The quest of efficiency and certification in polynomial optimization

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SPOT, Toulouse
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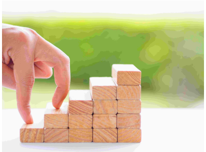
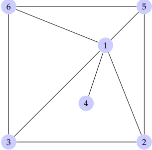
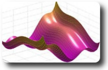




The Moment-Sums of Squares Hierarchy

NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Theory

(Primal)		(Dual)
$\inf \int f \, d\mu$		$\sup \lambda$
with μ proba $\Rightarrow$	INFINITE LP	$\Leftarrow$ with $f - \lambda \geqslant 0$

The Moment-Sums of Squares Hierarchy

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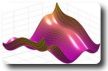
Practice

(Primal **Relaxation**)



(Dual **Strengthening**)

moments $\int \mathbf{x}^\alpha \, d\mu$

$f - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f^*$

[Lasserre/Parrilo 01]

degree d & n vars $\Longrightarrow \binom{n+2d}{n}$ **SDP** VARIABLES

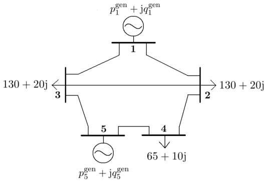
Numeric solvers $\Longrightarrow$ **Approx** Certificate

Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as ∞-*dimensional* LP over measures

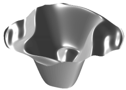
STATIC **Polynomial Optimization**

Optimal Powerflow $n \simeq \mathbf{10^3}$ [Josz et al 16]

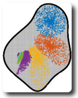


DYNAMICAL **Polynomial Optimization**

Regions of attraction [Henrion-Korda 14]



Reachable sets [Magron et al 17]

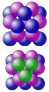


APPROXIMATE OPTIMIZATION BOUNDS!

Success Stories: Certified Optimization

Kepler's Conjecture(1611)

The max density of sphere packings is $\pi/\sqrt{18}$



Flyspeck : **F**ormalizing the **p**roof of **K**epler by T.Hales (1994)
Verification of thousands of "tight" nonlinear inequalities

Seminal Paper:

Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017

CONTRIBUTION:

(Non)-Polynomial optimization to verify **Flyspeck** inequalities

Exploiting Sparsity

Certified Polynomial Optimization

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Certified Polynomial Optimization

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Sums of squares (SOS) σ_j

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- Sums of squares (SOS) σ_j
- Bounded degree:
  $\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^{l} \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2r \right\}$

SDP for Polynomial Optimization

- **Hierarchy of SDP relaxations**:
$$\lambda_r := \sup_{\lambda} \Big\{ \lambda : f - \lambda \in \mathcal{Q}_r(\mathbf{X}) \Big\}$$

- Convergence guarantees $\lambda_r \uparrow f^*$ [Lasserre 01]

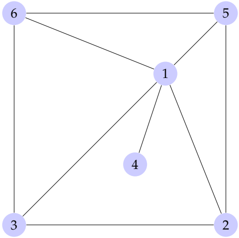
- Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

- **“No Free Lunch” Rule**: $\binom{n+2d}{n}$ SDP variables

Sparse Polynomial Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$



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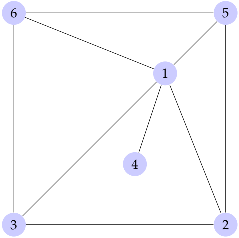

1. Index sets $I_1, \ldots, I_p$
2. Average size $\kappa \rightsquigarrow \binom{\kappa+2d}{\kappa}$ vars

$I_1 = \{1,4\}$
$I_2 = \{1,2,3,5\}$
$I_3 = \{1,3,5,6\}$
Dense SDP: 210 vars
Sparse SDP: 115 vars

Sparse Polynomial Optimization [Waki, Lasserre 06]

Sparse $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{R}[x, I_k]$

Sparse $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$ with $g_j \in \mathbb{R}[x, I_{k(j)}]$ for some $k(j)$

Additional constraints $n_k - \sum_{i \in I_k} x_i^2 \geqslant 0$ in $\mathbf{K}$

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RUNNING INTERSECTION PROPERTY (RIP)

$$\forall k = 1, \ldots, p-1 \quad I_{k+1} \cap \bigcup_{j \leqslant k} I_j \subseteq I_i \quad \text{for some } i \leqslant k$$

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Theorem: Sparse Putinar's representation [Lasserre 06]

$f > 0$ on $\mathbf{K}$ + RIP $\implies$ $\boxed{f = \sigma_{01} + \cdots + \sigma_{0p} + \sum_{j=1}^{m} \sigma_j g_j}$

with $\sigma_{0k} \in \Sigma[x, I_k]$, $\sigma_j \in \Sigma[x, I_{k(j)}]$

Sparse Examples

- Chained singular function:

$$f_{\text{cs}} = \sum_{i \in J} ((x_i + 10x_{i+1})^2 + 5(x_{i+2} - x_{i+3})^2 + (x_{i+1} - 2x_{i+2})^4 + 10(x_i - x_{i+3})^4)$$

where $J = \{1, 3, 4, \ldots, n-3\}$ and n is a multiple of 4

💡 $I_k = \{k, k+1, k+2, k+3\}$

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- Generalized Rosenbrock function:

$$f_{\mathsf{gR}} = 1 + \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i^2)^2 + (1 - x_{i+1})^2\right)$$

$I_k = \{k, k+1\}$

Roundoff Errors

- Exact:

$$f(\mathbf{x}) := x_1 x_2 + x_3 x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1 x_2 (1 + e_1) + x_3 x_4 (1 + e_2)](1 + e_3)$$

- $\mathbf{x} \in \mathbf{X}$, $\quad | e_i | \leqslant 2^{-\delta} \quad \delta = 24$ (single) or 53 (double)

Roundoff Errors

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \mathbf{e})$
Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}}(\mathbf{e}) \mathbf{x}^{\boldsymbol{\alpha}}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = \ell(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$, ℓ **linear** in $\mathbf{e}$

3: Bound $h(\mathbf{x}, \mathbf{e})$ with interval arithmetic

4: Bound $\ell(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

Exploiting Sparsity for Roundoff Error Bounds

$l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^{m} s_i(\mathbf{x}) e_i$

$I_1, \ldots, I_m$ correspond to $\{\mathbf{x}, e_1\}, \ldots, \{\mathbf{x}, e_m\}$

Dense relaxation: $\binom{n+m+2d}{n+m}$ SDP variables

Sparse relaxation: $m\binom{n+1+2d}{n+1}$ SDP variables

Preliminary Comparisons

$$f(\mathbf{x}) := x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

$$\mathbf{x} \in [4.00, 6.36]^6 , \quad \mathbf{e} \in [-\epsilon, \epsilon]^{15} , \quad \epsilon = 2^{-53}$$

- Dense SDP: $\binom{6+15+4}{6+15} = 12650$ variables $\rightsquigarrow$ **Out of memory**

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- SMT-based `rosa` tool: 762ϵ (19 $\times$ more CPU)

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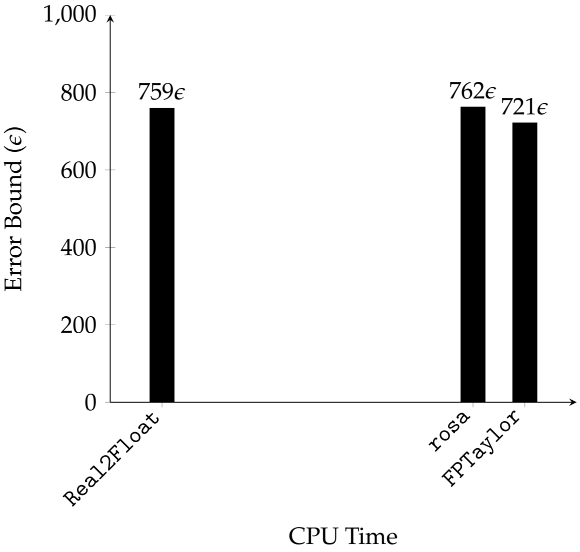
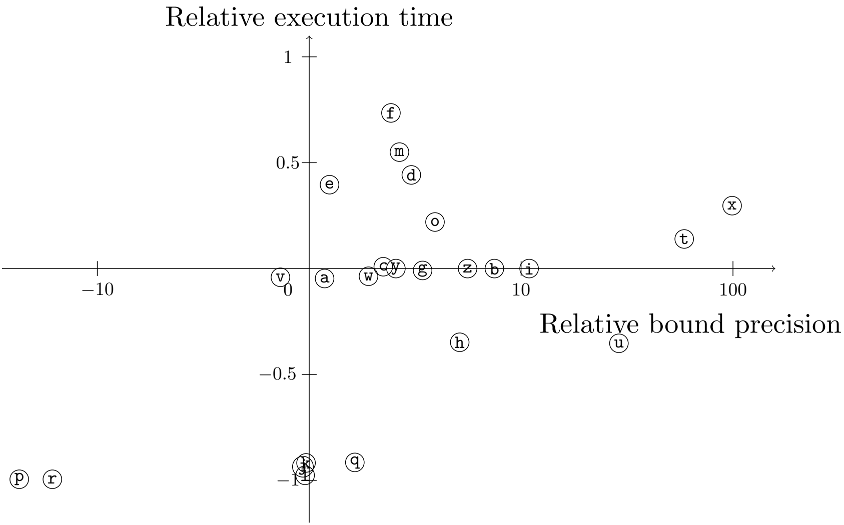
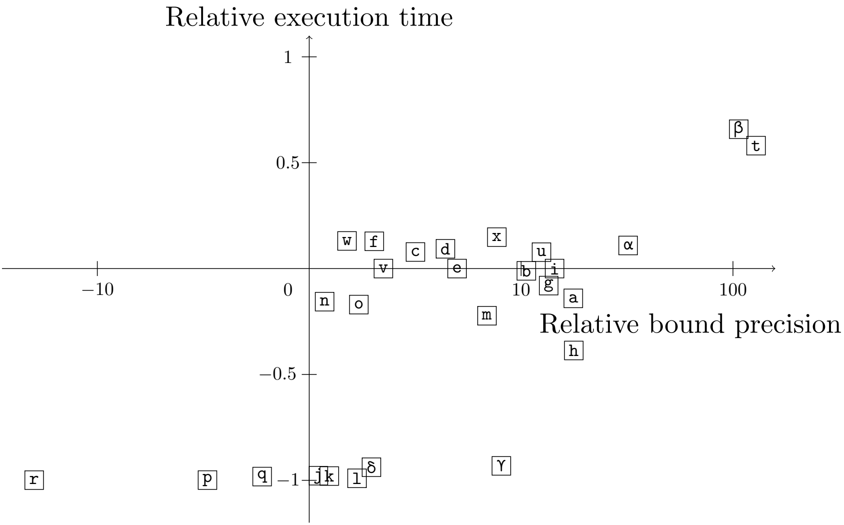


Comparison with `rosa`



Comparison with FPTaylor



Noncommutative (NC) Polynomials

Symmetric **Matrix** variables X_i, Y_j

$f = X_1(Y_1 + Y_2 + Y_3) + X_2(Y_1 + Y_2 - Y_3) + X_3(Y_1 - Y_2) - X_1 - 2Y_1 - Y_2$

with $X_1 X_2 \neq X_2 X_1$, **involution** $(X_1 Y_3)^\star = Y_3 X_1$

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Constraints $\mathbf{K} = \{(X, Y) : X_i, Y_j \succcurlyeq 0, X_i^2 = X_i, Y_j^2 = Y_j, X_i Y_j = Y_j X_i\}$

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MINIMAL EIGENVALUE OPTIMIZATION

$$\lambda_{\min} = \inf \{\langle f(X, Y)\mathbf{v}, \mathbf{v}\rangle : (X, Y) \in \mathbf{K}, \|\mathbf{v}\| = 1\}$$

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Minimal eigenvalue optimization

$$\begin{aligned}\lambda_{\min} &= \inf \{ \langle f(X,Y)\mathbf{v}, \mathbf{v} \rangle : (X,Y) \in \mathbf{K}, \|\mathbf{v}\| = 1 \} \\ &= \sup \quad \lambda \\ &\quad \text{s.t.} \quad f(X,Y) - \lambda \mathbf{I} \succcurlyeq 0 \,, \quad \forall (X,Y) \in \mathbf{K}\end{aligned}$$

Putinar's Representation

"**Archimedean**" constraint in $K = \{X : g_j(X) \succcurlyeq 0\}$:
$N - \sum_i X_i^2 \succcurlyeq 0$

Theorem: NC Putinar's representation [Helton-McCullough 02]

$f \succ 0$ on $\mathbf{K} \implies \boxed{f = \sum_i s_i^\star s_i + \sum_j \sum_i t_{ji}^\star g_j t_{ji}}$ with s_i, $t_{ji} \in \mathbb{R}\langle X \rangle$

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NC variant of Lasserre's Hierarchy for $\lambda_{\min}$:

replace "$f - \lambda \mathbf{I} \succcurlyeq 0$ on $\mathbf{K}$" by $f - \lambda \mathbf{I} = \sum_i s_i^\star s_i + \sum_j \sum_i t_{ji}^\star g_j t_{ji}$
with s_i, t_{ji} of **bounded** degrees

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$$\forall k = 1, \ldots, p-1 \quad I_{k+1} \cap \bigcup_{j \leqslant k} I_j \subseteq I_i \quad \text{for some } i \leqslant k$$

Theorem: Sparse Putinar's representation [Klep-M.-Povh 19]

$f \succ 0$ on $\mathbf{K}$ + RIP $\implies$ $\boxed{f = \sum_k \sum_i s_{ki}^\star s_{ki} + \sum_j \sum_i t_{ji}{}^\star g_j t_{ji}}$

with $s_{ki} \in \mathbb{R}\langle X, I_k \rangle$, $t_{ji} \in \mathbb{R}\langle X, I_{k(j)} \rangle$

Sparse Example: I_{3322} Bell Inequality

Entanglement in quantum mechanics

$\rightarrow$ **upper bounds** for violation levels of Bell inequalities

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6	0.250875397718	!!!!!

Exploiting Sparsity

Certified Polynomial Optimization

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$X = (X_1, \dots, X_n)$ **co-NP hard problem: check $f \geqslant 0$ on $\mathbf{K}$**

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1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

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   $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\} \quad g_j \in \mathbb{Q}[X]$

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[Collins 75] CAD **doubly exp. in** n **poly. in** d

[Grigoriev-Vorobjov 88, Basu-Pollack-Roy 98]
Critical points **singly exponential time** $(l+1)\, \tau\, d^{\mathcal{O}(n)}$

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Sums of squares (SOS)

$\sigma = {h_1}^2 + \cdots + {h_p}^2$

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HILBERT 17TH PROBLEM: f SOS of rational functions?

[Artin 27] **YES**!

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Semidefinite programming (SDP) $\leadsto$ **approximate** certificates

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Semidefinite programming (SDP) $\rightsquigarrow$ **approximate** certificates

$$\simeq \quad \to \quad =$$

The Question of Exact Certification

How to go from **approximate** to **exact** certification?

Motivation

Positivity certificates

- Stability proofs of critical control systems (Lyapunov)
- Certified function evaluation [Chevillard et. al 11]
- Formal verification of real inequalities [Hales et. al 15]:

COQ HOL-LIGHT

Decomposing Nonnegative Polynomials

1. **Polya**'s representation $f = \frac{\sigma}{(X_1^2+\cdots+X_n^2)^D}$
   positive definite form f
   [Reznick 95]

2. **Hilbert-Artin**'s representation $f = \frac{\sigma}{h^2}$
   $f \geqslant 0$
   [Artin 27]

3. **Putinar**'s representation $f = \sigma_0 + \sigma_1\, g_1 + \cdots + \sigma_l\, g_l$
   $f > 0$ on compact K $\quad \deg \sigma_i \leqslant 2D$
   [Putinar 93]

Decomposing Nonnegative Polynomials

- Deciding **polynomial nonnegativity**

$$f(a,b) = a^2 - 2ab + b^2 \geqslant 0$$

- $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$

- $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Decomposing Nonnegative Polynomials

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve **SDP**

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A}\,\mathbf{z} = \mathbf{d}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2$

- Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

From Approximate to Exact Solutions

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\boxed{\simeq \quad \rightarrow \quad =}\ ?$$

Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geqslant 0$ on $\mathbb{R}$ ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = {f_1}^2 + {f_2}^2$.

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2} X + \frac{1 + \sqrt{5}}{4} \right)^2 + \left(\frac{\sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}}}{4} X + \frac{\sqrt{10 - 2\sqrt{5}}}{4} \right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

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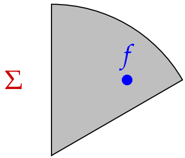
Round & Project Algorithm [Peyrl-Parrilo 08]



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

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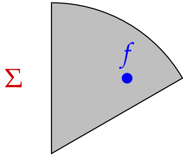

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Find $\tilde{\mathbf{G}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{G}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{G}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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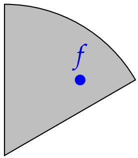
Σ



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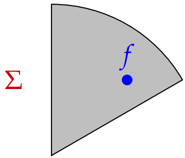

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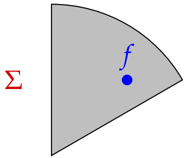

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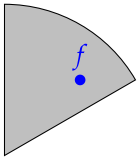

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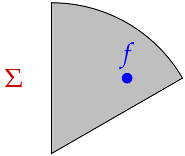

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Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = {\mathbf{v}_D}^T(X)\, \mathbf{G}\, \mathbf{v}_D(X)$ and $\mathbf{G} \succcurlyeq 0$

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$$

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$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1\, g_1 + \cdots + \tilde{\sigma}_m\, g_m$

Compact $\mathbf{K} \subseteq [0,1]^n$

$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1\, g_1 + \cdots + \tilde{\sigma}_m\, g_m$

$\boxed{\simeq \quad \to \quad =}$

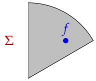
$\forall \mathbf{x} \in [0,1]^n, u(\mathbf{x}) \leqslant -\varepsilon$

$\min_{\mathbf{K}} f \geqslant \varepsilon$ when $\varepsilon \to 0$

COMPLEXITY?

From Approximate to Exact Solutions

Win TWO-PLAYER GAME

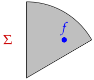


sum of squares of f?

≃ Output!

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



Hybrid Symbolic/Numeric Algorithms

sum of squares of $f - \varepsilon$?

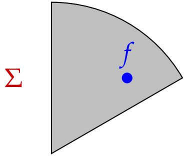
Error Compensation

$\simeq \quad \rightarrow \quad =$

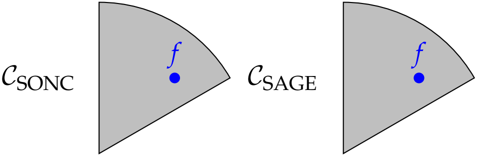
$\simeq$ Output!

From Approximate to Exact Solutions

Exact SOS



Exact SONC/SAGE



Software: RealCertify and POEM

Exact optimization via SOS: RealCertify

Maple & arbitrary precision SDP solver SDPA-GMP [Nakata 10]

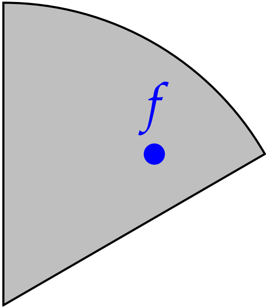
univsos	$\boxed{n = 1}$
multivsos	$\boxed{n > 1}$

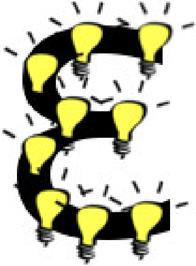
Exact optimization via SONC/SAGE: POEM

Python (SymPy) & geometric programming/relative entropy ECOS [Domahidi-Chu-Boyd 13]

`intsos` with $n \geqslant 1$: Perturbation

Σ





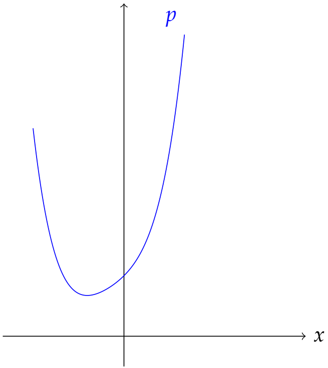
PERTURBATION idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$



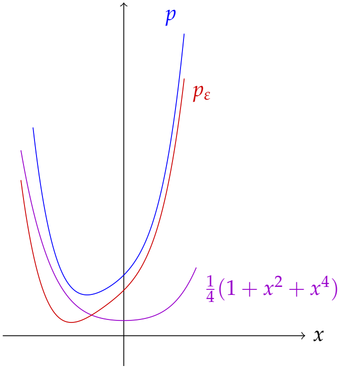
$p = 1 + X + X^2 + X^3 + X^4$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$

💡 **PERTURB**: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

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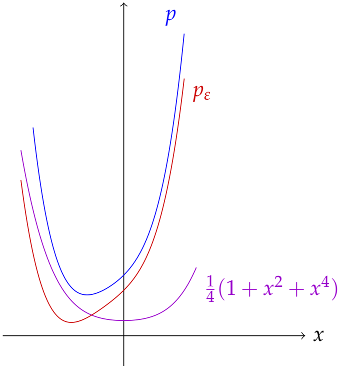
SDP Approximation:

$$p - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

ABSORB: small enough u_i
$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS



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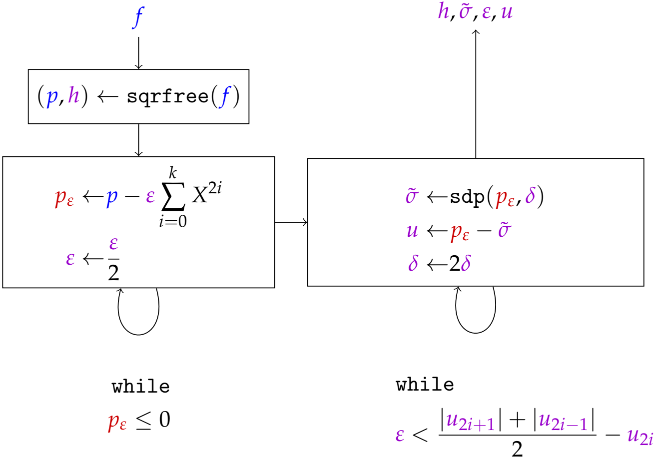
intsos with $n = 1$ and SDP Approximation

- **Input** $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



`intsos` with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

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$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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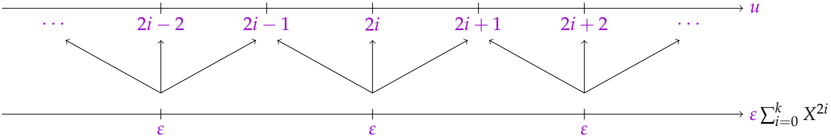

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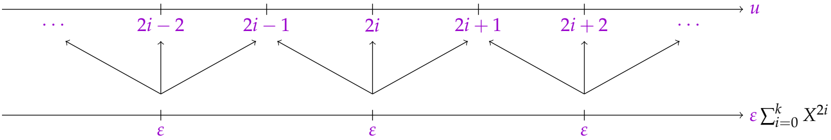
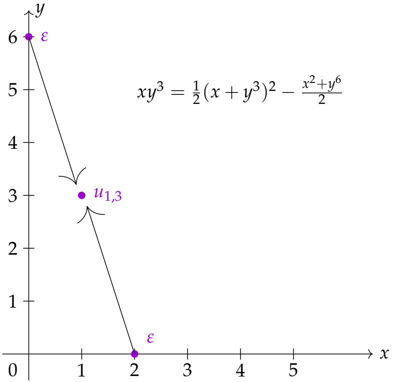


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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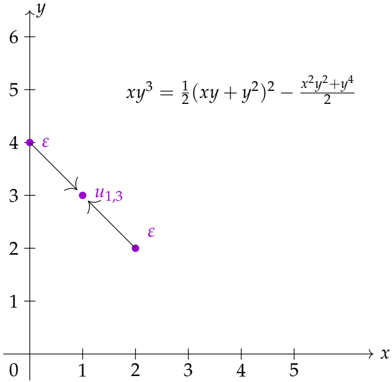

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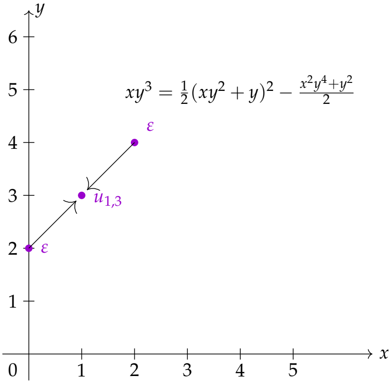

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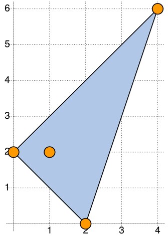
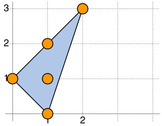
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\mathrm{spt}(f))$



Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm `intsos`

- **Input** $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
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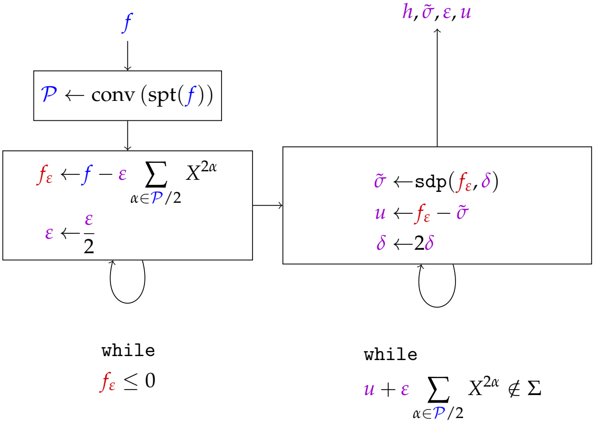

Algorithm `intsos`

Theorem (Exact Certification Cost in $\mathring{\Sigma}$)

$f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ with $\deg f = d = 2k$ and bit size τ

$\Longrightarrow$ `intsos` terminates with SOS output of bit size $\boxed{\tau\, d^{d^{\mathcal{O}(n)}}}$

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$f \in \mathbb{Q}[X]$ positive definite form with $\deg f = d$ and bit size τ

$$\implies D \leqslant 2^{\tau\, d^{\mathcal{O}(n)}} \quad \text{OUTPUT BIT SIZE} = \boxed{2^{2^{\tau^{\mathcal{O}(1)}} \cdot (4d+6)^{\mathcal{O}(n)}}}$$

Algorithm Putinarsos

Assumption: $\exists i$ s.t. $g_i = 1 - \|X\|_2^2$

$f > 0$ on $\mathbf{K} := \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$ has **Putinar**'s representation:

$$f = \sigma_0 + \sum_j \sigma_j \, g_j \quad \text{with } \sigma_j \in \Sigma[X] \, , \deg \sigma_j \leqslant 2D$$

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OUTPUT BIT SIZE = $D^{D^{\mathcal{O}(n)}}$ with $\log D = \mathcal{O}\left(2^{\tau \, d^{n \, C_{\mathbf{K}}}}\right)$

SOS Benchmarks

- Round & Project (SOS) [Peyrl-Parrilo]
- `RAGLib` (critical points) [Safey El Din]
- `SamplePoints` (CAD) [Moreno Maza-Alvandi et al.]

n	d	`RealCertify` τ_1 (bits)	`RealCertify` t_1 (s)	`RoundProject` τ_2 (bits)	`RoundProject` t_2 (s)	`RAGLib` t_3 (s)	`CAD` t_4 (s)
2	20	745 419	110.	78 949 497	141.	0.16	0.03
3	8	17 232	0.35	18 831	0.29	0.15	0.03
2	4	1 866	0.03	1 031	0.04	0.09	0.01
6	4	56 890	0.34	475 359	0.54	598.	–
10	4	344 347	2.45	8 374 082	4.59	–	–

Conclusion and Perspectives

1. Quest for efficiency
   Commutative: **Roundoff error** $n \simeq \mathbf{10^2}$
   Noncommutative: **Minimal eigenvalue** $n \simeq \mathbf{20 - 30}$

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APPLICATIONS IN QUANTUM PHYSICS

- Quantum games: number of mutually unbiased bases in dim 6, OPEN FOR SEVERAL DECADES!! — **symmetric**
- Ground state energy of hamiltonians — **symmetric** & **sparse**
- Inflation for quantum correlations — **symmetric** & **sparse**

Thank you for your attention!

`https://homepages.laas.fr/vmagron`

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