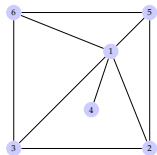


The quest of efficiency and certification in polynomial optimization

Victor Magron, CNRS–LAAS

Symbolic Computation seminar, NC State University
29 March 2021



The Moment-Sums of Squares Hierarchy

NP-hard NON CONVEX Problem $f^* = \inf f(x)$

Theory

(Primal)		(Dual)
$\inf \int f d\mu$		$\sup \lambda$
with μ proba \Rightarrow	INFINITE LP	\Leftarrow with $f - \lambda \geq 0$

The Moment-Sums of Squares Hierarchy

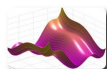
NP-hard NON CONVEX Problem $f^* = \inf f(x)$

Practice

(Primal **Relaxation**)

moments $\int x^\alpha d\mu$

finite number \Rightarrow



SDP

(Dual **Strengthening**)

$f - \lambda =$ sum of squares

\Leftarrow fixed degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f^*$

[Lasserre/Parrilo 01]

degree d & n vars $\Rightarrow \binom{n+2d}{n}$ **SDP** VARIABLES

Numeric solvers \Rightarrow **Approx Certificate**

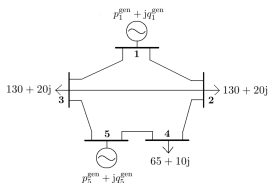


Success Stories: Lasserre's Hierarchy

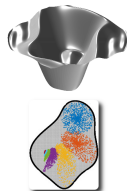
MODELING POWER: **Cast** as ∞ -dimensional LP over measures

💡 **STATIC** Polynomial Optimization

Optimal Powerflow $n \simeq 10^3$ [Josz et al 16]



💡 **DYNAMICAL** Polynomial Optimization
Regions of attraction [Henrion-Korda 14]



Reachable sets [Magron et al 17]



APPROXIMATE OPTIMIZATION BOUNDS!

Success Stories: Certified Optimization



Kepler's Conjecture(1611)

The max density of sphere packings is $\pi/\sqrt{18}$



Flyspeck : Formalizing the proof of Kepler by T.Hales (1994)
Verification of thousands of “tight” nonlinear inequalities

Seminal Paper:



Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017

CONTRIBUTION:



(Non)-Polynomial optimization to verify **Flyspeck** inequalities

Exploiting Sparsity

Certified Polynomial Optimization

Exploiting Sparsity

Certified Polynomial Optimization

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_l(\mathbf{x}) \geq 0\}$

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 $\mathbf{X} = [0, 1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geq 0, \quad x_2(1 - x_2) \geq 0\}$

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$$\underbrace{x_1 x_2}_f =$$

$$-\frac{1}{8} + \frac{1}{2} \overbrace{\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \frac{1}{2} \overbrace{x_1(1 - x_1)}^{g_1} + \frac{1}{2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Sums of squares (SOS) σ_j

- Bounded degree:

$$\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^l \sigma_j g_j, \text{ with } \deg \sigma_j g_j \leq 2r \right\}$$

SDP for Polynomial Optimization

- **Hierarchy of SDP relaxations:**

$$\lambda_r := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{Q}_r(\mathbf{X}) \right\}$$

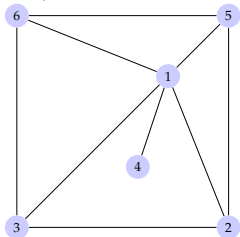
- Convergence guarantees $\lambda_r \uparrow f^*$ [Lasserre 01]
- Can be computed with SDP solvers (CSDP, SDPA, MOSEK)
- **“No Free Lunch” Rule:** $\binom{n+2r}{n}$ SDP variables

Sparse Polynomial Optimization [Waki, Lasserre 06]

■ Correlative (variable) sparsity

$$f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

Chordal graph

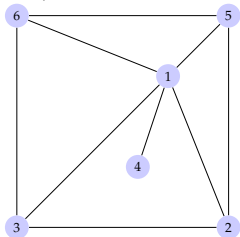


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Chordal graph



- 1 Maximal cliques I_k
- 2 Average size $\kappa \rightsquigarrow \kappa^{2d}$ variables

$$I_1 = \{1, 4\}$$

$$I_2 = \{1, 2, 3, 5\}$$

$$I_3 = \{1, 3, 5, 6\}$$

Dense: 210 variables

Sparse: 115 variables

Sparse Polynomial Optimization [Waki, Lasserre 06]

Sparse $f = f_1 + \dots + f_p$ with $f_k \in \mathbb{R}[x, I_k]$

Sparse $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$ with $g_j \in \mathbb{R}[x, I_{k(j)}]$ for some $k(j)$

Additional constraints $n_k - \sum_{i \in I_k} x_i^2 \geq 0$ in \mathbf{K}

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$$\forall k = 1, \dots, p-1 \quad I_{k+1} \cap \bigcup_{j \leq k} I_j \subseteq I_i \quad \text{for some } i \leq k$$

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Theorem: Sparse Putinar's representation [Lasserre 06]

$$f > 0 \text{ on } \mathbf{K} + \text{RIP} \implies f = \sigma_{01} + \dots + \sigma_{0p} + \sum_{j=1}^m \sigma_j g_j$$

with $\sigma_{0k} \in \Sigma[x, I_k]$, $\sigma_j \in \Sigma[x, I_{k(j)}]$

Sparse Examples

- Chained singular function:

$$f_{\text{CS}} = \sum_{i \in J} ((x_i + 10x_{i+1})^2 + 5(x_{i+2} - x_{i+3})^2 + (x_{i+1} - 2x_{i+2})^4 + 10(x_i - x_{i+3})^4)$$

where $J = \{1, 3, 4, \dots, n - 3\}$ and n is a multiple of 4

💡 $I_k = \{k, k + 1, k + 2, k + 3\}$

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- Generalized Rosenbrock function:

$$f_{\text{gR}} = 1 + \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i^2)^2 + (1 - x_{i+1})^2 \right)$$

💡 $I_k = \{k, k + 1\}$

Roundoff Errors

- Exact:

$$f(\mathbf{x}) := x_1x_2 + x_3x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1x_2(1 + e_1) + x_3x_4(1 + e_2)](1 + e_3)$$

- $\mathbf{x} \in \mathbf{X}$, $|e_i| \leq 2^{-\delta}$ $\delta = 24$ (single) or 53 (double)

Roundoff Errors

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \mathbf{e})$

Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\alpha} r_{\alpha}(\mathbf{e}) \mathbf{x}^{\alpha}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = \ell(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$, ℓ **linear** in \mathbf{e}

3: Bound $h(\mathbf{x}, \mathbf{e})$ with interval arithmetic

4: Bound $\ell(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

Exploiting Sparsity for Roundoff Error Bounds

$$l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^m s_i(\mathbf{x})e_i$$

I_1, \dots, I_m correspond to $\{\mathbf{x}, e_1\}, \dots, \{\mathbf{x}, e_m\}$

Dense relaxation: $\binom{n+m+2d}{n+m}$ SDP variables

Sparse relaxation: $m \binom{n+1+2d}{n+1}$ SDP variables

Preliminary Comparisons

$$f(\mathbf{x}) := x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

$$\mathbf{x} \in [4.00, 6.36]^6, \quad \mathbf{e} \in [-\epsilon, \epsilon]^{15}, \quad \epsilon = 2^{-53}$$

- **Dense SDP:** $\binom{6+15+4}{6+15} = 12650$ variables \leadsto **Out of memory**

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- **Symbolic Taylor** FPTaylor tool: 721ϵ (21 \times more CPU)

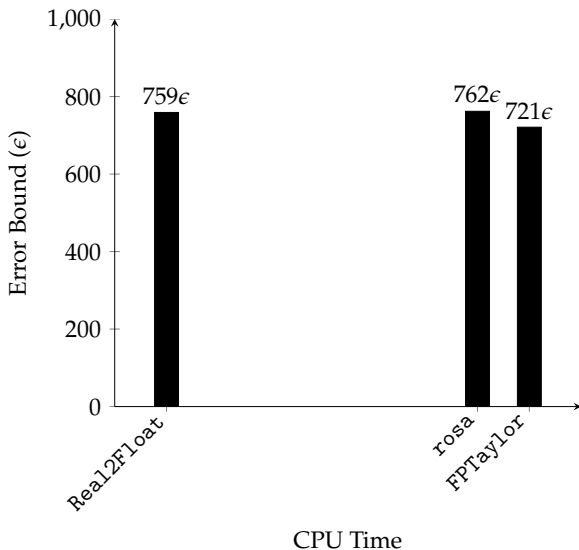
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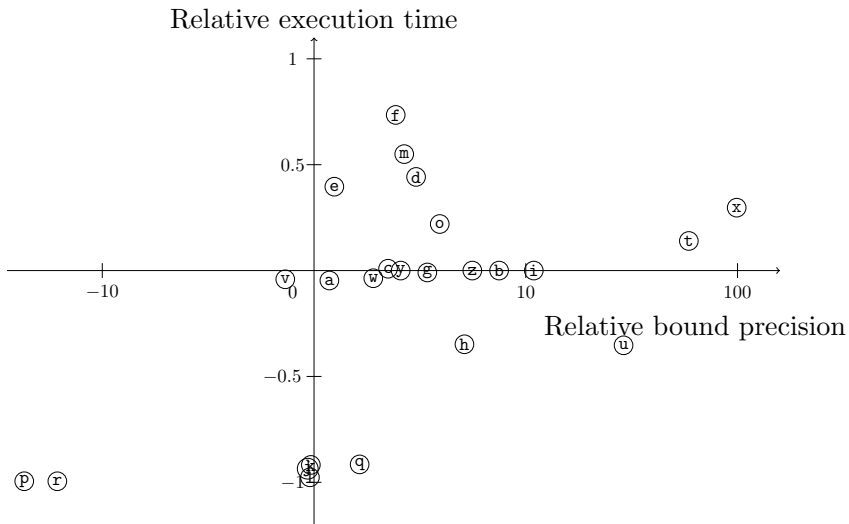
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- **SMT-based** rosa tool: 762ϵ (19 \times more CPU)

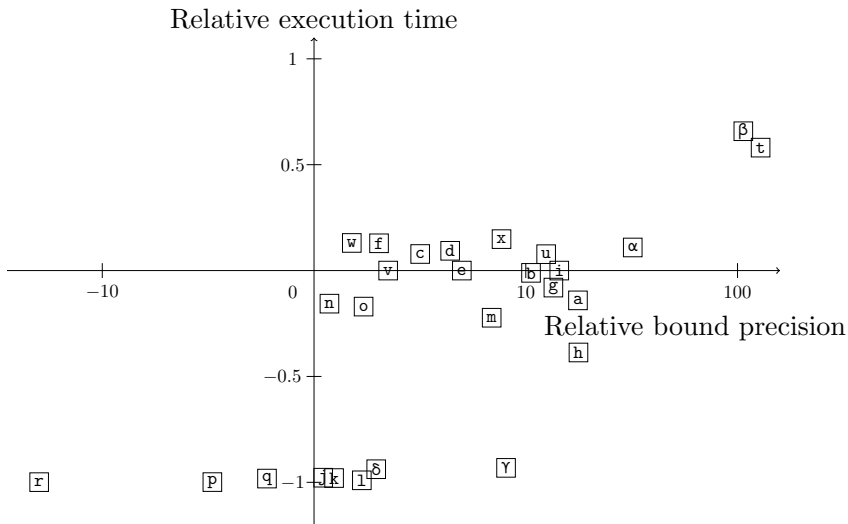
Preliminary Comparisons



Comparison with rosa



Comparison with FPTaylor



Noncommutative (NC) Polynomials

Symmetric **Matrix** variables X_i, Y_j

$$f = X_1(Y_1 + Y_2 + Y_3) + X_2(Y_1 + Y_2 - Y_3) + X_3(Y_1 - Y_2) - X_1 - 2Y_1 - Y_2$$

with $X_1X_2 \neq X_2X_1$, **involution** $(X_1Y_3)^* = Y_3X_1$

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Constraints $\mathbf{K} = \{(X, Y) : X_i, Y_j \succcurlyeq 0, X_i^2 = X_i, Y_j^2 = Y_j, X_iY_j = Y_jX_i\}$

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MINIMAL EIGENVALUE OPTIMIZATION

$$\lambda_{\min} = \inf \{ \langle f(X, Y) \mathbf{v}, \mathbf{v} \rangle : (X, Y) \in \mathbf{K}, \|\mathbf{v}\| = 1 \}$$

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Symmetric **Matrix** variables X_i, Y_j

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$$\begin{aligned}\lambda_{\min} &= \inf \{ \langle f(X, Y) \mathbf{v}, \mathbf{v} \rangle : (X, Y) \in \mathbf{K}, \|\mathbf{v}\| = 1 \} \\ &= \sup \lambda \\ &\text{s.t. } f(X, Y) - \lambda \mathbf{I} \succcurlyeq 0, \quad \forall (X, Y) \in \mathbf{K}\end{aligned}$$

Putinar's Representation

“Archimedean” constraint in $K = \{X : g_j(X) \succcurlyeq 0\}$:

$$N - \sum_i X_i^2 \succcurlyeq 0$$

Theorem: NC Putinar's representation [Helton-McCullough 02]

$$f \succcurlyeq 0 \text{ on } K \implies \boxed{f = \sum_i s_i^* s_i + \sum_j \sum_i t_{ji}^* g_j t_{ji}} \text{ with } s_i, t_{ji} \in \mathbb{R}\langle X \rangle$$

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NC variant of Lasserre's Hierarchy for λ_{\min} :

💡 replace “ $f - \lambda \mathbf{I} \succcurlyeq 0$ on \mathbf{K} ” by $f - \lambda \mathbf{I} = \sum_i s_i^* s_i + \sum_j \sum_i t_{ji}^* g_j t_{ji}$
with s_i, t_{ji} of **bounded** degrees

Sparse Putinar's Representation

Sparse $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{R}\langle X, I_k \rangle$

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$$\forall k = 1, \dots, p-1 \quad I_{k+1} \cap \bigcup_{j \leq k} I_j \subseteq I_i \quad \text{for some } i \leq k$$

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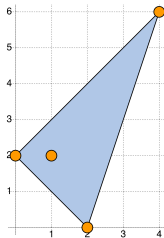
with $s_{ki} \in \mathbb{R}\langle X, I_k \rangle$, $t_{ji} \in \mathbb{R}\langle X, I_{k(j)} \rangle$

Exploiting sparsity via Newton polytope

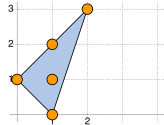
$$f = 4x^4y^6 + x^2 - xy^2 + y^2$$

$$\text{spt}(f) = \{(4, 6), (2, 0), (1, 2), (0, 2)\}$$

$$\text{Newton polytope } \mathcal{P} = \text{conv}(\text{spt}(f))$$



Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



$$f = \left(x \quad y \quad xy \quad xy^2 \quad x^2y^3 \right) \underbrace{\mathbf{G}}_{\succeq 0} \begin{pmatrix} x \\ y \\ xy \\ xy^2 \\ x^2y^3 \end{pmatrix}$$

Exploiting term sparsity [Magron-Wang 20-21]

$$f = x^2 - 2xy + 3y^2 - 2x^2y + 2x^2y^2 - 2yz \\ + 6z^2 + 18y^2z - 54yz^2 + 142y^2z^2$$

$$f = \left(1 \quad x \quad y \quad z \quad xy \quad yz \right) \underbrace{\mathbf{G}}_{\succeq 0}$$

$$\begin{pmatrix} 1 \\ x \\ y \\ z \\ xy \\ yz \end{pmatrix}$$

$\rightsquigarrow \frac{6 \times 7}{2} = 28$ “unknown” entries in \mathbf{G}

Exploiting term sparsity [Magron-Wang 20-21]

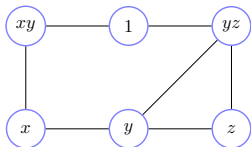
$$f = x^2 - 2xy + 3y^2 - 2x^2y + 2x^2y^2 - 2yz \\ + 6z^2 + 18y^2z - 54yz^2 + 142y^2z^2$$

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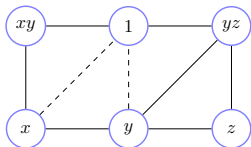
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Chordal extension



Replace \mathbf{Q} by $\mathbf{Q}_{\mathbf{G}}$ with nonzero entries at edges of \mathbf{G}

$\rightsquigarrow 6 + 9 = 15$ “unknown” entries in $\mathbf{Q}_{\mathbf{G}}$

Sparse Example: I_{3322} Bell Inequality

Entanglement in quantum mechanics

→ **upper bounds** for violation levels of Bell inequalities

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Exploiting Sparsity

Certified Polynomial Optimization

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$$X = (X_1, \dots, X_n)$$

$$f \in \mathbb{Q}[X]$$

co-NP hard problem: check $f \geq 0$ on \mathbf{K}

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$$\deg f, \deg g_j \leq d$$

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[Collins 75] 💡 CAD **doubly exp. in n poly. in d**



[Grigoriev-Vorobjov 88, Basu-Pollack-Roy 98]
💡 Critical points **singly exponential time** $(l+1) \tau d^{\mathcal{O}(n)}$

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HILBERT 17TH PROBLEM: f SOS of rational functions?



[Artin 27] **YES!**

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$$\boxed{\approx \quad \rightarrow \quad =}$$

The Question of Exact Certification

How to go from **approximate** to **exact** certification?

Motivation

Positivity certificates

- Stability proofs of critical control systems (Lyapunov)
- Certified function evaluation [Chevillard et. al 11]
- Formal verification of real inequalities [Hales et. al 15]:



COQ



HOL-LIGHT

Decomposing Nonnegative Polynomials

1 Reznick's representation

positive definite form f

[Reznick 95]

$$f = \frac{\sigma}{(X_1^2 + \dots + X_n^2)^D}$$

2 Hilbert-Artin's representation

$f \geq 0$

[Artin 27]

$$f = \frac{\sigma}{h^2}$$

3 Putinar's representation

$f > 0$ on compact K

[Putinar 93]

$$f = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_l g_l$$
$$\deg \sigma_i \leq 2D$$

Decomposing Nonnegative Polynomials

■ Deciding **polynomial nonnegativity**

$$f(a, b) = a^2 - 2ab + b^2 \geq 0$$

$$\blacksquare f(a, b) = (a \quad b) \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\geq 0} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\blacksquare a^2 - 2ab + b^2 = z_1 a^2 + 2z_2 ab + z_3 b^2 \quad (\mathbf{A} \mathbf{z} = \mathbf{d})$$

$$\blacksquare \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$$

Decomposing Nonnegative Polynomials

- Choose a cost \mathbf{c} e.g. $(1, 0, 1)$ and solve **SDP**

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0, \quad \mathbf{A} \mathbf{z} = \mathbf{d} \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = (a \ b) \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2$

- Solving **SDP** \implies Finding **SUMS OF SQUARES** certificates

From Approximate to Exact Solutions

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?



$(1.00001a - 0.99998b)^2!$



$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\simeq \rightarrow = ?$$

Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geq 0$ on \mathbb{R} ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = f_1^2 + f_2^2$.

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□

Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1 + \sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}}}{4}X + \frac{\sqrt{10 - 2\sqrt{5}}}{4}\right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \overset{\circ}{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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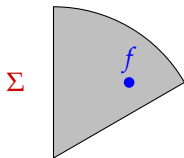
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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1 \left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

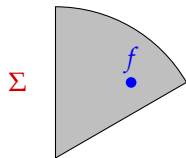
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Round & Project Algorithm [Peyrl-Parrilo 08]




$$f \in \mathring{\Sigma}[X] \text{ with } \deg f = 2D$$

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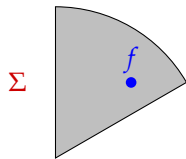
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$$f(X) \simeq \mathbf{v}_D^T(X) \tilde{\mathbf{G}} \mathbf{v}_D(X) \quad \tilde{\mathbf{G}} \succ 0$$

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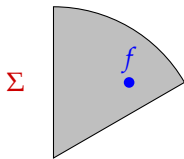
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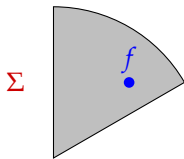
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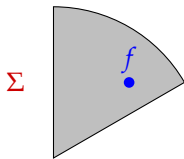
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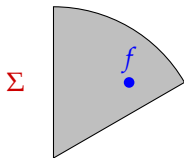
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Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = \mathbf{v}_D^T(X) \mathbf{G} \mathbf{v}_D(X)$ and $\mathbf{G} \succcurlyeq 0$

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\}$

💡 Hybrid **SYMBOLIC/NUMERIC** methods

📄 Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

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Compact $\mathbf{K} \subseteq [0, 1]^n$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

$$\boxed{\simeq \rightarrow =}$$

💡 $\forall \mathbf{x} \in [0, 1]^n, u(\mathbf{x}) \leq -\varepsilon$

$$\min_{\mathbf{K}} f \geq \varepsilon \text{ when } \varepsilon \rightarrow 0$$

COMPLEXITY?



From Approximate to Exact Solutions

Win TWO-PLAYER GAME



sum of squares of f ?



\approx Output!



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💡 **Hybrid** Symbolic/Numeric Algorithms

sum of squares of $f - \varepsilon$?

\approx Output!



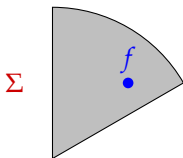
Error Compensation



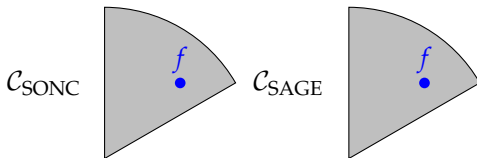
$\approx \rightarrow =$

From Approximate to Exact Solutions

Exact SOS



Exact SONC/SAGE



Exact optimization via SOS: RealCertify

Maple & arbitrary precision SDP solver SDPA-GMP

[Nakata 10]

univsos $n = 1$

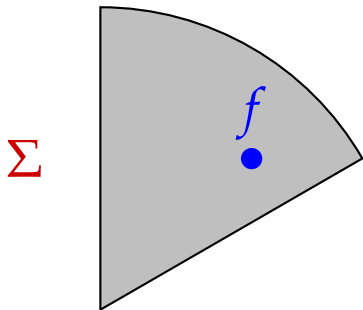
multivsos $n > 1$

Exact optimization via SONC/SAGE

POEM: Python (SymPy) & geometric programming/relative entropy ECOS

SONCSOCP: Julia & second order conic programming

intsos with $n \geq 1$: Perturbation



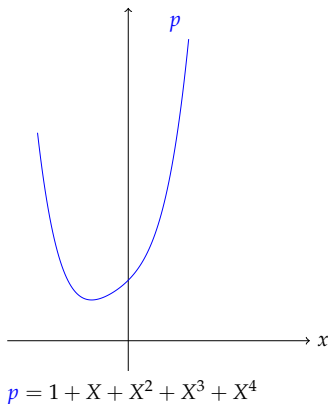
PERTURBATION idea

💡 Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$$p \in \mathbb{Q}[X], \deg p = d = 2k, p > 0$$

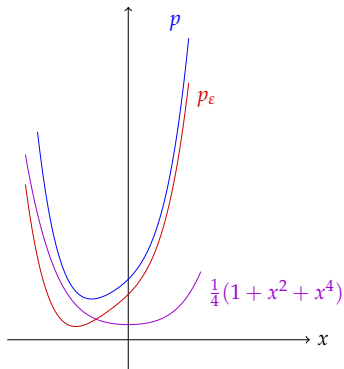


intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$

💡 **PERTURB:** find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^k X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ [Chevillard et. al 11]

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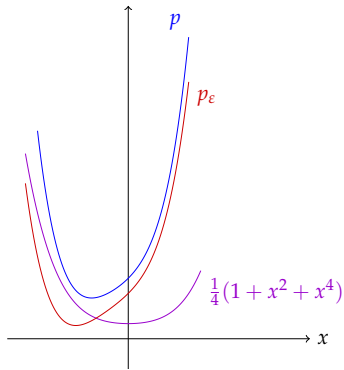
$$p_\varepsilon := p - \varepsilon \sum_{i=0}^k X^{2i} > 0$$

💡 **SDP Approximation**:

$$p - \varepsilon \sum_{i=0}^k X^{2i} = \tilde{\sigma} + u$$

💡 **ABSORB**: small enough u_i

$\implies \varepsilon \sum_{i=0}^k X^{2i} + u$ SOS



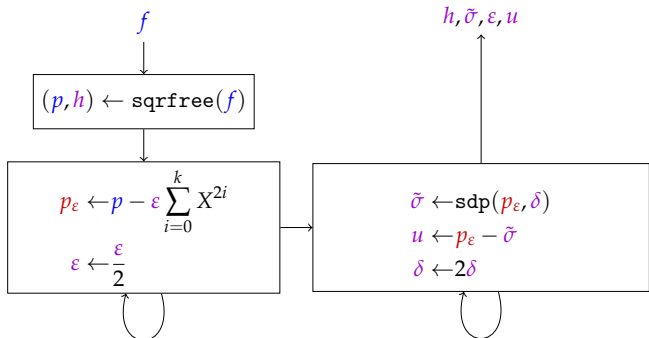
$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

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intsos with $n = 1$ and SDP Approximation

- **Input** $f \geq 0 \in \mathbb{Q}[X]$ of degree $d \geq 2$, $\epsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in \mathbb{Q}



while
 $p_\epsilon \leq 0$

while
 $\epsilon < \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i}$

intsos with $n = 1$: Absorbion

$$\text{💡 } X = \frac{1}{2}[(X+1)^2 - 1 - X^2]$$

$$\text{💡 } -X = \frac{1}{2}[(X-1)^2 - 1 - X^2]$$

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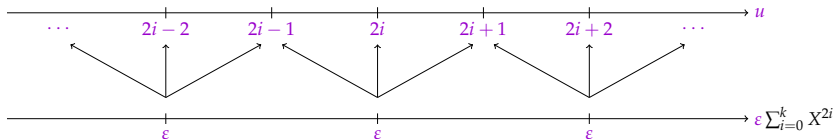
$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2} [(X^{i+1} + \text{sgn}(u_{2i+1})X^i)^2 - X^{2i} - X^{2i+2}]$$

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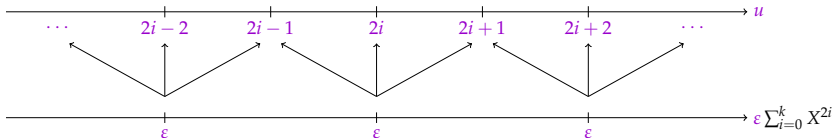


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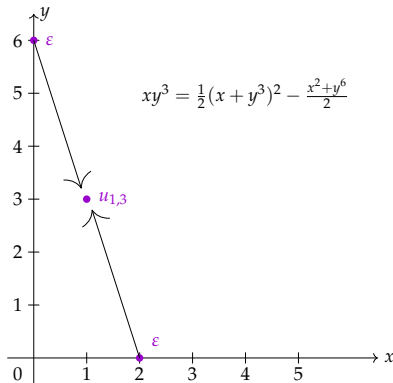


$$\epsilon \geq \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \epsilon \sum_{i=0}^k X^{2i} + u \quad \text{SOS}$$

intsos with $n \geq 1$: Absorption

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

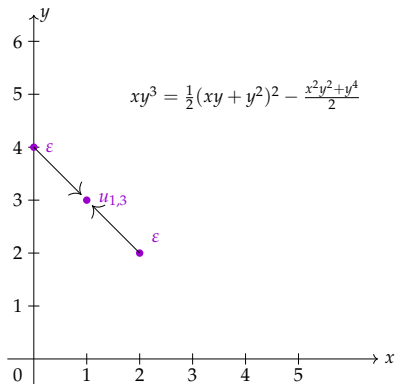
Choice of \mathcal{P} ?



intsos with $n \geq 1$: Absorbion

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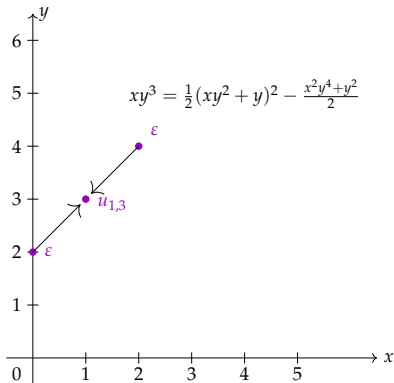
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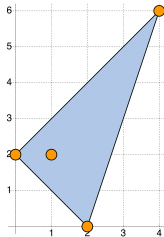
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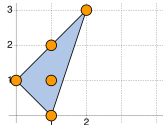
Choice of \mathcal{P} ?

$$f = 4x^4y^6 + x^2 - xy^2 + y^2$$
$$\text{spt}(f) = \{(4, 6), (2, 0), (1, 2), (0, 2)\}$$

Newton Polytope $\mathcal{P} = \text{conv}(\text{spt}(f))$

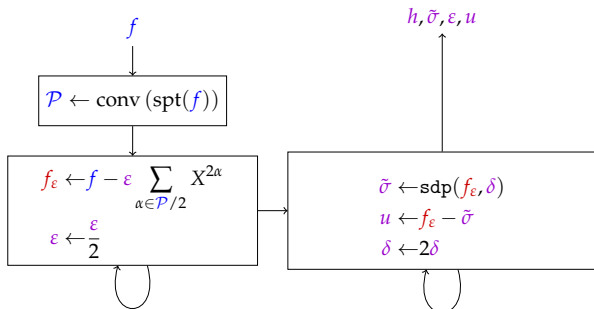


Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm intsos

- **Input** $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ of degree d , $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in \mathbb{Q}



while
 $f_\varepsilon \leq 0$

while
 $u + \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} \notin \Sigma$

Algorithm intsos

Theorem (Exact Certification Cost in $\overset{\circ}{\Sigma}$)

$f \in \mathbb{Q}[X] \cap \overset{\circ}{\Sigma}[X]$ with $\deg f = d = 2k$ and bit size τ

\implies intsos terminates with SOS output of bit size $\tau d^{d^{\mathcal{O}(n)}}$

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Algorithm Reznicksos

f positive definite form has **Reznick's** representation:

$$f = \frac{\sigma}{(X_1 + \cdots + X_n)^{2D}} \quad \text{with } \sigma \in \Sigma[X]$$

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Theorem (Exact Certification Cost of Polyá's representations)

$f \in \mathbb{Q}[X]$ positive definite form with $\deg f = d$ and bit size τ

$$\implies D \leq 2^{2^{\mathcal{O}(\tau \cdot (4d+6)3n+3)}}$$

Algorithm Putinarsos

Assumption: $\exists i$ s.t. $g_i = 1 - \|X\|_2^2$

$f > 0$ on $\mathbf{K} := \{\mathbf{x} : g_j(\mathbf{x}) \geq 0\}$ has **Putinar's** representation:

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Theorem [M.-Safey El Din 18]

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 **ABSORPTION** as in Algorithm intsos: $u = f_\epsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j g_j$

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💡 **ABSORPTION** as in Algorithm intsos: $u = f_\epsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j g_j$

$$\text{OUTPUT BIT SIZE} = \boxed{D^{D^{\mathcal{O}(n)}} \text{ with } \log D = \mathcal{O}(2^\tau d^{n c_K})}$$

SOS Benchmarks

- Round & Project (SOS) [Peyrl-Parrilo]
- RAGLib (critical points) [Safey El Din]
- SamplePoints (CAD) [Moreno Maza-Alvandi et al.]

n	d	RealCertify		RoundProject		RAGLib	CAD
		τ_1 (bits)	t_1 (s)	τ_2 (bits)	t_2 (s)	t_3 (s)	t_4 (s)
2	20	745 419	110.	78 949 497	141.	0.16	0.03
3	8	17 232	0.35	18 831	0.29	0.15	0.03
2	4	1 866	0.03	1 031	0.04	0.09	0.01
6	4	56 890	0.34	475 359	0.54	598.	—
10	4	344 347	2.45	8 374 082	4.59	—	—

Conclusion and Perspectives

1 Quest for efficiency

Commutative: **Roundoff error** $n \simeq 10^2$

Noncommutative: **Minimal eigenvalue** $n \simeq 20 - 30$

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Input f of deg d & bitsize $\tau \Rightarrow$ **Output** SOS of bitsize $\tau d^{d^{O(n)}}$

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APPLICATIONS IN QUANTUM PHYSICS

- Quantum games: number of mutually unbiased bases in dim 6, OPEN FOR SEVERAL DECADES!! 💡 **symmetric**
- Ground state energy of hamiltonians 💡 **symmetric & sparse**
- Inflation for quantum correlations 💡 **symmetric & sparse**

Thank you for your attention!

<https://homepages.laas.fr/vmagron>



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[SONCSOCP](#)