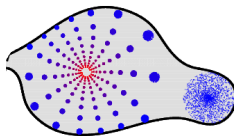
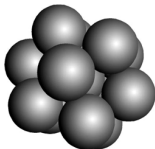


Certifying Non-negativity with Lasserre's Hierarchy and Semidefinite Programming

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5 November 2018

ENS Rennes SPA



Introduction

VERIFICATION OF NONLINEAR SYSTEMS ...

SAFETY of critical parts for **computing** \oplus **physical** devices



**Smart
Grids**



**Space
Systems**



... **CAST AS CERTIFIED OPTIMIZATION** \rightsquigarrow **SOLVE OFFLINE**

Input: linear  semidefinite  polynomial 

Output: value + numerical/symbolic/formal **certificate**

SDP for Polynomial Optimization

NP-hard NON CONVEX Problem $p^* = \inf p(x)$

Theory

(Primal)		(Dual)
$\inf \int p d\mu$		$\sup \lambda$
with μ proba \Rightarrow	INFINITE LP	\Leftarrow with $p - \lambda \geq 0$

SDP for Polynomial Optimization

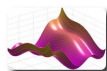
NP-hard NON CONVEX Problem $p^* = \inf p(x)$

Practice

(Primal **Relaxation**)

moments $\int x^\alpha d\mu$

finite number \Rightarrow



SDP

(Dual **Strengthening**)

$p - \lambda =$ sum of squares

\Leftarrow fixed degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow p^*$

[Lasserre/Parrilo 01]

degree d

n vars

$\Rightarrow \binom{n+d}{n}$ **SDP** VARIABLES

**Numeric
Solvers**

\Rightarrow **Approx Certificate**

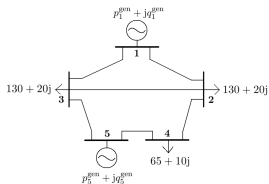


Success Stories: Lasserre's Hierarchy

MODELING POWER: Cast as ∞ -dimensional LP over measures

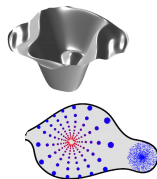
💡 **STATIC Polynomial Optimization**

Optimal Powerflow $n \simeq 10^3$ [Josz et al 16]



Roundoff Error $n \simeq 10^2$ [Magron et al 17]

💡 **DYNAMICAL Polynomial Optimization**
Regions of attraction [Henrion et al 14]



Reachable sets [Magron et al 17]



APPROXIMATE OPTIMIZATION BOUNDS!

Success Stories: Certified Optimization



Kepler's Conjecture(1611)

The max density of sphere packings is $\pi/\sqrt{18}$



Flyspeck : Formalizing the proof of Kepler by T.Hales (1994)
Verification of thousands of “tight” nonlinear inequalities

Seminal Paper:



Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017 ~ **120 citations**

MY CONTRIBUTION:



(Non)-Polynomial optimization to verify **Flyspeck** inequalities

Certification Challenges

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?



$(1.00001a - 0.99998b)^2!$



$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\simeq \rightarrow = ?$$

Certification Challenges

SCALABILITY

[Joswig et al 16] **20000+** terms, $d = 39$, $n = 6$

$$\begin{aligned} & 2d_1^2 d_2^2 k_1^{12} k_5^4 x_1^6 x_9^{13} + 46d_1^2 d_2^2 k_1^{12} k_5^5 x_1^5 x_9^{14} + 2d_1^2 d_2^2 k_1^{11} k_5^5 x_1^5 x_9^{14} + 46d_1^2 d_2^2 k_1^{11} k_5^5 x_1^4 x_9^{15} + 11d_1 d_2 k_1^{13} k_5^4 x_1^6 x_9^{14} + \\ & 297d_1 d_2 k_1^{13} k_5^4 x_1^5 x_9^{15} + 11d_1 d_2 k_1^{12} k_5^5 x_1^5 x_9^{15} + 297d_1 d_2 k_1^{12} k_5^5 x_1^4 x_9^{16} + 242k_1^{14} k_5^4 x_1^5 x_9^{16} + 242k_1^{13} k_5^4 x_1^4 x_9^{17} + 2d_1^3 d_2^3 k_1^{11} k_5^5 x_1^6 x_9^{11} + \\ & 46d_1^3 d_2^3 k_1^{11} k_5^5 x_1^5 x_9^{12} + 2d_1^3 d_2^3 k_1^{10} k_5^5 x_1^5 x_9^{12} + 46d_1^3 d_2^3 k_1^{10} k_5^5 x_1^4 x_9^{13} + 6d_1^3 d_2^3 k_1^{11} k_5^4 x_1^5 x_9^{13} + 138d_1^3 d_2^3 k_1^{11} k_5^4 x_1^4 x_9^{14} + \\ & 4d_1^3 d_2^3 k_1^{10} k_5^5 x_1^4 x_9^{14} + 92d_1^3 d_2^3 k_1^{10} k_5^5 x_1^3 x_9^{15} + 8d_1^2 d_2^3 k_1^{11} k_5^5 x_1^6 x_9^{12} + 184d_1^2 d_2^3 k_1^{11} k_5^5 x_1^5 x_9^{13} + 6d_1^2 d_2^3 k_1^{10} k_5^5 x_1^5 x_9^{13} + \\ & 138d_1^2 d_2^3 k_1^{10} k_5^5 x_1^4 x_9^{14} + 2d_1^2 d_2^3 k_1^{12} k_5^4 x_1^7 x_9^{11} + 73d_1^2 d_2^3 k_1^{12} k_5^4 x_1^6 x_9^{12} + 617d_1^2 d_2^3 k_1^{12} k_5^4 x_1^5 x_9^{13} + 2d_1^2 d_2^3 k_1^{12} k_5^4 x_1^6 x_9^{13} + \\ & 46d_1^2 d_2^3 k_1^{12} k_5^5 x_1^5 x_9^{14} + 2d_1^2 d_2^3 k_1^{11} k_5^5 x_1^6 x_9^{12} + 73d_1^2 d_2^3 k_1^{11} k_5^5 x_1^5 x_9^{13} + 617d_1^2 d_2^3 k_1^{11} k_5^5 x_1^4 x_9^{14} + 4d_1^2 d_2^3 k_1^{11} k_5^5 x_1^5 x_9^{14} + \\ & 92d_1^2 d_2^3 k_1^{11} k_5^4 x_1^4 x_9^{15} + 2d_1^2 d_2^3 k_1^{10} k_5^5 x_1^4 x_9^{15} + 46d_1^2 d_2^3 k_1^{10} k_5^5 x_1^3 x_9^{16} + 45d_1^2 d_2^3 k_1^{12} k_5^4 x_1^5 x_9^{14} + 1215d_1^2 d_2^3 k_1^{12} k_5^4 x_1^4 x_9^{15} + \\ & 34d_1^2 d_2^3 k_1^{11} k_5^5 x_1^4 x_9^{15} + 918d_1^2 d_2^3 k_1^{11} k_5^5 x_1^3 x_9^{16} + d_1 d_2^2 k_1^{12} k_5^4 x_1^7 x_9^{12} + 91d_1 d_2^2 k_1^{12} k_5^4 x_1^6 x_9^{13} + 1760d_1 d_2^2 k_1^{12} k_5^4 x_1^5 x_9^{14} + \\ & d_1 d_2^2 k_1^{11} k_5^5 x_1^6 x_9^{13} + 80d_1 d_2^2 k_1^{11} k_5^5 x_1^5 x_9^{14} + 1463d_1 d_2^2 k_1^{11} k_5^5 x_1^4 x_9^{15} + 12d_1 d_2 k_1^{13} k_5^4 x_1^7 x_9^{12} + 467d_1 d_2 k_1^{13} k_5^4 x_1^6 x_9^{13} + \\ & 3575d_1 d_2 k_1^{13} k_5^4 x_1^5 x_9^{14} + 11d_1 d_2 k_1^{13} k_5^4 x_1^6 x_9^{14} + 297d_1 d_2 k_1^{13} k_5^3 x_1^5 x_9^{15} + 12d_1 d_2 k_1^{12} k_5^5 x_1^6 x_9^{13} + 467d_1 d_2 k_1^{12} k_5^5 x_1^5 x_9^{14} + \\ & 3575d_1 d_2 k_1^{12} k_5^5 x_1^4 x_9^{15} + 22d_1 d_2 k_1^{12} k_5^5 x_1^5 x_9^{15} + 594d_1 d_2 k_1^{12} k_5^4 x_1^4 x_9^{16} + 11d_1 d_2 k_1^{11} k_5^5 x_1^4 x_9^{16} + 297d_1 d_2 k_1^{11} k_5^5 x_1^3 x_9^{17} + \\ & 1254d_1 k_1^{13} k_5^4 x_1^4 x_9^{16} + 1012d_1 k_1^{12} k_5^5 x_1^3 x_9^{17} + 43d_2 k_1^{13} k_5^4 x_1^6 x_9^{14} + 1834d_2 k_1^{13} k_5^4 x_1^5 x_9^{15} + 43d_2 k_1^{12} k_5^5 x_1^5 x_9^{15} + \\ & 1592d_2 k_1^{12} k_5^4 x_1^4 x_9^{16} + 286k_1^{14} k_5^4 x_1^6 x_9^{14} + 2904k_1^{14} k_5^4 x_1^5 x_9^{15} + 242k_1^{14} k_5^3 x_1^5 x_9^{16} + 286k_1^{13} k_5^5 x_1^5 x_9^{15} + \dots \end{aligned}$$

Certification Challenges

“In theory, theory and practice are the same. In practice, they are different.” - *A. Einstein*

CONVERGENCE RATE

THEORY $\frac{1}{\sqrt[c]{\log \frac{\text{STAIRS}}{c}}}$

[Nie-Schweighofer 07]

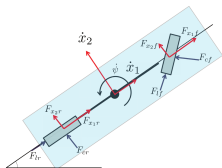


↑ PRACTICE ?

Scientific challenge: bridge THEORY & PRACTICE gap

Modeling Challenges Cyber-Physical

CONTROL SYSTEMS



Vehicles



Collisions



Fluid mechanics

PARTIAL DIFFERENTIAL EQUATIONS

MIXING DISCRETE/CONTINUOUS EQUATIONS

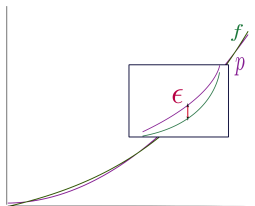
Discrete $x_{t+1} = f(x_t)$	$\implies \mu_T = \mu_0 + f_{\#} \mu$	} Liouville Transport Equation
Continuous $\dot{x} = f(x)$	$\implies \mu_T = \mu_0 + \operatorname{div} f \mu$	

Modeling Challenges Cyber-Physical

FINITE-PRECISION SOFTWARE/HARDWARE



Tuned Precision
FPGAs



Approx Math
Functions

$$a + (b + c) \neq (a + b) + c$$

Optimize Programs

PERFORMANCE



VS



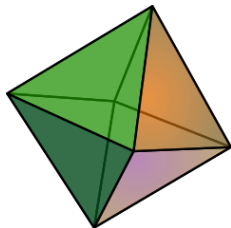
ACCURACY

MIXED PRECISION \oplus **Scalability** \oplus **Loops**

What is Semidefinite Optimization?

- Linear Programming (LP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \mathbf{A} \mathbf{z} \geq \mathbf{d} . \end{aligned}$$



- Linear cost \mathbf{c}
- Linear inequalities “ $\sum_i A_{ij} z_j \geq d_i$ ”

Polyhedron

What is Semidefinite Optimization?

- Semidefinite Programming (SDP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 . \end{aligned}$$

- Linear cost \mathbf{c}
- Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities “ $\mathbf{F} \succcurlyeq 0$ ”
(\mathbf{F} has nonnegative eigenvalues)



Spectrahedron

What is Semidefinite Optimization?

- Semidefinite Programming (SDP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0, \quad \mathbf{A} \mathbf{z} = \mathbf{d}. \end{aligned}$$

- Linear cost \mathbf{c}
- Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
- Linear matrix inequalities “ $\mathbf{F} \succcurlyeq 0$ ”
(\mathbf{F} has nonnegative eigenvalues)



Spectrahedron

Applications of SDP

- Combinatorial optimization
- Control theory
- Matrix completion
- Unique Games Conjecture (Khot '02) :
“A *single concrete algorithm* provides **optimal guarantees** among all efficient algorithms for a large class of computational problems.”
(Barak and Steurer survey at ICM'14)
- Solving polynomial optimization (Lasserre '01)

Lasserre's Hierarchy

- Prove **polynomial inequalities** with SDP:

$$f(a, b) := a^2 - 2ab + b^2 \geq 0 .$$

- Find \mathbf{z} s.t. $f(a, b) = \underbrace{\begin{pmatrix} a & b \\ z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$.

- Find \mathbf{z} s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2z_2 ab + z_3 b^2 \quad (\mathbf{A}\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Lasserre's Hierarchy

- Choose a cost \mathbf{c} e.g. $(1, 0, 1)$ and solve:

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0, \quad \mathbf{A} \mathbf{z} = \mathbf{d}. \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = (a \quad b) \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2.$

- Solving **SDP** \implies Finding **SUMS OF SQUARES** certificates

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

- Semialgebraic set

$$\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_m(\mathbf{x}) \geq 0\}$$

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

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- $:= [0, 1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geq 0, \quad x_2(1 - x_2) \geq 0\}$

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

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$$\underbrace{x_1 x_2}_f + \frac{1}{8} = \frac{1}{2} \overbrace{\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \frac{1}{2} \overbrace{x_1(1 - x_1)}^{\sigma_1} + \frac{1}{2} \overbrace{x_2(1 - x_2)}^{\sigma_2}$$

Lasserre's Hierarchy

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- Sums of squares (SOS) σ_i

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

- Semialgebraic set

$$\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_m(\mathbf{x}) \geq 0\}$$

- $:= [0, 1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geq 0, \quad x_2(1 - x_2) \geq 0\}$

$$\underbrace{x_1 x_2}_f + \frac{1}{8} = \frac{1}{2} \overbrace{\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \frac{1}{2} \overbrace{x_1(1 - x_1)}^{\sigma_1} + \frac{1}{2} \overbrace{x_2(1 - x_2)}^{\sigma_2}$$

- Sums of squares (SOS) σ_i

- Bounded degree:

$$\mathcal{Q}_d(\mathbf{K}) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \text{ with } \deg \sigma_j g_j \leq 2d \right\}$$

Lasserre's Hierarchy

- **Hierarchy of SDP relaxations:**

$$\lambda_d := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{Q}_d(\mathbf{K}) \right\}$$



- Convergence guarantees $\lambda_d \uparrow f^*$ [Lasserre 01]
- Can be computed with SDP solvers (CSDP, SDPA)
- **“No Free Lunch” Rule:** $\binom{n+2d}{n}$ SDP variables

SDP for Polynomial Optimization

Success Stories

Challenges

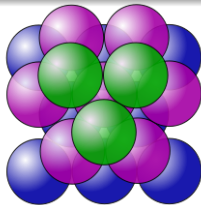
SDP for Nonlinear Optimization

RealCertify: Certify Non-negativity

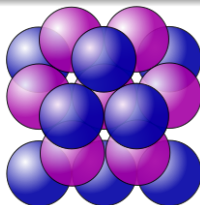
Oranges Stack

Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is $\frac{\pi}{\sqrt{18}}$



Face-centered cubic Packing



Hexagonal Compact Packing

A “Simple” Example

In the computational part:

- Multivariate Polynomials:

$$\Delta \mathbf{x} := x_1 x_4 (-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2 x_5 (x_1 - x_2 + x_3 + x_4 - x_5 + x_6) + x_3 x_6 (x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2 (x_3 x_4 + x_1 x_6) - x_5 (x_1 x_3 + x_4 x_6)$$

A “Simple” Example

In the computational part:

- **Semialgebraic** functions: composition of polynomials with $|\cdot|, \sqrt{\cdot}, +, -, \times, /, \sup, \inf, \dots$

$$p(\mathbf{x}) := \partial_4 \Delta \mathbf{x} \quad q(\mathbf{x}) := 4x_1 \Delta \mathbf{x}$$

$$r(\mathbf{x}) := p(\mathbf{x}) / \sqrt{q(\mathbf{x})}$$

$$l(\mathbf{x}) := -\frac{\pi}{2} + 1.6294 - 0.2213 (\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913 (\sqrt{x_4} - 2.52) + 0.728 (\sqrt{x_1} - 2.0)$$

A “Simple” Example

In the computational part:

- **Transcendental** functions \mathcal{T} : composition of semialgebraic functions with \arctan , \exp , \sin , $+$, $-$, \times , \dots

A “Simple” Example

In the computational part:

- Feasible set $\mathbf{K} := [4, 6.3504]^3 \times [6.3504, 8] \times [4, 6.3504]^2$

Lemma₉₉₂₂₆₉₉₀₂₈ from Flyspeck:

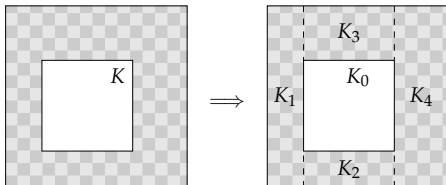
$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{p(\mathbf{x})}{\sqrt{q(\mathbf{x})}}\right) + l(\mathbf{x}) \geq 0$$

Existing Certified Frameworks

Lemma₉₉₂₂₆₉₉₀₂₈ from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{\partial_4 \Delta \mathbf{x}}{\sqrt{4x_1 \Delta \mathbf{x}}}\right) + l(\mathbf{x}) \geq 0$$

- Dependency issue using Interval Calculus:
 - One can bound $\partial_4 \Delta \mathbf{x} / \sqrt{4x_1 \Delta \mathbf{x}}$ and $l(\mathbf{x})$ separately
 - Too coarse lower bound: -0.87
 - Subdivide \mathbf{K} to prove the inequality



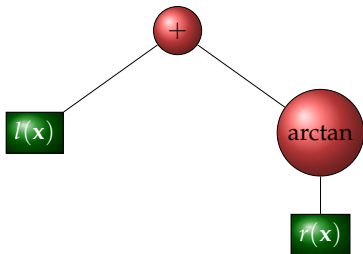
Nonlinear Function Representation

Tree representations of multivariate functions:

- leaves are **semialgebraic** functions
- nodes are **univariate** functions or binary operations

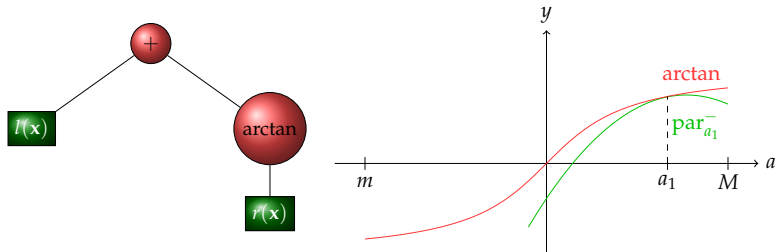
Nonlinear Function Representation

- For the “Simple” Example from Flyspeck:



Maxplus Optimization Algorithm

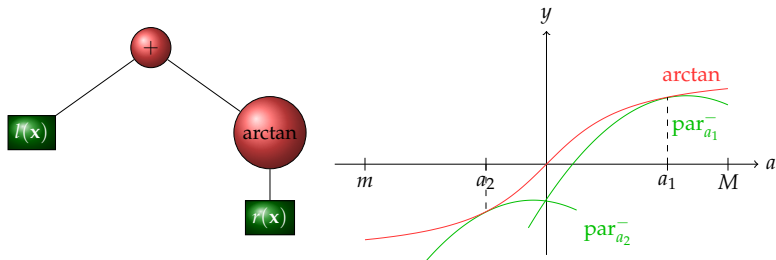
First iteration:



- 1 control point $\{a_1\}$: $m_1 = -4.7 \times 10^{-3} < 0$

Maxplus Optimization Algorithm

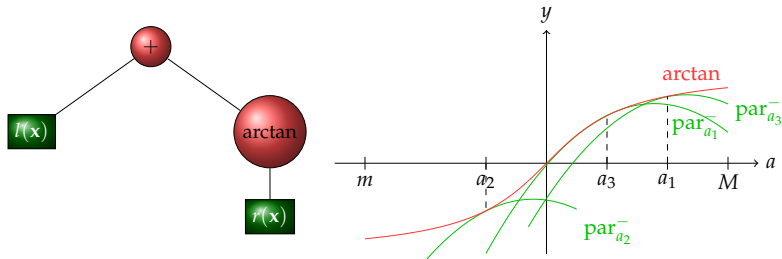
Second iteration:



2 control points $\{a_1, a_2\}$: $m_2 = -6.1 \times 10^{-5} < 0$

Maxplus Optimization Algorithm

Third iteration:



3 control points $\{a_1, a_2, a_3\}$: $m_3 = 4.1 \times 10^{-6} > 0$

OK!

Contribution: Publications and Software



M., Allamigeon, Gaubert, Werner.
Formal Proofs for Nonlinear Optimization,
Journal of Formalized Reasoning 8(1):1–24, 2015.



Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M.,
McLaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute,
Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of
Mathematics, Pi*, 5 2017

Software Implementation NLCertify:



15 000 lines of OCAML code



4000 lines of COQ code



M. NLCertify: A Tool for Formal Nonlinear Optimization, *ICMS*,
2014.

Roundoff Error Bounds

- Exact:

$$f(\mathbf{x}) := x_1x_2 + x_3x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1x_2(1 + e_1) + x_3x_4(1 + e_2)](1 + e_3)$$

- $\mathbf{x} \in \mathbf{X}$, $|e_i| \leq 2^{-p}$ $p = 24$ (single) or 53 (double)

Roundoff Error Bounds

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \mathbf{e})$

Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\alpha} r_{\alpha}(\mathbf{e}) \mathbf{x}^{\alpha}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = l(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$, l **linear** in \mathbf{e}

3: Bound $h(\mathbf{x}, \mathbf{e})$ with interval arithmetic

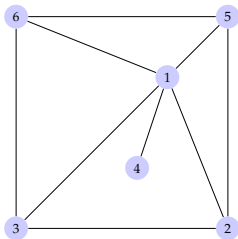
4: Bound $l(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

Roundoff Error Bounds

Sparse SDP Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$

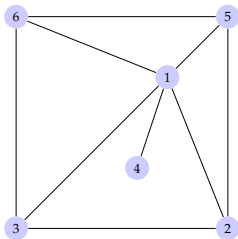


Roundoff Error Bounds

Sparse SDP Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

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1 Maximal cliques C_1, \dots, C_l

2 Average size $\kappa \rightsquigarrow \binom{\kappa+2k}{\kappa}$ vars

$$C_1 := \{1, 4\}$$

$$C_2 := \{1, 2, 3, 5\}$$

$$C_3 := \{1, 3, 5, 6\}$$

Dense SDP: 210 vars

Sparse SDP: 115 vars

Contributions

$$l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^m s_i(\mathbf{x})e_i$$

Maximal cliques correspond to $\{\mathbf{x}, e_1\}, \dots, \{\mathbf{x}, e_m\}$



M., Constantinides, Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming, *Trans. Math. Soft.*, 2016

Reachable Sets of Polynomial Systems

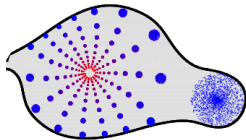
Iterations $\mathbf{x}_{t+1} = f(\mathbf{x}_t)$

Uncertain $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u})$

💡 **Converging** SDP hierarchies

💡 Image measure

💡 Liouville equation (conservation)



$$\mu_t + \mu = f_{\#} \mu + \mu_0$$

Reachable Sets of Polynomial Systems

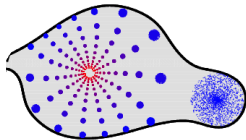
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M., Garoche, Henrion, Thirioux. Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems, 2017.

Invariant Measures of Polynomial Systems

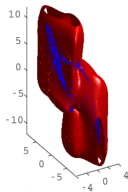
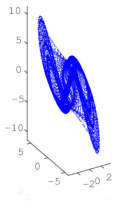
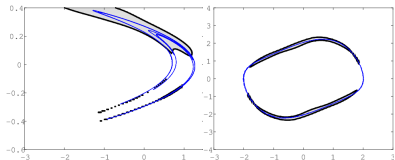
Discrete $\mathbf{x}_{t+1} = f(\mathbf{x}_t) \implies f_{\#} \mu - \mu = 0$

Continuous $\dot{\mathbf{x}} = f(\mathbf{x}) \implies \operatorname{div} f \mu = 0$

💡 **Converging** SDP hierarchies

💡 measures with density in L_p

💡 singular measures \implies chaotic attractors



Invariant Measures of Polynomial Systems

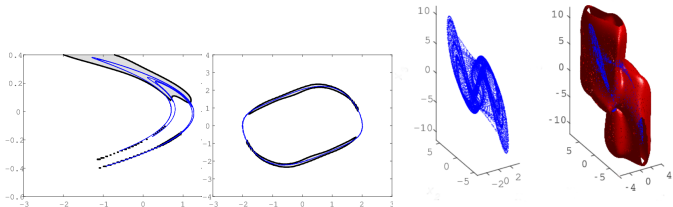
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💡 measures with density in L_p

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M., Forets, Henrion. Semidefinite Characterization of Invariant Measures for Polynomial Systems. 2018.

SDP for Polynomial Optimization

Success Stories

Challenges

SDP for Nonlinear Optimization

RealCertify: Certify Non-negativity

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$$X = (X_1, \dots, X_n)$$

$$f \in \mathbb{Q}[X]$$

co-NP hard problem: check $f \geq 0$ on \mathbb{K}

RealCertify: Certify Non-negativity

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$$f \in \mathbb{Q}[X]$$

1 Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

$$\boxed{n = 1} \quad f = 1 + X + X^2 + X^3 + X^4$$

$$\boxed{n > 1} \quad f = 4X_1^4 + 4X_1^3X_2 - 7X_1^2X_2^2 - 2X_1X_2^3 + 10X_2^4$$

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2 Constrained $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \dots, g_m(\mathbf{x}) \geq 0\}$

$$g_j \in \mathbb{Q}[X]$$

$$f = -X_1^2 - 2X_1X_2 - 2X_2^2 + 6 \quad \mathbf{K} = \{1 - X_1^2 \geq 0, 1 - X_2^2 \geq 0\}$$



RealCertify: Certify Non-negativity

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1 $f \in \Sigma =$ sums of squares (SOS)

$$f = \sigma = h_1^2 + \dots + h_p^2 \geq 0$$

2 Weighted SOS $f = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_m g_m \geq 0$ on \mathbf{K}

From Approximate to Exact Solutions

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?



$(1.00001a - 0.99998b)^2!$



$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\simeq \rightarrow = ?$$

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



sum of squares of f ?



\approx Output!



From Approximate to Exact Solutions

Win TWO-PLAYER GAME



 **Hybrid** Symbolic/Numeric Algorithms

sum of squares of $f + \epsilon$?

\approx Output!



Error Compensation

$\approx \rightarrow =$



Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geq 0$ on \mathbb{R} ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = f_1^2 + f_2^2$.

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□

Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1 + \sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10 + 2\sqrt{5}} + \sqrt{10 - 2\sqrt{5}}}{4}X + \frac{\sqrt{10 - 2\sqrt{5}}}{4}\right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \overset{\circ}{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1 \left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

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Existing Frameworks

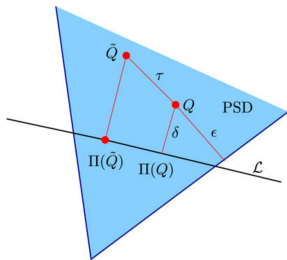
- project & round [Peyrl-Parrilo 08] [Kaltofen-Yang-Zhi 08]
 $f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

$$f(X) \simeq \mathbf{v}_D^T(X) \tilde{\mathbf{Q}} \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leq D$

 Find $\tilde{\mathbf{Q}}$ with semidefinite programming

$$f(X) = \mathbf{v}_D^T(X) \Pi(\mathbf{Q}) \mathbf{v}_D(X)$$



Existing Frameworks

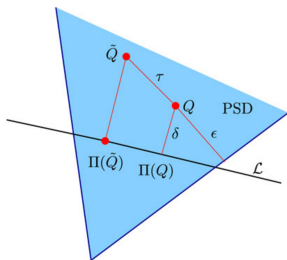
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- RAGLib (critical points) [Safey El Din]
- SamplePoints (CAD) [Moreno Maza-Alvandi et al.]

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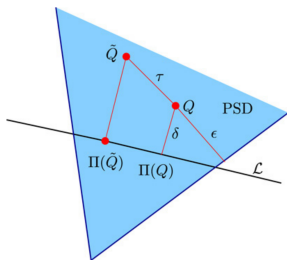
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Demo 1

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\}$

💡 Hybrid **SYMBOLIC/NUMERIC** methods

📄 Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

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$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

Compact $\mathbf{K} \subseteq [0, 1]^n$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 g_1 + \cdots + \tilde{\sigma}_m g_m$$

$$\boxed{\simeq \rightarrow =}$$

💡 $\forall \mathbf{x} \in [0, 1]^n, u(\mathbf{x}) \leq -\varepsilon$

$$\min_{\mathbf{K}} f \geq \varepsilon \text{ when } \varepsilon \rightarrow 0$$

COMPLEXITY?



Modules & Install

`gricad-gitlab:RealCertify`

Depends on Maple &

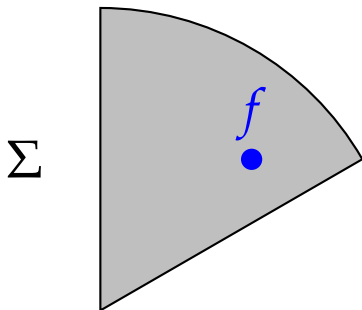
`univsos` $n = 1$

- Square free decomposition with `sqrfree`
- PARI/GP for complex zero isolation

`multivos` $n > 1$

- arbitrary precision SDP solver SDPA-GMP [Nakata 10]
- Newton Polytope with `convex` package [Franz 99]
- Cholesky's decomposition with `LUDecomposition`

intsos with $n \geq 1$: Perturbation



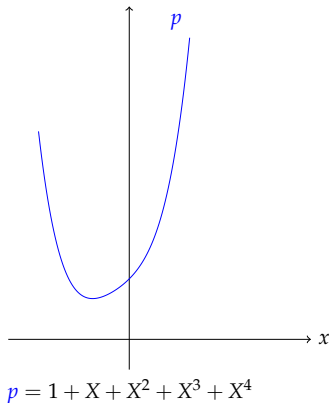
PERTURBATION idea

💡 Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$$p \in \mathbb{Q}[X], \deg p = d = 2k, p > 0$$

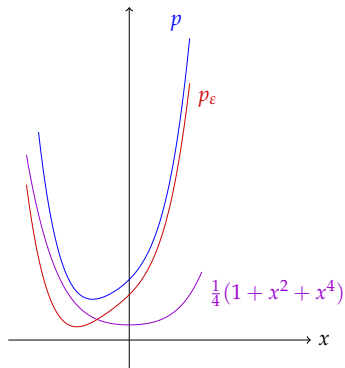


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💡 **PERTURB:** find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^k X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ [Chevillard et. al 11]

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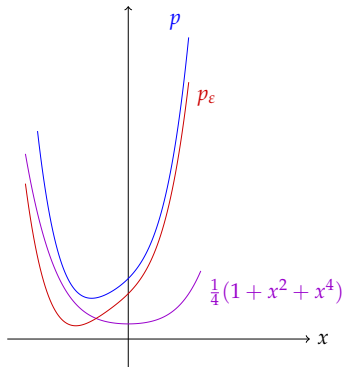
$$p_\varepsilon := p - \varepsilon \sum_{i=0}^k X^{2i} > 0$$

💡 **SDP Approximation:**

$$p - \varepsilon \sum_{i=0}^k X^{2i} = \tilde{\sigma} + u$$

💡 **ABSORB:** small enough u_i

$$\implies \varepsilon \sum_{i=0}^k X^{2i} + u \text{ SOS}$$



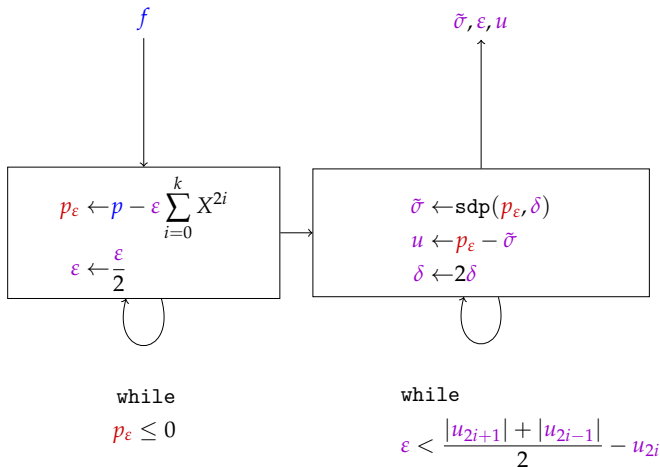
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intsos with $n = 1$ and SDP Approximation

- **Input:** $f \geq 0 \in \mathbb{Q}[X]$ of degree $d \geq 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output:** SOS decomposition with coefficients in \mathbb{Q}



intsos with $n = 1$: Absorbion

$$\text{💡 } X = \frac{1}{2}[(X+1)^2 - 1 - X^2]$$

$$\text{💡 } -X = \frac{1}{2}[(X-1)^2 - 1 - X^2]$$

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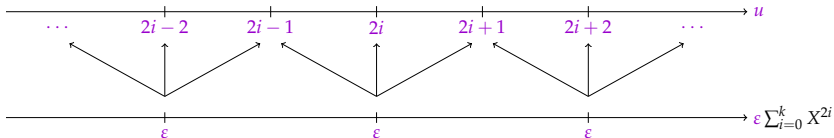
$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2} [(X^{i+1} + \text{sgn}(u_{2i+1})X^i)^2 - X^{2i} - X^{2i+2}]$$

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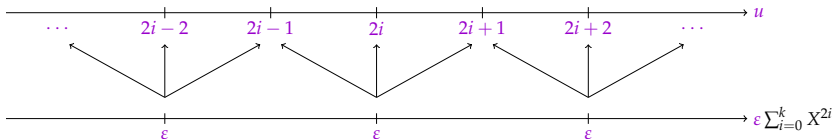


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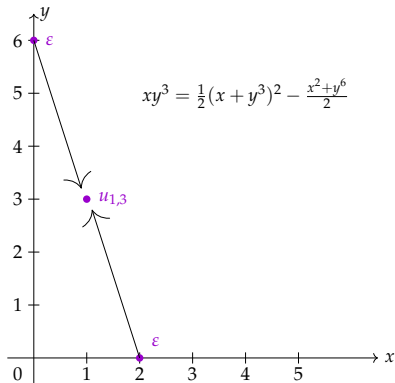


$$\epsilon \geq \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \epsilon \sum_{i=0}^k X^{2i} + u \quad \text{SOS}$$

intsos with $n \geq 1$: Absorbion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

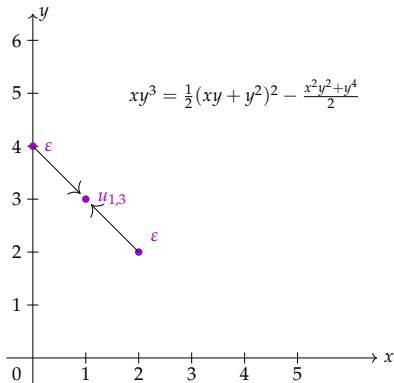
Choice of \mathcal{P} ?



intsos with $n \geq 1$: Absorption

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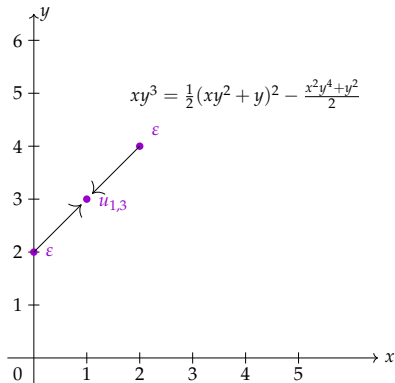
Choice of \mathcal{P} ?



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Choice of \mathcal{P} ?



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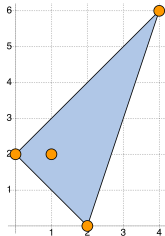
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Choice of \mathcal{P} ?

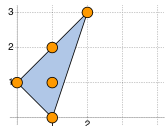
$$f = 4x^4y^6 + x^2 - xy^2 + y^2$$

$$\text{spt}(f) = \{(4, 6), (2, 0), (1, 2), (0, 2)\}$$

Newton Polytope $\mathcal{P} = \text{conv}(\text{spt}(f))$

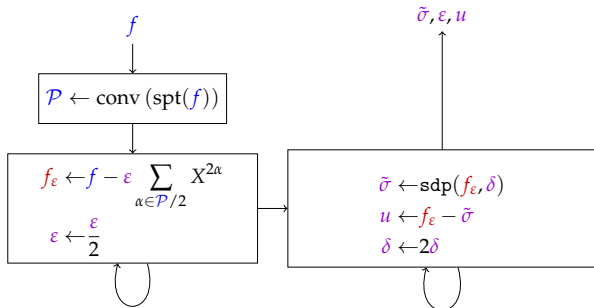


Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm intsos

- **Input:** $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d , $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output:** SOS decomposition with coefficients in \mathbb{Q}

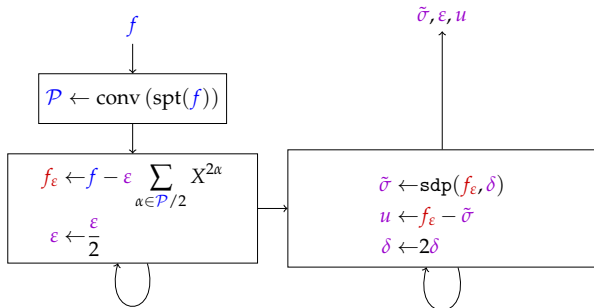


while
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 $u + \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} \notin \Sigma$

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Demo 2

Algorithm Putinarsos

Assumption: $\exists i$ s.t. $g_i = 1 - \|X\|_2^2$
 $f > 0$ on $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\}$ has **Putinar's** representation:

$$f = \sigma_0 + \sum_j \sigma_j g_j \quad \text{with } \sigma_j \in \Sigma[X], \deg \sigma_j \leq 2D$$

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 **ABSORPTION** as in Algorithm intsos: $u = f_\epsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j g_j$

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Demo 3

Unconstrained Benchmarks

Id	n	d	multivsos		RoundProject		RAGLib	CAD
			τ_1 (bits)	t_1 (s)	τ_2 (bits)	t_2 (s)	t_3 (s)	t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	598.	—
f_{10}	10	4	344 347	2.45	8 374 082	4.59	—	—

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Demo 4

Constrained Benchmarks

Id	n	d	multivsos			RAGLib	CAD
			D	τ_1 (bits)	t_1 (s)	t_2 (s)	t_3 (s)
f_{260}	6	3	2	114 642	2.72	4.19	—
f_{491}	6	3	2	108 359	9.65	0.01	0.05
f_{752}	6	2	2	10 204	0.26	0.07	—
f_{859}	6	7	4	6 355 724	303.	0.05	—
f_{863}	4	2	1	5 492	0.14	0.01	0.01
f_{884}	4	4	3	300 784	25.1	113.	—
butcher	6	3	2	247 623	1.32	231.	—
heart	8	4	2	618 847	2.94	24.7	—

Demo 5

Conclusion and Perspectives

Input f on \mathbf{K} with $\deg f = d$ and bit size τ

Algo	Input	\mathbf{K}	OUTPUT BIT SIZE
intsos	$\overset{\circ}{\Sigma}$	\mathbb{R}^n	$\tau d^{\mathcal{O}(n)}$
Putinarsos	> 0	$\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0\}$ compact	$\mathcal{O}(2^{\tau d^n c_{\mathbf{K}}})$

 How to handle degenerate situations?

Conclusion and Perspectives

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💡 How to handle degenerate situations?

💡 Better arbitrary-precision SDP solvers

💡 Extension to other relaxations, sums of hermitian squares

Crucial need for polynomial systems certification
Available M2/PhD Positions



End

Thank you for your attention!

`gricad-gitlab:RealCertify`

`http://www-verimag.imag.fr/~magron`



Magron, Safey El Din & Schweighofer. Algorithms for Weighted Sums of Squares Decomposition of Non-negative Univariate Polynomials, *JSC*. arxiv:1706.03941



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