Exact Polynomial Optimization via SOS Decompositions

Victor Magron, LAAS CNRS

Joint work with
Mohab Safey El Din (Sorbonne Université)
Markus Schweighofer (Konstanz University)

SIAM Applied Algebraic Geometry

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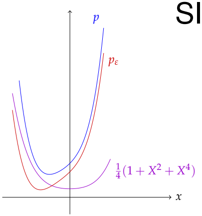

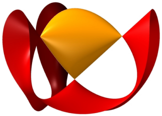

Deciding Nonnegativity & Exact Optimization

$X = (X_1, \dots, X_n)$ **co-NP hard problem: check $f \geqslant 0$ on $\mathbf{K}$**

$f \in \mathbb{Q}[X]$ **NP hard problem:** $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$

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1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

2. Constrained
   $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\} \quad g_j \in \mathbb{Q}[X]$

$\deg f, \deg g_j \leqslant d$

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[Collins 75] CAD **doubly exp. in** n **poly. in** d

[Grigoriev-Vorobjov 88, Basu-Pollack-Roy 98]
Critical points **singly exponential time** $(m+1)\,\tau\, d^{\mathcal{O}(n)}$

Deciding Nonnegativity & Exact Optimization

Sums of squares (SOS)

$\sigma = {h_1}^2 + \cdots + {h_p}^2$

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Hilbert 17th Problem: f SOS of rational functions?

[Artin 27] **YES**!

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[Lasserre/Parrilo 01] **Numerical** solvers compute σ

Semidefinite programming (SDP) $\rightsquigarrow$ **approximate** certificates

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$$\boxed{\simeq \quad \rightarrow \quad =}$$

The Question of Exact Certification

How to go from **approximate** to **exact** certification?

Decomposing Nonnegative Polynomials

1. **Reznick**'s representation $f = \frac{\sigma}{(X_1^2 + \cdots + X_n^2)^D}$
   positive definite form f
   [Reznick 95]

2. **Hilbert-Artin**'s representation $f = \frac{\sigma}{h^2}$
   $f \geqslant 0$
   [Artin 27]

3. **Putinar**'s representation $f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_m g_m$
   $f > 0$ on compact K
   $\deg \sigma_i \leqslant 2D$
   [Putinar 93]

Decomposing Nonnegative Polynomials

- Deciding **polynomial nonnegativity**

$$f(a,b) = a^2 - 2ab + b^2 \geqslant 0$$

- $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$

- $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Decomposing Nonnegative Polynomials

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve **SDP**

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A} \, \mathbf{z} = \mathbf{d} \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2$

- Solving **SDP** $\Longrightarrow$ Finding **SUMS OF SQUARES** certificates

From Approximate to Exact Solutions

Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$
$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$?

Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geqslant 0$ on $\mathbb{R}$ ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = {f_1}^2 + {f_2}^2$.

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Proof.

$f = h^2(q + ir)(q - ir)$ □

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1 + \sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}}}{4}X + \frac{\sqrt{10 - 2\sqrt{5}}}{4}\right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

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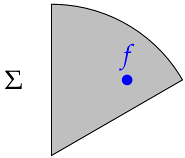
Round & Project Algorithm [Peyrl-Parrilo 08]



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

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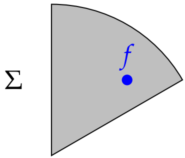

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Find $\tilde{\mathbf{G}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{G}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{G}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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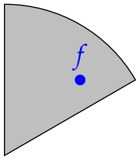
Σ



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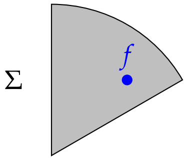

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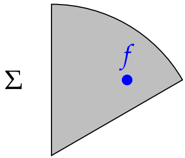

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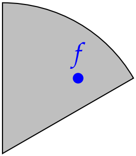

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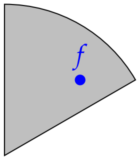

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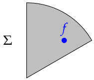
Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = \mathbf{v}_D{}^T(X)\, \mathbf{G}\, \mathbf{v}_D(X)$ and $\mathbf{G} \succcurlyeq 0$

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



sum of squares of f?

≃ Output!

From Approximate to Exact Solutions

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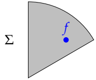


Hybrid Symbolic/Numeric Algorithms

sum of squares of $f - \varepsilon$?

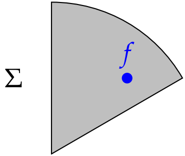
$\simeq$ Output!

Error Compensation

$\simeq \quad \rightarrow \quad =$

From Approximate to Exact Solutions

Exact SOS



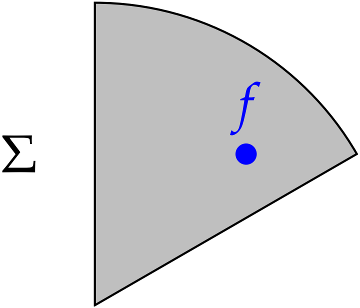
Software: `RealCertify`

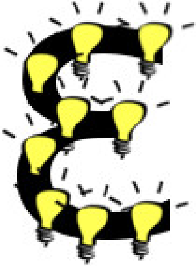
`Maple` & arbitrary precision SDP solver `SDPA-GMP` [Nakata et al. ’10]

`univsos` $\boxed{n = 1}$

`multivsos` $\boxed{n > 1}$

intsos with $n \geqslant 1$: Perturbation





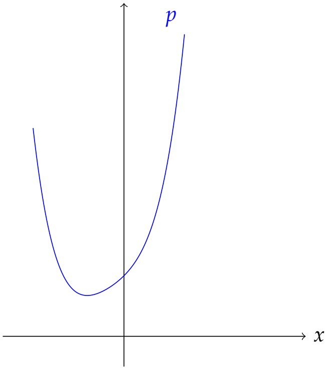
PERTURBATION idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$



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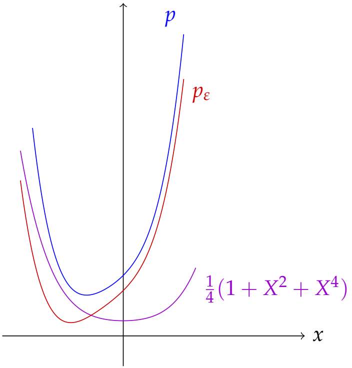
$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$

💡 **PERTURB**: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

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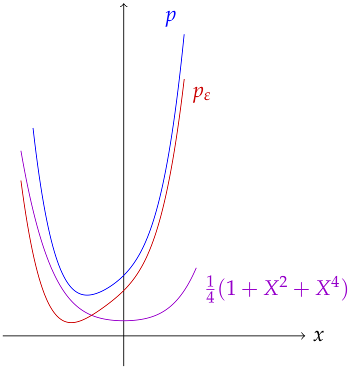
💡 SDP Approximation:

$$p - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

💡 **ABSORB**: small enough u_i

$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS



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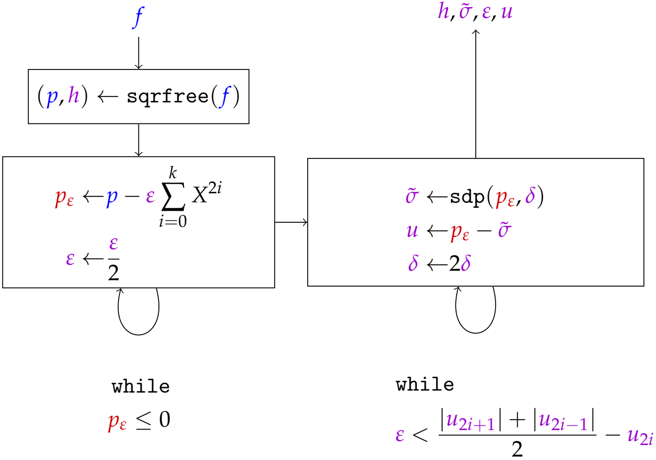
intsos with $n = 1$ and SDP Approximation

- **Input** $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



`intsos` with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

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$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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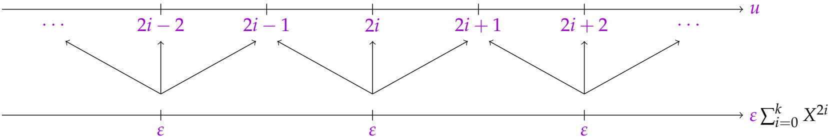

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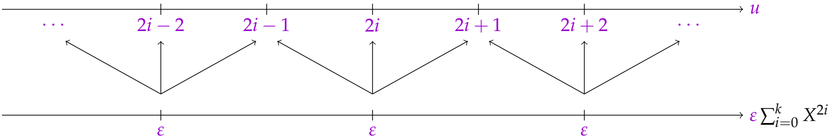
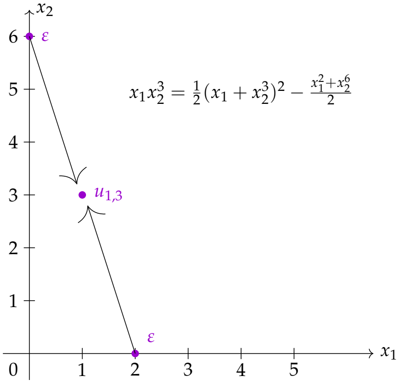


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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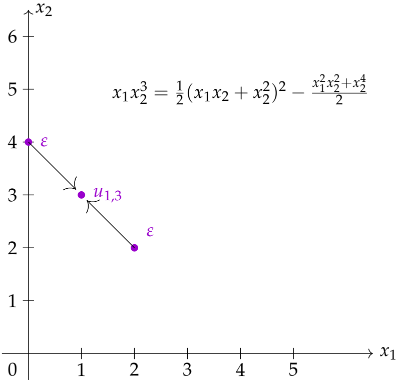

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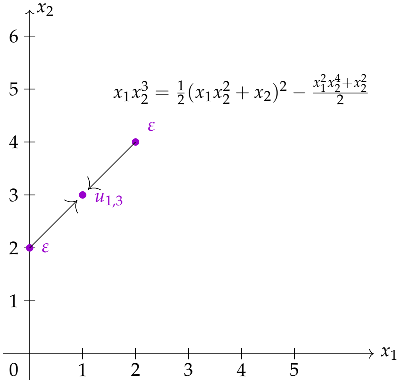


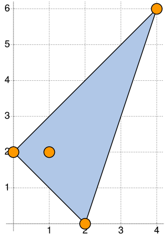
`intsos` with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

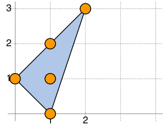
Choice of $\mathcal{P}$?

$f = 4X_1^4 X_2^6 + X_1^2 - X_1 X_2^2 + X_2^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\text{spt}(f))$

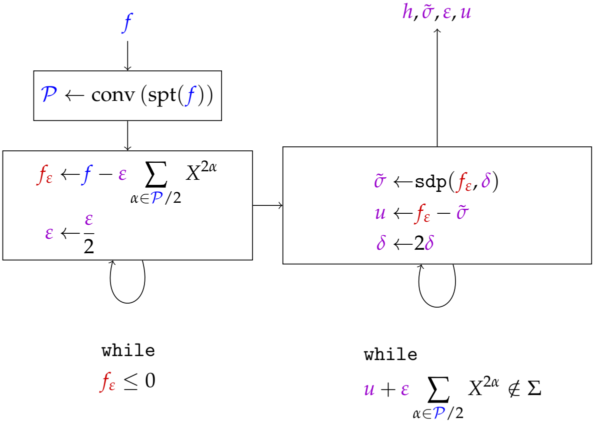


Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm `intsos`

- **Input** $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



SOS Benchmarks

- rounding-projection (SOS) [Peyrl-Parrilo '08]
- `RAGLib` (critical points) [Safey El Din]
- `SamplePoints` (CAD) [Moreno Maza-Alvandi et al.]

Id	n	d	`RealCertify` τ_1 (bits)	`RealCertify` t_1 (s)	`RoundProject` τ_2 (bits)	`RoundProject` t_2 (s)	`RAGLib` t_3 (s)	`CAD` t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	598.	–
f_{10}	10	4	344 347	2.45	8 374 082	4.59	–	–

Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	Output Bit Size
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau^2 \, d^{\mathcal{O}\,(d^n)}$

Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	OUTPUT BIT SIZE
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau^2 \, d^{\mathcal{O}\,(d^n)}$

- How to handle degenerate situations?
- Better arbitrary precision SDP solvers?
- Extension to other relaxations, e.g., SONC

Thank you for your attention!

RealCertify SONCSOCP

https://homepages.laas.fr/vmagron

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