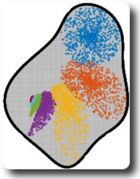
Applications of the Moment-Sums of Squares Hierarchy

Victor Magron, CNRS–LAAS



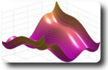
TU Chemnitz
22 October 2019

The Moment-Sums of Squares Hierarchy

NP-hard NON CONVEX Problem $p^\star = \inf p(\mathbf{x})$

Theory

(Primal)		(Dual)



$$\inf \int p \, d\mu \qquad \sup \lambda$$

with μ proba $\Rightarrow$ **INFINITE LP** $\Leftarrow$ with $p - \lambda \geqslant 0$

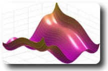
The Moment-Sums of Squares Hierarchy

NP-hard NON CONVEX Problem $p^\star = \inf p(\mathbf{x})$

Practice

(Primal **Relaxation**)

moments $\int \mathbf{x}^\alpha \, d\mu$



(Dual **Strengthening**)

$p - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow p^*$ [Lasserre/Parrilo 01]

degree d
n vars $\Longrightarrow \binom{n+d}{n}$ **SDP** VARIABLES

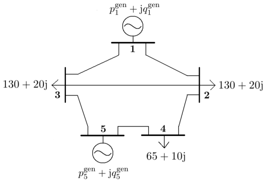
Numeric Solvers $\Longrightarrow$ **Approx** Certificate

Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as ∞-*dimensional* LP over measures

STATIC **Polynomial Optimization**

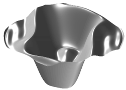
Optimal Powerflow $n \simeq \mathbf{10^3}$ [Josz et al 16]



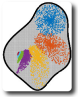
Roundoff Error $n \simeq \mathbf{10^2}$ [Magron et al 17]

DYNAMICAL **Polynomial Optimization**

Regions of attraction [Henrion et al 14]



Reachable sets [Magron et al 17]

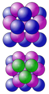


⚠ **APPROXIMATE** OPTIMIZATION BOUNDS!

Success Stories: Certified Optimization

Kepler's Conjecture(1611)

The max density of sphere packings is $\pi/\sqrt{18}$



Flyspeck : **F**ormalizing the **p**roof of **K**epler by T.Hales (1994)
Verification of thousands of "tight" nonlinear inequalities

Seminal Paper:

Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017

CONTRIBUTION:

(Non)-Polynomial optimization to verify **Flyspeck** inequalities

Reachability in Discrete Time

Invariant Measures of Polynomial Systems

Oranges Stack and Kepler's Conjecture

Roundoff Error Bounds

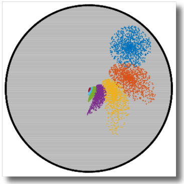
Reachability in Discrete Time

Invariant Measures of Polynomial Systems

Oranges Stack and Kepler's Conjecture

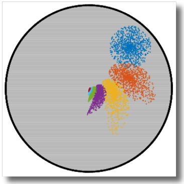
Roundoff Error Bounds

Reachability in Discrete Time



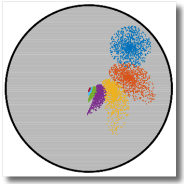
Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$

Reachability in Discrete Time



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$
Polynomial map $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

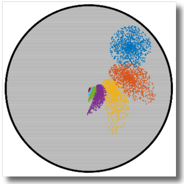
Reachability in Discrete Time



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$
Polynomial map $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

Reachable Set (RS) of admissible trajectories
$\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t \in \mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$

Reachability in Discrete Time

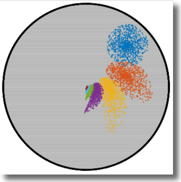


Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$
Polynomial map $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

Reachable Set (RS) of admissible trajectories
$\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t \in \mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$
$\mathbf{X}^\infty = \bigcup_{t \in \mathbb{N}} f^t(\mathbf{X}_0) \subseteq \mathbf{X} \subset \mathbb{R}^n$ (box or ball)

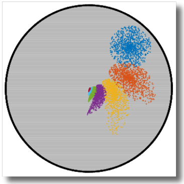
Tractable approximations of RS $\mathbf{X}^\infty$?

Reachability in Continuous Time



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\}$
Polynomial map $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

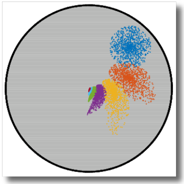
Reachability in Continuous Time



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\}$
Polynomial map $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

Reachable Set (RS) of admissible trajectories
$\mathbf{X}^T := \{(\mathbf{x}(t)) : \dot{\mathbf{x}} = f(\mathbf{x}), \forall t \in [0, T], \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{X}_0\}$

Reachability in Continuous Time



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\}$
Polynomial map $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$

Reachable Set (RS) of admissible trajectories
$\mathbf{X}^T := \{(\mathbf{x}(t)) : \dot{\mathbf{x}} = f(\mathbf{x}), \forall t \in [0, T], \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{X}_0\}$
$\mathbf{X}^\infty := \lim_{t \to \infty} \mathbf{X}^T \subseteq \mathbf{X} \subset \mathbb{R}^n$ (box or ball)

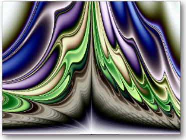
Tractable approximations of RS $\mathbf{X}^\infty$?

Motivations

- Occurs in several contexts :
  1. program analysis: fixpoint computation

```
toyprogram (x1,x2)
    requires (0.25 ⩽ x1 ⩽ 0.75 && 0.25 ⩽ x2 ⩽ 0.75)
        ;
    while (x1^2 + x2^2 ⩽ 1){
        x1 = x1 + 2x1x2;
           x2 = 0.5(x2 − 2x1^3);
     }
```

  2. hybrid systems, biology: Neuron Model, Growth Model
  3. control: integrator, Hénon map



Related work: RS

1. Contractive methods based on LP relaxations and polyhedra projection [Bertsekas 72]
2. Extension to nonlinear systems [Harwood et al. 16]
3. Bernstein/Krivine-Handelman representations [Ben Sassi- et al. 15, Ben Sassi et al. 12]

$\oplus$ LP relaxations $\implies$ scalability

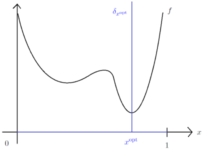
$\ominus$ Convex approximations of nonconvex sets $\implies$ coarse

$\ominus$ No convergence guarantees (very often)

Related work: Lasserre hierarchy

Cast a polynomial optimization problem as an *infinite-dimensional* LP over measures [Lasserre 2001]

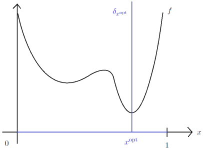
$$f^\star := \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f(\mathbf{x}) d\mu$$

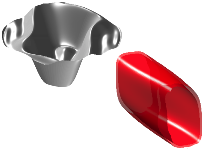


Related work: Lasserre hierarchy

Cast a polynomial optimization problem as an *infinite-dimensional* LP over measures [Lasserre 2001]

$$f^\star := \inf_{\mathbf{x}\in\mathbf{X}} f(\mathbf{x}) = \inf_{\mu\in\mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f(\mathbf{x})d\mu$$

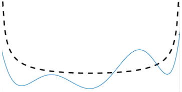




⇝ Regions of attraction [Henrion-Korda 14]

⇝ Maximum invariants [Korda et al. 13]

⇝ Invariant 1D densities [Henrion 2012]



⇝ Maximal positively invariants [Oustry-Tacchi-Henrion 2019]

Related work: Lasserre hierarchy

4. SDP approximation of polynomial images of semialgebraic sets [M.-Henrion-Lasserre 15]

- $\mathbf{X}_1 := f(\mathbf{X}_0) \subseteq \mathbf{X}$, with $\mathbf{X} \subset \mathbb{R}^n$ a box or a ball
  $\implies$ Discrete-time system with a single iteration

- 💡 Approximation of image measure supports
  $\implies$ certified SDP over approximations of $\mathbf{X}_1$

- $\mathbf{X}_t := f^t(\mathbf{X}_0)$

  ⊖ $\deg f^t = d \times t \implies$ very expensive computation

  ⊖ Would only approximate $\mathbf{X}_t$ and not $\mathbf{X}^\infty$

Contributions

- General framework to approximate $\mathbf{X}^\infty$

  $\oplus$ **No discretization** is required

Contributions

- General framework to approximate $\mathbf{X}^\infty$
  $\oplus$ **No discretization** is required

- Infinite-dimensional LP formulation
  💡 support of measures solving Liouville's Equation

Contributions

- General framework to approximate $\mathbf{X}^\infty$
  $\oplus$ **No discretization** is required

- Infinite-dimensional LP formulation
  💡 support of measures solving Liouville's Equation

- Finite-dimensional SDP relaxations

- $\mathbf{X}^\infty \subseteq \mathbf{X}_r^\infty = \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geqslant 1\}$
  $\oplus$ Strong convergence guarantees
  $\lim_{r\to\infty} \text{vol}(\mathbf{X}_r^\infty \backslash \mathbf{X}^\infty) = 0$
  $\oplus$ Compute w_r by solving one **semidefinite program**

Contributions

- General framework to approximate $\mathbf{X}^\infty$
  $\oplus$ **No discretization** is required

- Infinite-dimensional LP formulation
  support of measures solving Liouville's Equation

- Finite-dimensional SDP relaxations

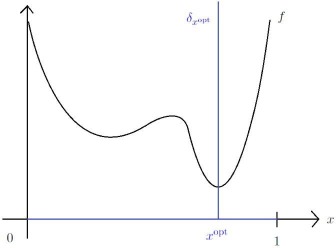
- $\mathbf{X}^\infty \subseteq \mathbf{X}_r^\infty = \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geqslant 1\}$
  $\oplus$ Strong convergence guarantees
  $\lim_{r \to \infty} \mathrm{vol}(\mathbf{X}_r^\infty \backslash \mathbf{X}^\infty) = 0$
  $\oplus$ Compute w_r by solving one **semidefinite program**

M., Garoche, Henrion and Thirioux. Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems. *SICON*, 2019

Characterizing the RS

Characterize a value

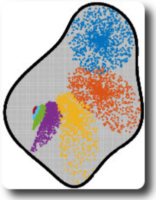
$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$



Dirac measure $\mu^\star = \delta_{\mathbf{x}^{\text{opt}}}$

Characterize a set

?

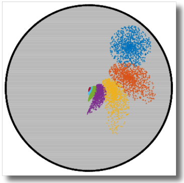


Lebesgue measure $\mu^\star = \lambda_{\mathbf{X}^\infty}$

Occupation Measures and Liouville's Equation

$\mathbf{x}_{t+1} = f(\mathbf{x}_t) \quad \mathbf{x}_0 \in \mathbf{X}_0$

$\mathbf{x}_1 = f(\mathbf{x}_0) \ldots \mathbf{x}_t = f(\mathbf{x}_{t-1})$



- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$
- **Pushforward** $f_\# : \mathcal{M}_+(\mathbf{X}_0) \to \mathcal{M}_+(\mathbf{X})$

$$\mu_1(\mathbf{A}) = f_\# \mu_0(\mathbf{A}) := \mu_0(f^{-1}(\mathbf{A}))$$

- $f_\# \mu_0$ is the **image measure** of μ_0 under f

Occupation Measures and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ and

$$\mu_1 = f_\# \mu_0$$
$$\dots$$
$$\mu_t = f_\# \mu_{t-1}$$
$$\nu_t = \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} f_\#^i \mu_0$$

- The measures μ_t, ν_t, μ_0 satisfy **Liouville's Equation**:

$$\mu_t + \nu_t = f_\# \nu_t + \mu_0$$

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\lambda_{\mathbf{X}_t} = f^t_\# \mu_{0,t}$
  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\lambda_{\mathbf{X}_t} = f^t_{\#}\, \mu_{0,t}$
  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**
- Lebesgue measure $\lambda_{\mathbf{X}^T}$ on $\mathbf{X}^T := \bigcup_{t=0}^T \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\lambda_{\mathbf{X}_t} = f^t_{\#}\, \mu_{0,t}$
  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**
- Lebesgue measure $\lambda_{\mathbf{X}^T}$ on $\mathbf{X}^T := \bigcup_{t=0}^T \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

$$\lambda_{\mathbf{X}^T} + \nu^T = f_{\#}\, \nu^T + \mu_0^T$$

average **occupation measure** ν^T: measures time spent in $\mathbf{X}^T$

Volume Assumption

Discrete Time

Define $\mathbf{Y}^0 := \mathbf{X}^0$ and $\mathbf{Y}^t := \mathbf{X}_t \backslash \mathbf{X}^{t-1}$.

$$\lim_{T \to \infty} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t < \infty .$$

Volume Assumption

Discrete Time

Define $\mathbf{Y}^0 := \mathbf{X}^0$ and $\mathbf{Y}^t := \mathbf{X}_t \backslash \mathbf{X}^{t-1}$.

$$\lim_{T \to \infty} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t < \infty .$$

Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Volume Assumption

Continuous Time

Define $\tau(\mathbf{x})$ = minimal time to reach $\mathbf{x}$.

$$\frac{1}{\mathrm{vol}(\mathbf{X})} \int_{\mathbf{X}^\infty} \tau(\mathbf{x}) d\mathbf{x} < \infty.$$

Volume Assumption

Continuous Time

Define $\tau(\mathbf{x})$ = minimal time to reach $\mathbf{x}$.

$$\frac{1}{\mathrm{vol}(\mathbf{X})} \int_{\mathbf{X}^\infty} \tau(\mathbf{x}) d\mathbf{x} < \infty.$$

Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Infinite Primal LP for Discrete RS

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \nu = f_{\#} \nu + \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0), \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Infinite Primal LP for Continuous RS

$$
\begin{aligned}
p^T := \sup_{\mu_0,\mu,\nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \operatorname{div}(f\nu) = \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0)\,, \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Infinite Primal LP for Continuous RS

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \operatorname{div}(f \nu) = \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0), \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

$$
\int_{\mathbf{X}} v(\mathbf{x}) \operatorname{div} f \nu = - \int_{\mathbf{X}} \operatorname{grad} v(\mathbf{x}) \cdot f(\mathbf{x}) d\nu
$$

Infinite Primal LP for Discrete/Continuous RS

Lemma

Volume Assumption $\implies$ optimal solution $\mu^\star = \lambda_{\mathbf{X}^\infty}$

Primal-dual LP in Discrete Time

Primal LP

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \nu = f_\# \nu + \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \\
& \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^T := \inf_{u, v, w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{C}_+(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{C}_+(\mathbf{X}) \\
& w \in \mathcal{C}_+(\mathbf{X}) \\
& u + v \circ f - v \in \mathcal{C}_+(\mathbf{X}) \\
& u \geqslant 0 \\
& u \in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X})
\end{aligned}
$$

Primal-dual LP in Continuous Time

Primal LP

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \operatorname{div}(f \nu) = \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \\
& \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^T := \inf_{u, v, w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{C}_+(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{C}_+(\mathbf{X}) \\
& w \in \mathcal{C}_+(\mathbf{X}) \\
& u + \operatorname{grad} v \cdot f \in \mathcal{C}_+(\mathbf{X}) \\
& u \geqslant 0 \\
& u \in \mathbb{R}, (v, w) \in \mathcal{C}^1(\mathbf{X}) \times \mathcal{C}(\mathbf{X})
\end{aligned}
$$

Zero Duality Gap

Lemma

1. $p^T = d^T$ and $\exists$ minimizing sequence (u_k, v_k, w_k) for dual LP.
2. Volume Assumption $\Longrightarrow p^T = d^T = \operatorname{vol} \mathbf{X}^\infty$

SDP Strengthening of the Dual LP

Discrete Time

$$
\begin{aligned}
d_r^T := \inf_{u,v,w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + Tu) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{Q}_r(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \\
& u + v \circ f - v \in \mathcal{Q}_{rd}(\mathbf{X}) \\
& w \in \mathcal{Q}_r(\mathbf{X}) \\
& u \geqslant 0
\end{aligned}
$$

SDP Strengthening of the Dual LP

Continuous Time

$$
\begin{aligned}
d_r^T := \inf_{u,v,w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + Tu) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{Q}_r(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \\
& u + \operatorname{grad} v \cdot f \in \mathcal{Q}_{r+d}(\mathbf{X}) \\
& w \in \mathcal{Q}_r(\mathbf{X}) \\
& u \geqslant 0
\end{aligned}
$$

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \backslash \mathbf{X}^\infty$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \backslash \mathbf{X}^\infty$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.
2. Dual SDP has optimal solution (u_r, v_r, w_r):

$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r + u_r T - \mathbf{1}_{\mathbf{X}^\infty}| \, d\mathbf{x} = 0 \, .$$

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \backslash \mathbf{X}^\infty$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.
2. Dual SDP has optimal solution (u_r, v_r, w_r):

$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r + u_r T - \mathbf{1}_{\mathbf{X}^\infty}| \, d\mathbf{x} = 0 \, .$$

3. Let $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \backslash \mathbf{X}^\infty$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.
2. Dual SDP has optimal solution (u_r, v_r, w_r):

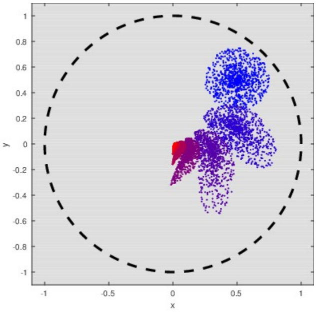
$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r + u_r T - \mathbf{1}_{\mathbf{X}^\infty}| \, d\mathbf{x} = 0 \, .$$

3. Let $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.
4. Volume Assumption $\Rightarrow \lim_{r \to \infty} \mathrm{vol}(\mathbf{X}_r^\infty \backslash \mathbf{X}^\infty) = 0$.

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

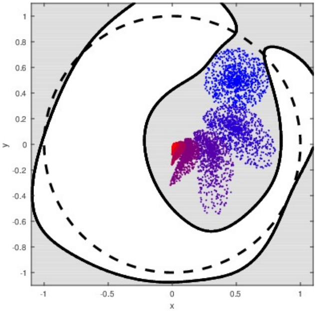


$\mathbf{X}_2^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$



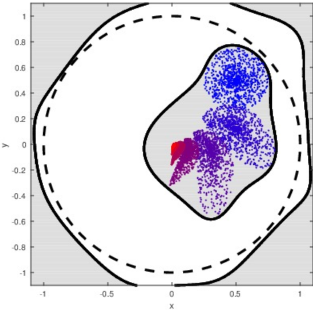
$\mathbf{X}_3^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$

$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

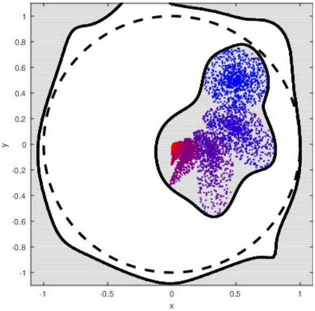


$\mathbf{X}_4^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2),$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3),$$

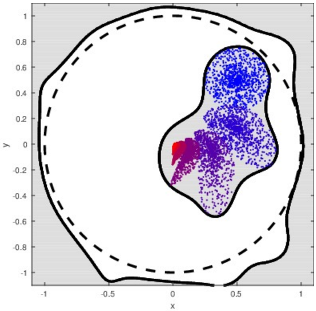


$\mathbf{X}_5^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

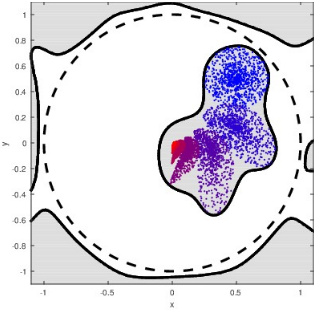


$\mathbf{X}_6^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2)\,,$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3)\,,$$

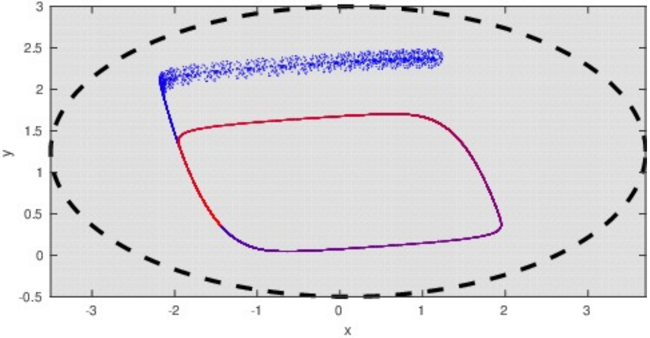


$\mathbf{X}_7^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

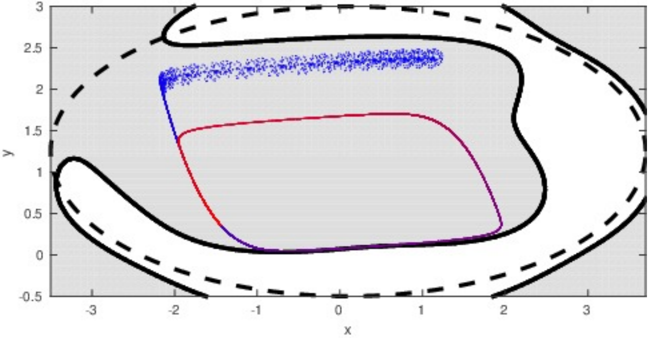


$\mathbf{X}_2^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

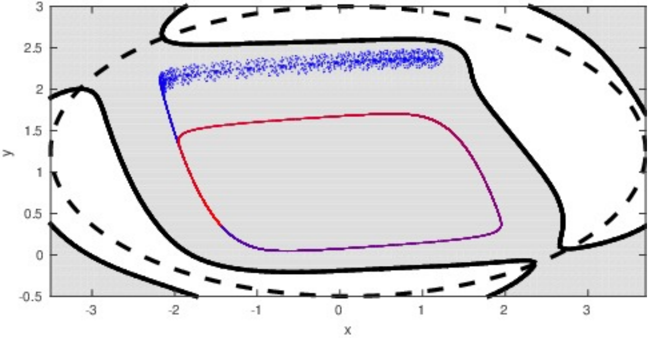


$\mathbf{X}_3^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

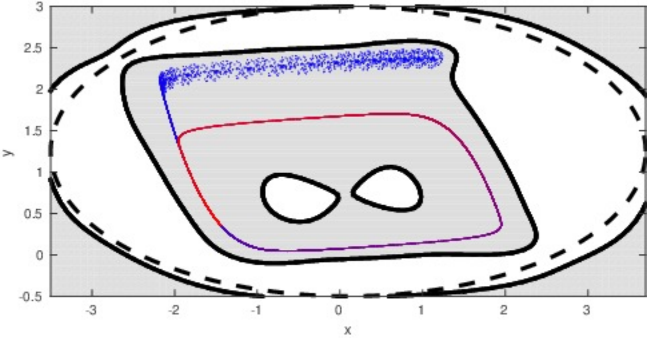


$\mathbf{X}_4^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

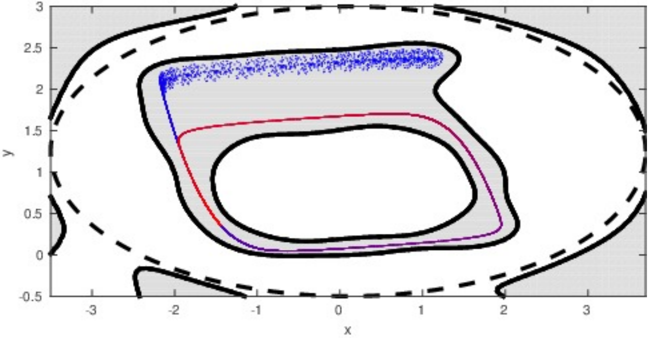


$\mathbf{X}_5^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

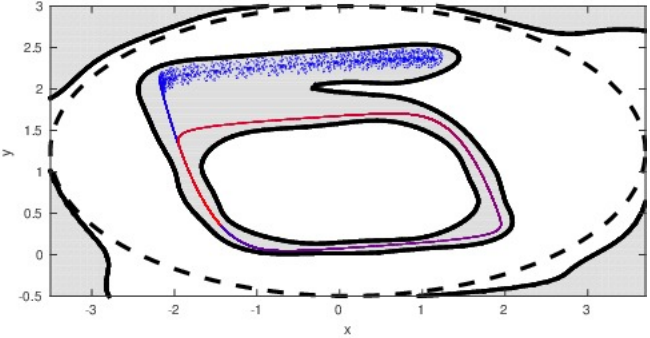


$\mathbf{X}_6^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$

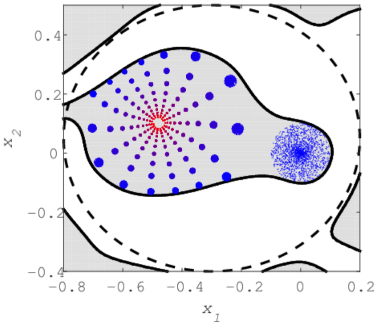


$\mathbf{X}_7^\infty$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
$$x_2^+ := 2x_1x_2 + c_2 \,,$$

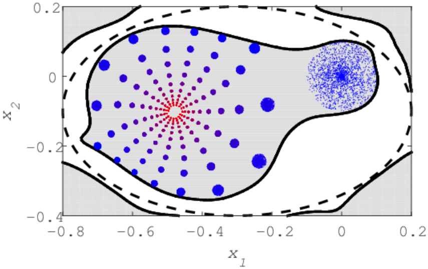


$\mathbf{X}_5^\infty$ with $c_1 = -0.7$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
$$x_2^+ := 2x_1x_2 + c_2 \,,$$

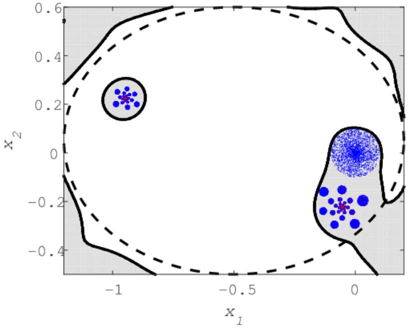


$\mathbf{X}_5^\infty$ with $c_1 = -0.7$ and $c_2 = -0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1\,,$$
$$x_2^+ := 2x_1x_2 + c_2\,,$$

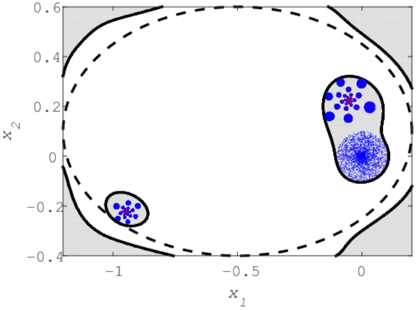


$\mathbf{X}_5^\infty$ with $c_1 = -0.9$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$
\begin{aligned}
x_1^+ &:= x_1^2 - x_2^2 + c_1 \,, \\
x_2^+ &:= 2x_1 x_2 + c_2 \,,
\end{aligned}
$$



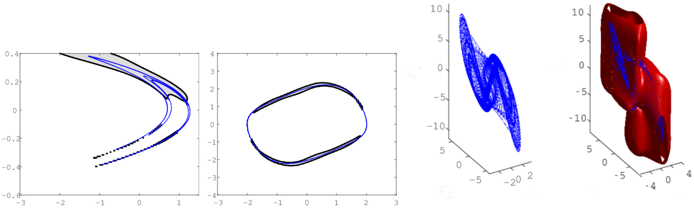
$\mathbf{X}_5^\infty$ with $c_1 = -0.9$ and $c_2 = -0.2$

Invariant Measures of Polynomial Systems

Discrete $\mathbf{x}_{t+1} = f(\mathbf{x}_t) \implies f_{\#}\mu - \mu = 0$

Continuous $\dot{\mathbf{x}} = f(\mathbf{x}) \implies \operatorname{div} f \mu = 0$

- **Converging** SDP hierarchies
- measures with density in L_p
- singular measures $\implies$ chaotic attractors

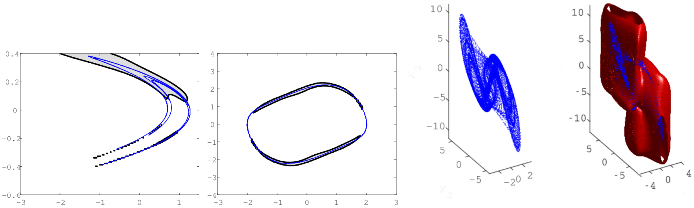


Invariant Measures of Polynomial Systems

Discrete $\mathbf{x}_{t+1} = f(\mathbf{x}_t) \implies f_{\#}\,\mu - \mu = 0$

Continuous $\dot{\mathbf{x}} = f(\mathbf{x}) \implies \operatorname{div} f\,\mu = 0$

- **Converging** SDP hierarchies
- measures with density in L_p
- singular measures $\implies$ chaotic attractors

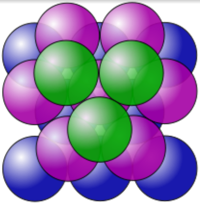


M., Forets, Henrion. Semidefinite Characterization of Invariant Measures for Polynomial Systems. *Discrete and Continuous Dynamical Systems B*, 2018.

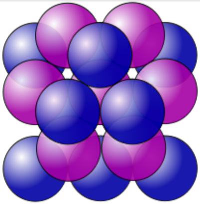
Oranges Stack and Kepler's Conjecture

Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is $\frac{\pi}{\sqrt{18}}$



Face-centered cubic Packing



Hexagonal Compact Packing

A “Simple” Example

In the computational part:

- Multivariate Polynomials:
  $\Delta\mathbf{x} := x_1x_4(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2x_5(x_1 - x_2 + x_3 + x_4 - x_5 + x_6) + x_3x_6(x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2(x_3x_4 + x_1x_6) - x_5(x_1x_3 + x_4x_6)$

A “Simple” Example

In the computational part:

- Semialgebraic functions: composition of polynomials with $|\cdot|, \sqrt{}, +, -, \times, /, \sup, \inf, \ldots$

$$p(\mathbf{x}) := \partial_4 \Delta \mathbf{x} \qquad q(\mathbf{x}) := 4 x_1 \Delta \mathbf{x}$$
$$r(\mathbf{x}) := p(\mathbf{x}) / \sqrt{q(\mathbf{x})}$$

$$l(\mathbf{x}) := -\frac{\pi}{2} + 1.6294 - 0.2213\,(\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913\,(\sqrt{x_4} - 2.52) + 0.728\,(\sqrt{x_1} - 2.0)$$

A "Simple" Example

In the computational part:

- Transcendental functions $\mathcal{T}$: composition of semialgebraic functions with $\arctan$, $\exp$, $\sin$, $+, -, \times, \ldots$

A "Simple" Example

In the computational part:

- Feasible set $\mathbf{K} := [4, 6.3504]^3 \times [6.3504, 8] \times [4, 6.3504]^2$

Lemma$_{9922699028}$ from Flyspeck:

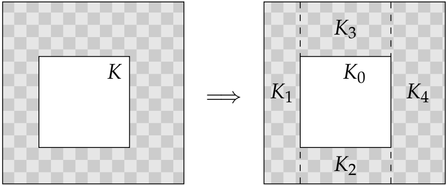
$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{p(\mathbf{x})}{\sqrt{q(\mathbf{x})}}\right) + l(\mathbf{x}) \geqslant 0$$

Existing Certified Frameworks

Lemma$_{9922699028}$ from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{\partial_4 \Delta \mathbf{x}}{\sqrt{4x_1 \Delta \mathbf{x}}}\right) + l(\mathbf{x}) \geqslant 0$$

- Dependency issue using Interval Calculus:
  - One can bound $\partial_4 \Delta \mathbf{x} / \sqrt{4x_1 \Delta \mathbf{x}}$ and $l(\mathbf{x})$ separately
  - Too coarse lower bound: -0.87
  - Subdivide $\mathbf{K}$ to prove the inequality



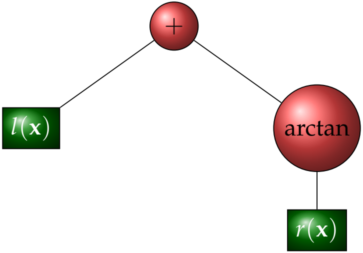
Nonlinear Function Representation

Tree representations of multivariate functions:

- leaves are semialgebraic functions
- nodes are univariate functions or binary operations

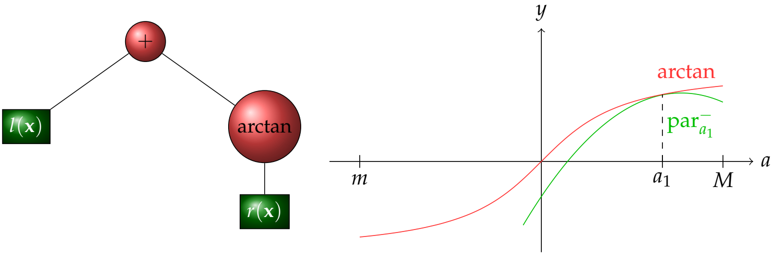
Nonlinear Function Representation

- For the “Simple” Example from Flyspeck:



Maxplus Optimization Algorithm

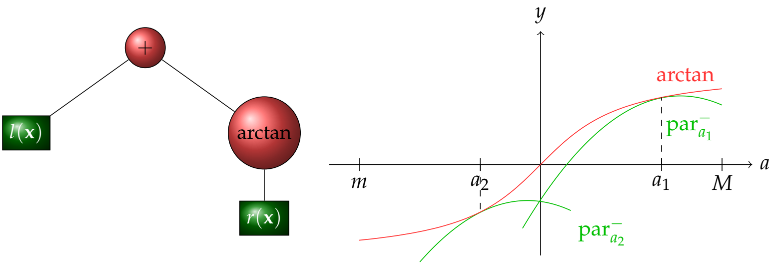
First iteration:



1. control point $\{a_1\}$: $m_1 = -4.7 \times 10^{-3} < 0$

Maxplus Optimization Algorithm

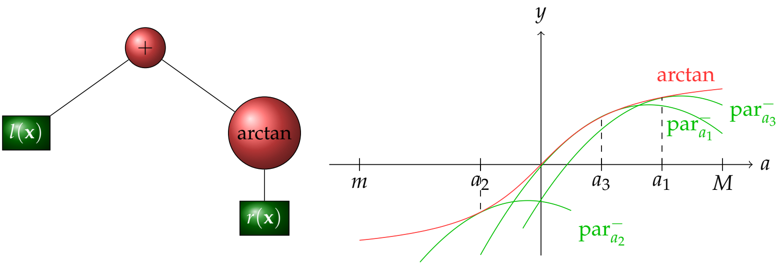
Second iteration:



2 control points $\{a_1, a_2\}$: $m_2 = -6.1 \times 10^{-5} < 0$

Maxplus Optimization Algorithm

Third iteration:



3 control points $\{a_1, a_2, a_3\}$: $m_3 = 4.1 \times 10^{-6} > 0$

OK!

Contribution: Publications and Software

- M., Allamigeon, Gaubert, Werner.
  Formal Proofs for Nonlinear Optimization,
  Journal of Formalized Reasoning 8(1):1–24, 2015.

- Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017

Software Implementation `NLCertify`**:**

- 15 000 lines of OCaml code

- 4000 lines of Coq code

- M. NLCertify: A Tool for Formal Nonlinear Optimization, *ICMS*, 2014.

Reachability in Discrete Time

Invariant Measures of Polynomial Systems

Oranges Stack and Kepler's Conjecture

Roundoff Error Bounds

Roundoff Error Bounds

- Exact:

$$f(\mathbf{x}) := x_1 x_2 + x_3 x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1 x_2 (1 + e_1) + x_3 x_4 (1 + e_2)](1 + e_3)$$

- $\mathbf{x} \in \mathbf{X}$, $\quad | e_i | \leqslant 2^{-p} \quad p = 24$ (single) or 53 (double)

Roundoff Error Bounds

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \mathbf{e})$
Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}}(\mathbf{e}) \mathbf{x}^{\boldsymbol{\alpha}}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = l(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$, l **linear** in $\mathbf{e}$

3: Bound $h(\mathbf{x}, \mathbf{e})$ with interval arithmetic

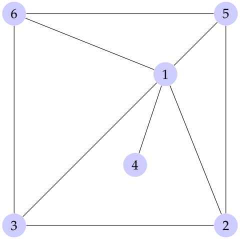
4: Bound $l(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

Roundoff Error Bounds

Sparse SDP Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

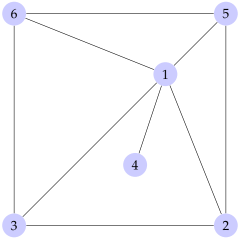


Roundoff Error Bounds

Sparse SDP Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$



1. Maximal cliques $C_1, \ldots, C_l$
2. Average size $\kappa \rightsquigarrow \binom{\kappa+2k}{\kappa}$ vars

$C_1 := \{1,4\}$

$C_2 := \{1,2,3,5\}$

$C_3 := \{1,3,5,6\}$

Dense SDP: 210 vars

Sparse SDP: 115 vars

Contributions

$l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^{m} s_i(\mathbf{x}) e_i$

Maximal cliques correspond to $\{\mathbf{x}, e_1\}, \ldots, \{\mathbf{x}, e_m\}$

M., Constantinides, Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming, *Trans. Math. Soft.*, 2016

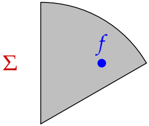
Conclusion and Perspectives

Moment-SOS powerful to handle **NONLINEAR VERIFICATION**:

- Optimize values/curves/sets
- Analysis of **NONLINEAR SYSTEMS** (Reachability, Invariants)
- Formal nonlinear optimization: `NLCertify`

Next talk

Exact certificates of positivity



Thank you for your attention!

homepages.laas.fr/vmagron