polynomial Approximations

---



Degree 8

# Contributions

---

- Numerical schemes that **avoid computing finitely many points**.
- Pareto curve approximation with polynomials, **convergence guarantees** in  $L_1$ -norm



V. Magron, D. Henrion, J.B. Lasserre. Approximating Pareto Curves using Semidefinite Relaxations. *Operations Research Letters*. arxiv:1404.4772, April 2014.

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# Polynomial Images of Semialgebraic Sets

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- Semialgebraic set  $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_l(\mathbf{x}) \geq 0\}$
- A polynomial map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  
 $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$
- $\deg f = d := \max\{\deg f_1, \dots, \deg f_m\}$
- $\mathbf{F} := f(\mathbf{S}) \subseteq \mathbf{B}$ , with  $\mathbf{B} \subset \mathbb{R}^m$  a box or a ball
- Tractable approximations of  $\mathbf{F}$  ?

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- Includes important special cases:

- 1  $m = 1$ : polynomial optimization

$$\mathbf{F} \subseteq [\inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}), \sup_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})]$$

- 2 Approximate **projections** of  $\mathbf{S}$  when  $f(\mathbf{x}) := (x_1, \dots, x_m)$

- 3 **Pareto curve** approximations

For  $f_1, f_2$  two conflicting criteria:  $(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x}) \ f_2(\mathbf{x}))^\top \right\}$



# Method 1: Existential Quantifier Elimination

---

Another point of view:

$$\mathbf{F} = \{ \mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } f(\mathbf{x}) = \mathbf{y} \} ,$$

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$$\mathbf{F} = \{ \mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } \|\mathbf{y} - f(\mathbf{x})\|_2^2 = 0 \} ,$$

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Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } h_f(\mathbf{x}, \mathbf{y}) \geq 0\} ,$$

with

$$h_f(\mathbf{x}, \mathbf{y}) := -\|\mathbf{y} - f(\mathbf{x})\|_2^2 .$$

# Method 1: Existential Quantifier Elimination

---

Existential QE: approximate  $\mathbf{F}$  as closely as desired [Lasserre 14]

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\} ,$$

for some polynomials  $q_k \in \mathbb{R}_{2k}[\mathbf{y}]$ .

# Method 1: Outer Approximations of F

---

■ Let  $\mathbf{K} = \mathbf{S} \times \mathbf{B}$ ,  $\mathcal{Q}_k(\mathbf{K})$  be the  $k$ -truncated quadratic module

■ **REMEMBER:**

$$q - h_f \in \mathcal{Q}_k(\mathbf{K}) \implies \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geq 0$$

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- Define  $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$
- Hierarchy of Semidefinite programs:

$$\inf_q \left\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K}) \right\} .$$

# Method 1: Outer Approximations of $F$

---

Assuming the existence of solution  $q_k$ , the sublevel sets

$$F_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geq 0\} \supseteq F ,$$

provide a sequence of certified outer approximations of  $F$ .



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provide a sequence of certified outer approximations of  $F$ .

It comes from the following:

- $q_k$  feasible solution,  $q_k - h_f \in \mathcal{Q}_k(\mathbf{K})$
- $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q_k(\mathbf{y}) \geq h_f(\mathbf{x}, \mathbf{y}) \iff \forall \mathbf{y}, q_k(\mathbf{y}) \geq h(\mathbf{y})$  .

# Method 1: Strong Convergence Property

---

## Theorem

Assuming that  $\overset{\circ}{\mathbf{S}} \neq \emptyset$  and  $\mathcal{Q}_k(\mathbf{K})$  is Archimedean,

- 1 The sequence of optimal solutions  $(q_k)$  converges to  $h$  w.r.t the  $L_1(\mathbf{B})$ -norm:

$$\lim_{k \rightarrow \infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0, (q_k \rightarrow_{L_1} h)$$

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$$\lim_{k \rightarrow \infty} \text{vol}(\mathbf{F}_k^1 \setminus \mathbf{F}) = 0.$$

## Method 2: Support of Image Measures

---

- **Pushforward**  $f_{\#} : \mathcal{M}(\mathbf{S}) \rightarrow \mathcal{M}(\mathbf{B})$ :

$$f_{\#}\mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{S} : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{B}), \forall \mu_0 \in \mathcal{M}(\mathbf{S})$$

- $f_{\#}\mu_0$  is the **image measure** of  $\mu_0$  under  $f$

## Method 2: Support of Image Measures

---

$$p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int_{\mathbf{B}} \mu_1$$

s.t.  $\mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}},$   
 $\mu_1 = f_{\#} \mu_0,$   
 $\mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}).$

Lebesgue measure on  $\mathbf{B}$  is  $\lambda_{\mathbf{B}}(d\mathbf{y}) := \mathbf{1}_{\mathbf{B}}(\mathbf{y}) d\mathbf{y}$

## Method 2: Support of Image Measures

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$$\begin{aligned} p^* &:= \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int_{\mathbf{B}} \mu_1 \\ \text{s.t. } & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}}, \\ & \mu_1 = f_{\#} \mu_0, \\ & \mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}). \end{aligned}$$

### Lemma

Let  $\mu_1^*$  be an optimal solution of the above LP.  
Then  $\mu_1^* = \lambda_{\mathbf{F}}$  and  $p^* = \text{vol } \mathbf{F}$ .

## Method 2: Primal-dual LP Formulation

---

Primal LP

$$\begin{aligned} p^* &:= \sup_{\mu_0, \mu_1, \hat{\mu}_1} \int \mu_1 \\ \text{s.t. } \quad &\mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}}, \\ &\mu_1 = f_{\#} \mu_0, \\ &\mu_0 \in \mathcal{M}_+(\mathbf{S}), \\ &\mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}). \end{aligned}$$

Dual LP

$$\begin{aligned} d^* &:= \inf_{v, w} \int w(\mathbf{y}) \lambda_{\mathbf{B}}(d\mathbf{y}) \\ \text{s.t. } \quad &v(f(\mathbf{x})) \geq 0, \quad \forall \mathbf{x} \in \mathbf{S}, \\ &w(\mathbf{y}) \geq 1 + v(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbf{B}, \\ &w(\mathbf{y}) \geq 0, \quad \forall \mathbf{y} \in \mathbf{B}, \\ &v, w \in \mathcal{C}(\mathbf{B}). \end{aligned}$$

## Method 2: Strong Convergence Property

---

Strengthening of the dual LP:

$$\begin{aligned} d_k^* &:= \inf_{v, w} \sum_{\beta \in \mathbb{N}_{2k}^m} w_{\beta} z_{\beta}^{\mathbf{B}} \\ \text{s.t. } & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\ & w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), \\ & w \in \mathcal{Q}_k(\mathbf{B}), \\ & v, w \in \mathbb{R}_{2k}[\mathbf{y}]. \end{aligned}$$



## Method 2: Strong Convergence Property

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### Theorem

Assuming that  $\mathring{\mathbf{F}} \neq \emptyset$  and  $\mathcal{Q}_k(\mathbf{S})$  is Archimedean,

- 1 The sequence  $(w_k)$  converges to  $\mathbf{1}_{\mathbf{F}}$  w.r.t the  $L_1(\mathbf{B})$ -norm:

$$\lim_{k \rightarrow \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 .$$

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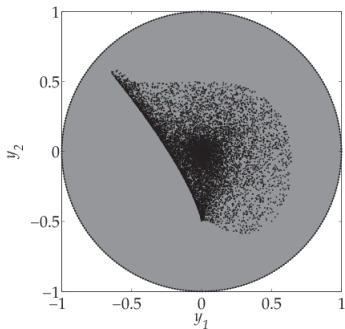
- 2 Let  $\mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geq 1\}$ . Then,

$$\lim_{k \rightarrow \infty} \text{vol}(\mathbf{F}_k^2 \setminus \mathbf{F}) = 0 .$$

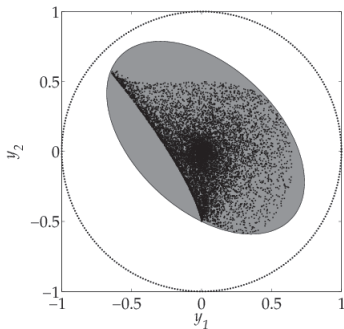
# Polynomial Image of the Unit Ball

Image of the unit ball  $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leq 1\}$  by

$$f(\mathbf{x}) := (x_1 + x_1x_2, x_2 - x_1^3)/2$$



$F_1^1$

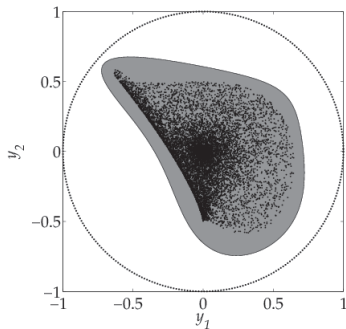


$F_1^2$

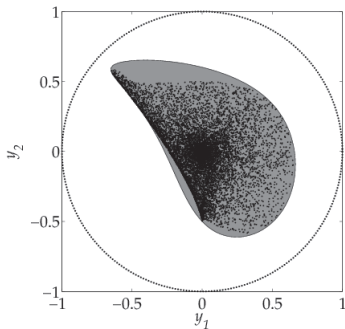
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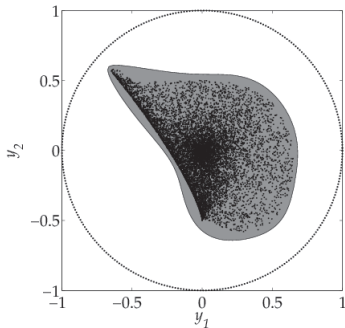


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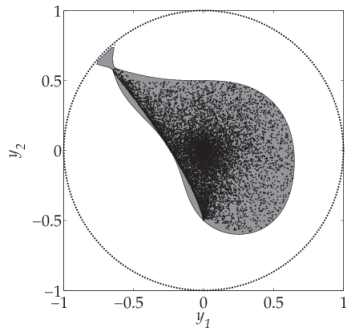
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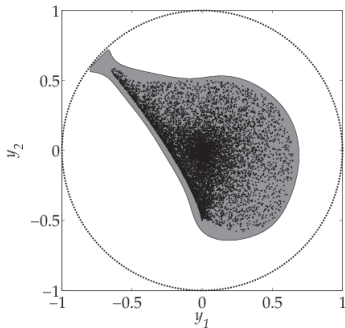


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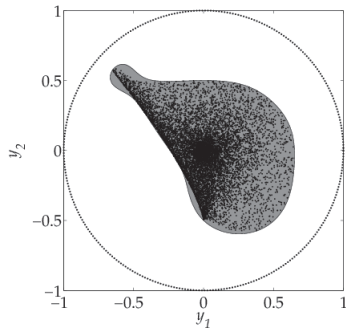
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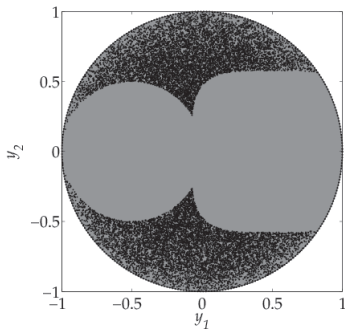


$F_4^2$

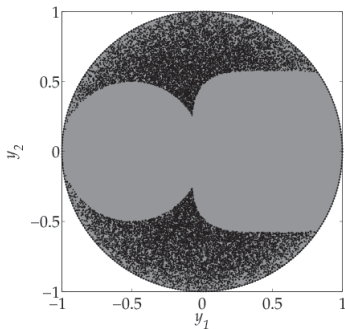
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$f(\mathbf{x}) = (x_1, x_2)$ : projection on  $\mathbb{R}^2$  of the semialgebraic set

$$\mathbf{S} := \{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leq 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geq 0, \\ 1/9 - (x_1 - 1/2)^4 - x_2^4 \geq 0 \}$$



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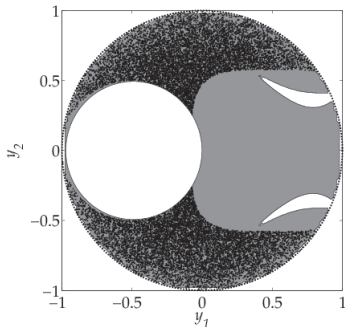


$F_2^2$

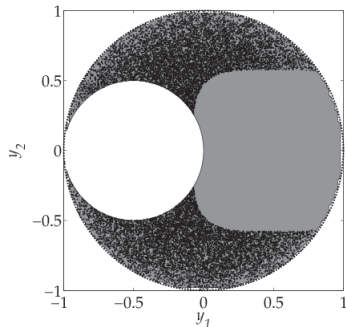
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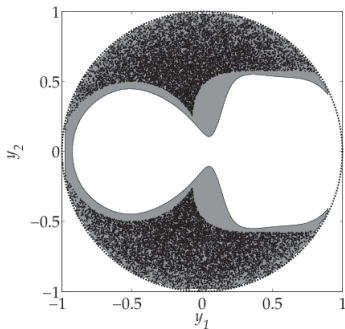
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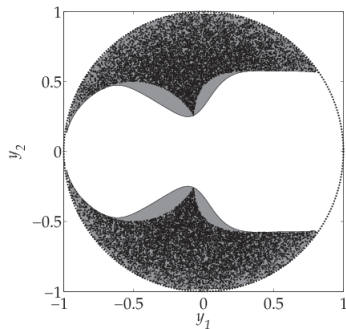
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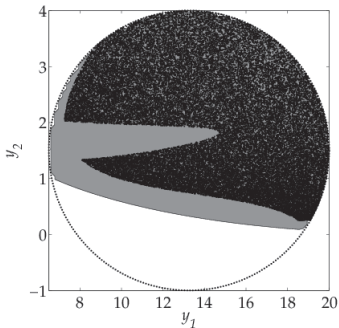


$F_4^2$

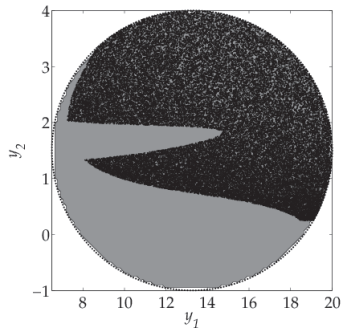
# Approximating Pareto Curves

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Back on our previous nonconvex example:



$F_1^1$

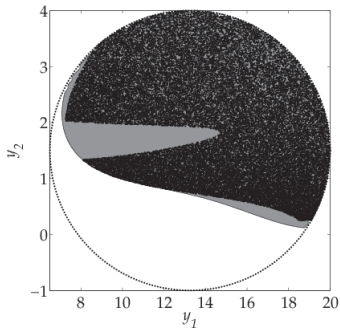


$F_1^2$

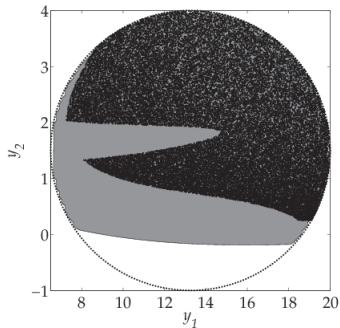
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$F_2^1$

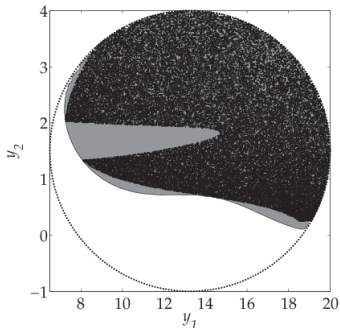


$F_2^2$

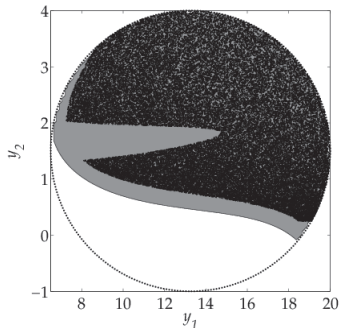
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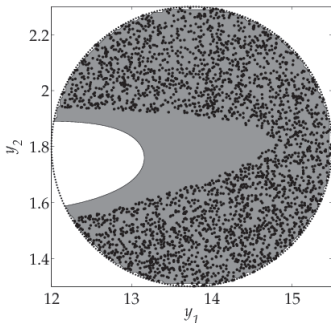


$F_3^2$

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“Zoom” on the region which is hard to approximate:

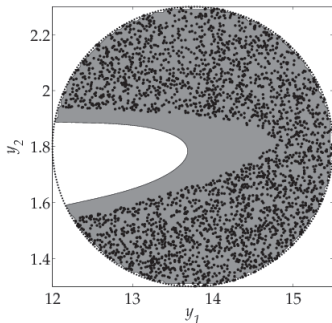


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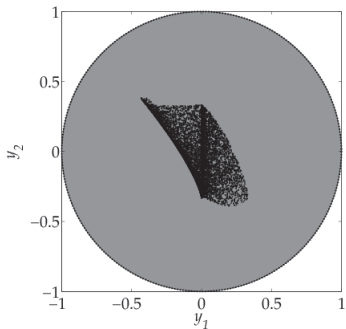


$F_5^1$

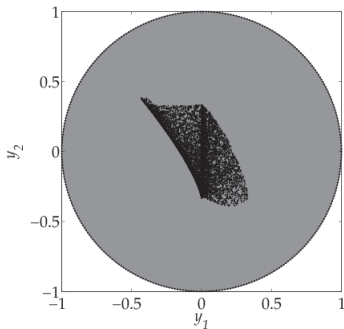
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$F_1^1$

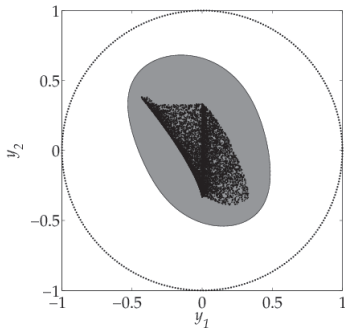


$F_1^2$

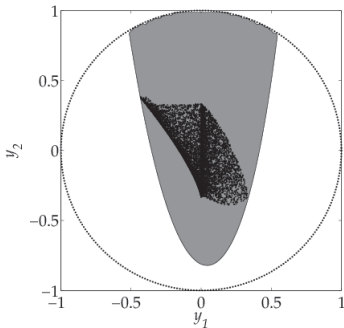
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$F_2^1$



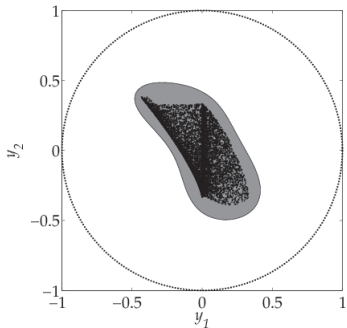
$F_2^2$



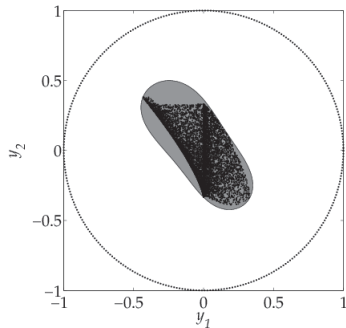
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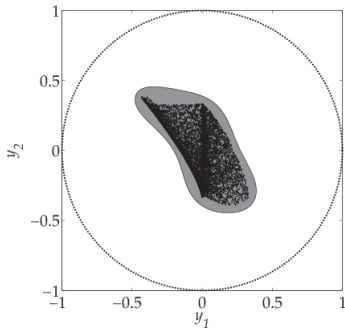


$F_3^2$

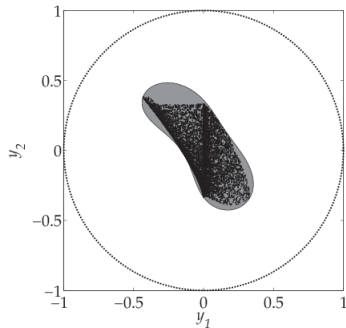
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$F_4^1$



$F_4^2$

# Contributions

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V. Magron, D. Henrion, J.B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. [oo:2014.10.4606](https://arxiv.org/abs/2014.10.4606), October 2014.

## Moment-SOS Hierarchies for Polynomial Optimization

### New Applications of Moment-SOS Hierarchies

Semialgebraic Maxplus Optimization

Formal Nonlinear Optimization

Pareto Curves

Polynomial Images of Semialgebraic Sets

**Program Analysis with Polynomial Templates**

### Conclusion

# One-loop with Conditional Branching

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- $r, s, T^i, T^e \in \mathbb{R}[\mathbf{x}]$
- $\mathbf{x}_0 \in \mathbf{X}_0$ , with  $\mathbf{X}_0$  semialgebraic set

```
 $\mathbf{x} = \mathbf{x}_0$ ;  
while ( $r(\mathbf{x}) \leq 0$ ) {  
  if ( $s(\mathbf{x}) \leq 0$ ) {  
     $\mathbf{x} = T^i(\mathbf{x})$ ;  
  }  
  else {  
     $\mathbf{x} = T^e(\mathbf{x})$ ;  
  }  
}
```

# Well-representative Templates w.r.t. Properties

Sufficient condition to get inductive invariant:

$$\begin{aligned} \alpha &:= \min_{q \in \mathbb{R}[\mathbf{x}]} \sup_{\mathbf{x} \in \mathbf{X}_0} q(\mathbf{x}) \\ \text{s.t. } & q - q \circ T^i \geq 0, \text{ if } s(\mathbf{x}) \leq 0, \\ & q - q \circ T^e \geq 0, \text{ if } s(\mathbf{x}) \geq 0, \\ & q - \kappa \geq 0. \end{aligned}$$

■  $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k \subseteq \{\mathbf{x} \in \mathbb{R}^n : q(\mathbf{x}) \leq \alpha\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \kappa(\mathbf{x}) \leq \alpha\}$

# Bounding Template using SOS

Sufficient condition to get bounding inductive invariant:

$$\begin{aligned} \alpha &:= \min_{q \in \mathbb{R}[\mathbf{x}]} \sup_{\mathbf{x} \in \mathbf{X}_0} q(\mathbf{x}) \\ \text{s.t. } & q - q \circ T^i \geq 0, \text{ if } s(\mathbf{x}) \leq 0, \\ & q - q \circ T^e \geq 0, \text{ if } s(\mathbf{x}) \geq 0, \\ & q - \|\cdot\|_2^2 \geq 0. \end{aligned}$$

■  $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k \subseteq \{\mathbf{x} \in \mathbb{R}^n : q(\mathbf{x}) \leq \alpha\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^2 \leq \alpha\}$

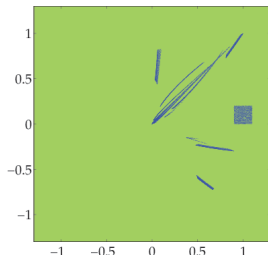
# Bounds for $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k$

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$$\mathbf{X}_0 := [0.9, 1.1] \times [0, 0.2] \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$$

$$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := \left(\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2\right)$$

$$\kappa(\mathbf{x}) = \|\mathbf{x}\|^2$$



Degree 6



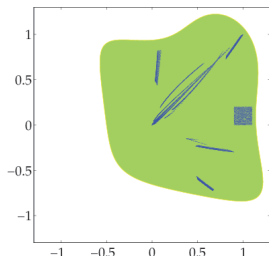
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Degree 8

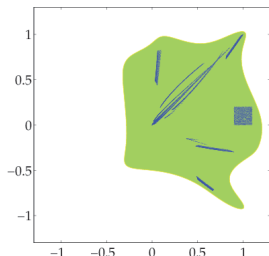
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Degree 10

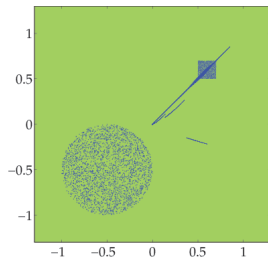
# Does $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k$ avoid unsafe region?

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$$\mathbf{X}_0 := [0.5, 0.7]^2 \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$$

$$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := \left(\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2\right)$$

$$\kappa(\mathbf{x}) = \frac{1}{4} - \left(x_1 + \frac{1}{2}\right)^2 - \left(x_2 + \frac{1}{2}\right)^2$$



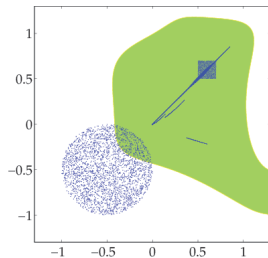
Degree 6

# Does $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k$ avoid unsafe region?

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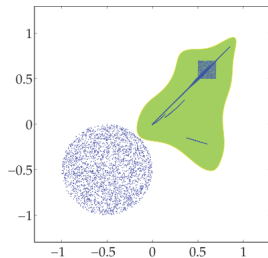
Degree 8

# Does $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k$ avoid unsafe region?

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Degree 10

# Contributions

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A. Adjé, V. Magron. Polynomial template generation using sum-of-squares programming. *Submitted*. arxiv:1409.3941, October 2014.

Moment-SOS Hierarchies for Polynomial Optimization

New Applications of Moment-SOS Hierarchies

**Conclusion**

# Conclusion

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With **MOMENT-SOS HIERARCHIES**, you can

- Optimize nonlinear (transcendental) functions
- Approximate Pareto Curves, images and projections of semialgebraic sets
- Analyze programs



# Conclusion

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## Further research:

- Alternative polynomial bounds using geometric programming (T. de Wolff, S. Ilman)
- Mixed LP/SOS certificates (trade-off CPU/precision)

# End

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Thank you for your attention!

`cas.ee.ic.ac.uk/people/vmagron`

# Hidden Details

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$$\ell_{\mathbf{z}}(q) : q \in \mathbb{R}[\mathbf{x}] \mapsto \sum_{\alpha} q_{\alpha} \mathbf{z}_{\alpha}$$

- Moment matrix

$$\mathbf{M}(\mathbf{z})_{\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}} := \ell_{\mathbf{z}}(\mathbf{x}^{\alpha} \mathbf{x}^{\beta}) = \mathbf{z}_{\alpha+\beta}$$

- Localizing matrix  $\mathbf{M}(\mathbf{g}_j \mathbf{z})$  associated with  $\mathbf{g}_j$

$$\mathbf{M}(\mathbf{g}_j \mathbf{z})_{\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}} := \ell_{\mathbf{z}}(\mathbf{g}_j \mathbf{x}^{\alpha} \mathbf{x}^{\beta}) = \sum_{\gamma} \mathbf{g}_{j, \gamma} \mathbf{z}_{\alpha+\beta+\gamma}$$

## Hidden Details

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■  $\mathbf{M}_k(\mathbf{z})$  contains  $\binom{n+2k}{n}$  variables, has size  $\binom{n+k}{n}$

■ Truncated matrix of order  $k = 2$  with variables  $x_1, x_2$ :

$$\mathbf{M}_2(\mathbf{z}) = \begin{array}{c} 1 \\ - \\ x_1 \\ x_2 \\ - \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} \left( \begin{array}{c|cc|cc|ccc} 1 & & & & & & & & \\ \hline & x_1 & x_2 & & & x_1^2 & x_1x_2 & x_2^2 & \\ \hline 1 & z_{1,0} & z_{0,1} & & & z_{2,0} & z_{1,1} & z_{0,2} & \\ - & - & - & - & - & - & - & - & \\ x_1 & z_{1,0} & & z_{2,0} & z_{1,1} & & z_{3,0} & z_{2,1} & z_{1,2} \\ x_2 & z_{0,1} & & z_{1,1} & z_{0,2} & & z_{2,1} & z_{1,2} & z_{0,3} \\ - & - & - & - & - & - & - & - & \\ x_1^2 & & z_{2,0} & & z_{3,0} & z_{2,1} & & z_{4,0} & z_{3,1} & z_{2,2} \\ x_1x_2 & & z_{1,1} & & z_{2,1} & z_{1,2} & & z_{3,1} & z_{2,2} & z_{1,3} \\ x_2^2 & & z_{0,2} & & z_{1,2} & z_{0,3} & & z_{2,2} & z_{1,3} & z_{0,4} \end{array} \right)$$

# Hidden Details

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- Consider  $g_1(\mathbf{x}) := 2 - x_1^2 - x_2^2$ . Then  $v_1 = \lceil \deg g_1 / 2 \rceil = 1$ .

$$\mathbf{M}_1(g_1 \mathbf{z}) = \begin{matrix} & \color{blue}{1} & & \color{blue}{x_1} & & \color{blue}{x_2} \\ \color{blue}{1} & & & & & \\ \color{blue}{x_1} & \left( \begin{array}{ccc} 2 - z_{2,0} - z_{0,2} & 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{0,1} - z_{2,1} - z_{0,3} \\ 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{2,0} - z_{4,0} - z_{2,2} & 2z_{1,1} - z_{3,1} - z_{1,3} \\ 2z_{0,1} - z_{2,1} - z_{0,3} & 2z_{1,1} - z_{3,1} - z_{1,3} & 2z_{0,2} - z_{2,2} - z_{0,4} \end{array} \right) & & & \\ \color{blue}{x_2} & & & & & \end{matrix}$$

$$\begin{aligned} \mathbf{M}_1(g_1 \mathbf{z})(3,3) &= \ell(g_1(\mathbf{x}) \cdot x_2 \cdot x_2) = \ell(2x_2^2 - x_1^2x_2^2 - x_2^4) \\ &= 2z_{0,2} - z_{2,2} - z_{0,4} \end{aligned}$$

# Hidden Details

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- Truncation with moments of order at most  $2k$
- $v_j := \lceil \deg g_j / 2 \rceil$
- Hierarchy of semidefinite relaxations:

$$\left\{ \begin{array}{l} \inf_{\mathbf{z}} \ell_{\mathbf{z}}(f) = \sum_{\alpha} \int_{\mathbf{S}} f_{\alpha} \mathbf{x}^{\alpha} \mu(d\mathbf{x}) = \sum_{\alpha} f_{\alpha} \mathbf{z}_{\alpha} \\ \mathbf{M}_k(\mathbf{z}) \succeq 0, \\ \mathbf{M}_{k-v_j}(g_j \mathbf{z}) \succeq 0, \quad 1 \leq j \leq l, \\ \mathbf{z}_1 = 1. \end{array} \right.$$