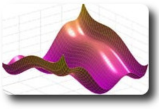
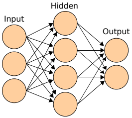
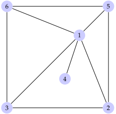
Polynomial optimization methods for machine learning

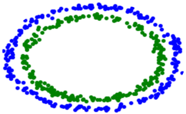
Victor Magron, CNRS LAAS
Chair of Polynomial optimization for Machine Learning

Spot ANITI Days, 12 November 2022

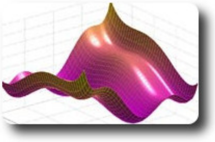


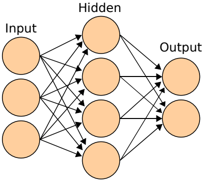






Context: what and who?





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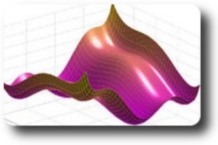

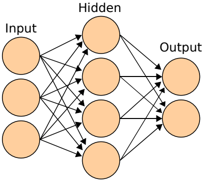

⇝ use **polynomial optimization** (POP) for machine learning

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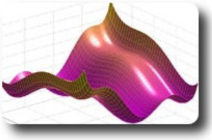

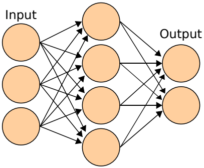

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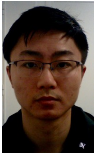

Tong Chen

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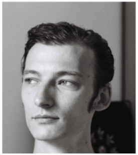

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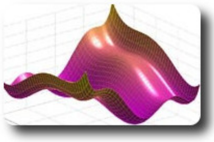

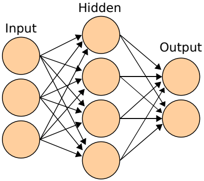

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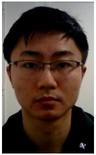

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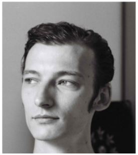

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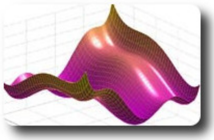

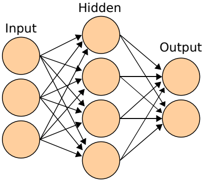

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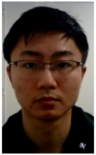

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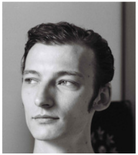

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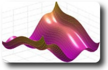
The POP hammer

NP-hard NON CONVEX Problem $f_{\min} = \inf f(\mathbf{x})$

Theory

(Primal)		(Dual)
$\inf \int f \, d\mu$		$\sup \; b$
with μ proba $\Rightarrow$	**infinite-dim**	$\Leftarrow$ with $f - b \geqslant 0$

The POP hammer

NP-hard NON CONVEX Problem $f_{\min} = \inf f(\mathbf{x})$

Practice

(Primal **Relaxation**)  (Dual **Strengthening**)

moments $\int \mathbf{x}^\alpha \, d\mu$ $f - b =$ **sum of squares**

finite number $\Rightarrow$ **finite-dim** $\Leftarrow$ **fixed** degree

[Lasserre '01] HIERARCHY of **CONVEX Problems** $\uparrow f_{\min}$
Based on representation of positive polynomials [Putinar '93]

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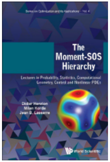

Emerging applications: quantum information theory, deep learning & power systems

Sparse hierarchies

Structure exploitation POP $\inf f(\mathbf{x})$ with "SPARSE" f

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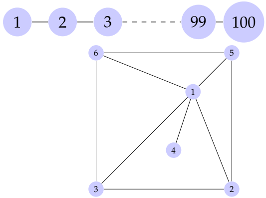

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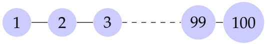
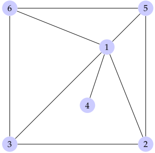


Term sparsity: few terms in f



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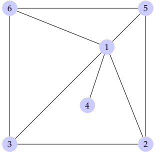

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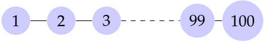

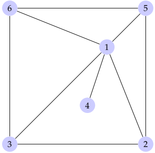

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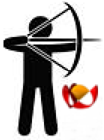
PERFORMANCE

VS



ACCURACY

Sparse hierarchies

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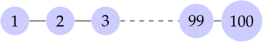

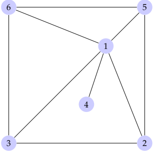

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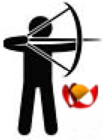
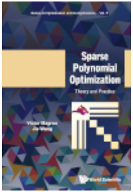


ACCURACY

Tons of applications: roundoff error bounds, entanglement, optimal power-flow, analysis of dynamical systems



Koopman operators

Koopman operators for nonlinear dynamical system $\mathbf{x}^+ = f(\mathbf{x})$

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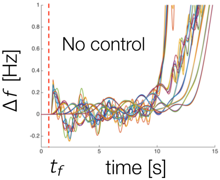
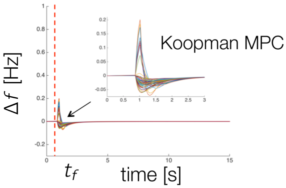
Applications to model predictive control





Christoffel-Darboux kernels

Old tool well-known in approximation theory & orthogonal polynomials

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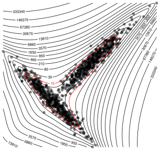
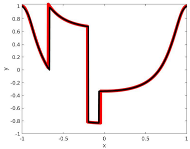
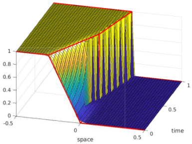
Outlier detection



Interpolation



Recovery



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Old tool well-known in approximation theory & orthogonal polynomials

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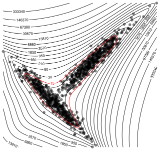

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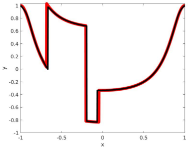

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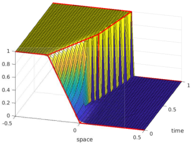

💡 Defined easily from the input data

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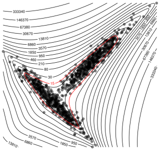

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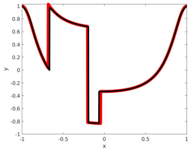

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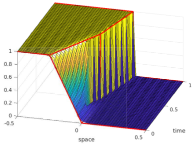

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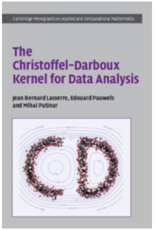
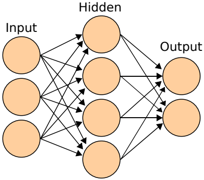


Chair activities

ROBUSTNESS CERTIFICATION OF NEURAL NETWORKS

💡 sparse polynomial optimization



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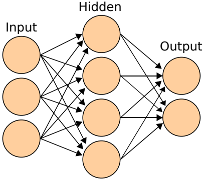
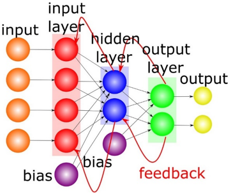


Stability analysis of recurrent networks

💡 copositive programming/integral constraints, Koopman



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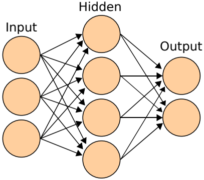

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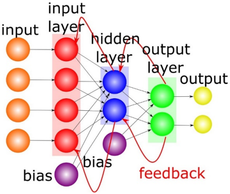

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Training/classification

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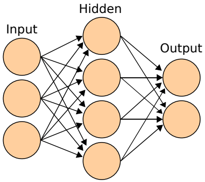

STABILITY ANALYSIS OF RECURRENT NETWORKS

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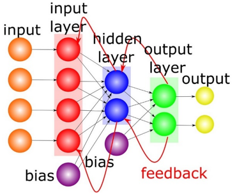
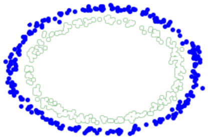


TRAINING/CLASSIFICATION

💡 Christoffel-Darboux kernels



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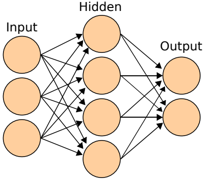

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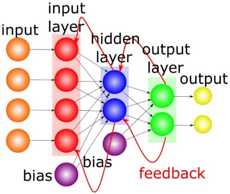

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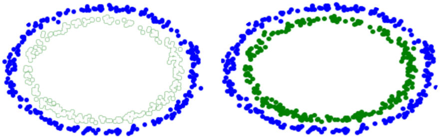


Zoom: robustness of NN [PhD Chen '19-22]

[SIAM News March '21]

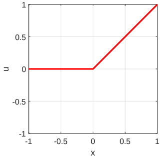
"Yet DL has an Achilles' heel. Current implementations can be highly unstable, meaning that a certain small perturbation to the input of a trained neural network can cause substantial change in its output. This phenomenon is both a nuisance and a major concern for the safety and robustness of DL-based systems in critical applications—like healthcare—where reliable computations are essential"

Zoom: robustness of NN [PhD Chen '19-22]

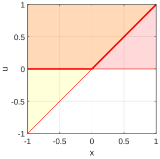
$$\mathbf{x}_0 \in \mathbb{R}^p \longrightarrow \mathbf{z}_0 \in \mathbb{R}^p \longrightarrow \mathbf{z}_1 \in \mathbb{R}^{p_1} \longrightarrow \cdots \longrightarrow \mathbf{z}_m \in \mathbb{R}^{p_m}$$

$$\mathbf{z}_i = \mathbf{A}_i \mathrm{ReLU}(\mathbf{z}_{i-1}) + \mathbf{b}_i$$

ReLU (left) & its "semialgebraicity" (right)



$u = \max\{x, 0\}$



$u(u - x) = 0, u \geq x, u \geq 0$

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"Direct" certification of a classifier with 1 hidden layer

$$\max_{\mathbf{x},\mathbf{z}} \quad (\mathbf{C}^{i,:} - \mathbf{C}^{k,:})\mathbf{z}$$

$$\text{s.t.} \quad \begin{cases} \mathbf{z} = \mathrm{ReLU}(\mathbf{A}\mathbf{x} + \mathbf{b}) \\ ||\mathbf{x} - \mathbf{x}_0|| \leq \epsilon \end{cases}$$

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$$\mathbf{z} = \mathrm{ReLU}(\mathbf{A}\mathbf{x} + \mathbf{b}) \to \mathbf{z} = \mathrm{ReLU}(\mathbf{W}\mathbf{z} + \mathbf{A}\mathbf{x} + \mathbf{b})$$

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- Go between 1ST & 2ND stair in SPARSE hierarchy

Collaborations & output

Polynomial optimization for ML

POP ANITI
MAC control
DISCO chair of Travé-Massuyès
UT3 Pauwels

TU Chemnitz formal verif
CREATE project Descartes
UCSB Koopman
CTU Prague control
CWI ideal sparsity
Kyushu University RNN

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Julia packages **github:InterRelax** and **github:TSSOS**

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$\sim$ 40 publications including 3 books, 2 NeurIPS, 20 journals