# Noncommutative polynomial optimization

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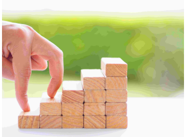
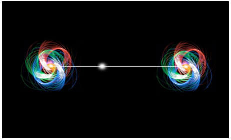

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Eigenvalue optimization

Trace optimization

SDP hierarchies

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Self-adjoint noncommutative variables $x_i, y_j$

$f = x_1(y_1 + y_2 + y_3) + x_2(y_1 + y_2 - y_3) + x_3(y_1 - y_2) - y_1 - 2y_1 - y_2$

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$$\begin{aligned} \lambda_{\min} &= \inf \{\langle f(x, y)\mathbf{v}, \mathbf{v}\rangle : (x, y) \in \mathbf{X}, \|\mathbf{v}\| = 1\} \\ &= \sup \quad \lambda \\ &\quad \text{s.t.} \quad f(x, y) - \lambda \mathbf{I} \succcurlyeq 0, \quad \forall (x, y) \in \mathbf{X} \end{aligned}$$

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$S \subset \operatorname{Sym} \mathbb{T}$ $\quad X_j$ operators from finite von Neumann algebra

Constraints $\mathcal{D}_S = \{\underline{X} = (X_1, \ldots, X_n) : s(\underline{X}) \succcurlyeq 0, \quad \forall s \in S\}$

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$\mathcal{A} = L^\infty(\mathcal{X}, \mu)$ is a vNa in $L^2(\mathcal{X}, \mu)$

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- More complicated ones!

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- Univ case [Klep-Pascoe-Volcic 21]: $a \succcurlyeq 0 \Rightarrow a =$ SOHS/SOHS
- Multilinear case [Huber 21]

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classical correlations = convex combinations of deterministic correlations

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Bell inequalities = linear inequalities in the correlations $P(a,b|s,t)$ and the marginals $P(a|s)$, $P(b|t)$, that are valid for all classical correlations and define the so-called **Bell polytope**

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Alice & Bob share a bipartite quantum state $\Psi$ and they answer $s, t$ by performing quantum measurements on their part of the quantum state:

$$P(a, b|s, t) = \Psi^\star X_s^a Y_t^b \Psi, \quad P(a|s) = \Psi^\star X_s^a \Psi, \quad P(b|t) = \Psi^\star Y_t^b \Psi$$

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$$P(a, b|s, t) = \Psi^\star X_s^a Y_t^b \Psi, \quad P(a|s) = \Psi^\star X_s^a \Psi, \quad P(b|t) = \Psi^\star Y_t^b \Psi$$

$X_s^a, Y_t^b$ are bounded operators on separable Hilbert spaces s.t.:

$$X_s^a Y_t^b = Y_t^b X_s^a, \quad X_s^a X_s^a = X_s^a, \qquad Y_t^b Y_t^b = Y_t^b$$

$$X_s^a X_s^{a'} = Y_t^b Y_t^{b'} = 0, \quad \sum_a X_s^a = \sum_b Y_t^b = I$$

# Motivation: Bell inequalities

Entanglement in quantum mechanics
$\rightarrow$ **upper bounds** for violation levels of Bell inequalities

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[Pozas et al 19] extension $\rightarrow$ identify correlations not attainable in entanglement-swapping scenario (quantum teleportation)

quantum physics operators $x_i, y_j$ satisfy causal constraints:

$$\operatorname{tr}(x_1 x_2 y_1 y_2) - \operatorname{tr}(x_1 x_2) \operatorname{tr}(y_1 y_2) = 0.$$

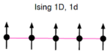
# Motivation: condensed matter

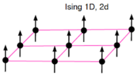
Ground-state energy ⇔ minimal eigenvalue of an Hamiltonian

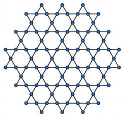
$$H = \sum_{\langle i,j \rangle} (x_i\, x_j + y_i\, y_j + z_i\, z_j)$$

spin states $(x_i, y_i, z_i)$, constraints

Lattices: 1D 2D Kagome



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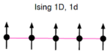


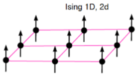


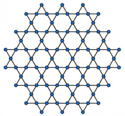

First neighbors interactions : $H = \sum_{i=1}^{n} x_i x_{i+1} + y_i y_{i+1} + z_i z_{i+1}$

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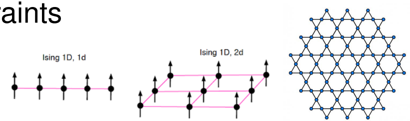


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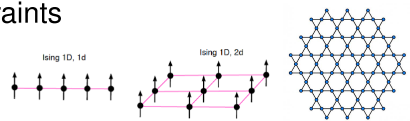


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Existing $\pm$ efficient techniques: quantum Monte Carlo & variational algorithms $\Rightarrow$ **upper bounds** on minimal energy

What is noncommutative optimization?

Eigenvalue optimization

Trace optimization

SDP hierarchies

# Eigenvalue optimization

$$\lambda_{\min} = \inf_{\mathbf{v},\mathcal{H}} \{ \langle a(\underline{X})\mathbf{v}, \mathbf{v} \rangle : \underline{X} \in \mathcal{D}_S, \|\mathbf{v}\| = 1 \, , \mathbf{v} \in \mathcal{H} \}$$

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**Theorem: NC Putinar's representation [Helton-McCullough 02]**

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$a - \lambda\mathbf{1} = \sum_i h_i^\star h_i + \sum_j \sum_i t_{ji}^\star s_j t_{ji}$ with $h_i$, $t_{ji}$ of **bounded** degrees

# Sparse eigenvalue optimization

Self-adjoint noncommutative (NC) variables $\underline{x} = (x_1, \ldots, x_n)$

**Theorem [Helton & McCullough '02]**

$f \succcurlyeq 0 \Leftrightarrow f$ SOS (all positive polynomials are sums of squares)

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**Theorem: Sparse NC Positivstellensatz [Klep Magron Povh '21]**

$f = \sum_k f_k$, $f_k$ depends on $\mathbf{x}(I_k)$

$f > 0$ on $\mathcal{D}_S$

Each $g_j$ depends on some $I_k$

RIP holds for $(I_k)$

ball constraints for each $\mathbf{x}(I_k)$

$\Longrightarrow$

$$f = \sum_{k,i} (s_{ki}^\star s_{ki} + \sum_{j \in J_k} t_{ji}{}^\star g_j t_{ji})$$

$s_{ki}$ "sees" vars in $I_k$

$t_{ji}$ "sees" vars from $g_j$

# Sparse eigenvalue optimization

$\mathbf{I}_{3322}$ Bell inequality (entanglement in quantum information)

$$f = x_1(y_1 + y_2 + y_3) + x_2(y_1 + y_2 - y_3) + x_3(y_1 - y_2) - x_1 - 2y_1 - y_2$$

Maximal violation levels $\rightarrow$ **upper bounds** on $\lambda_{\max}$ of $f$ on $\{(x,y) : x_i^2 = x_i, y_j^2 = y_j, x_i y_j = y_j x_i\}$

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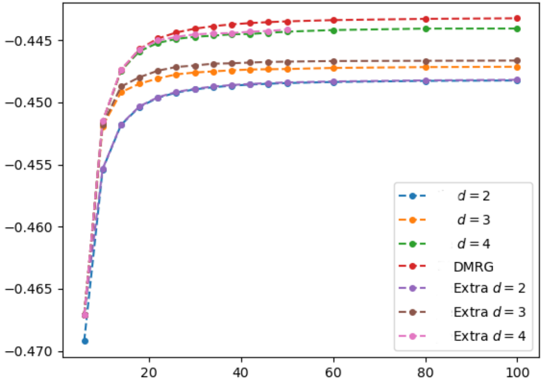
| 6 | 0.250875397718 0 | (1 hour) |

PERFORMANCE  VS  ACCURACY

# Sparse eigenvalue optimization

**Lower bounds** of the energy    1D lattice

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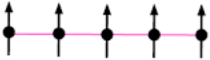

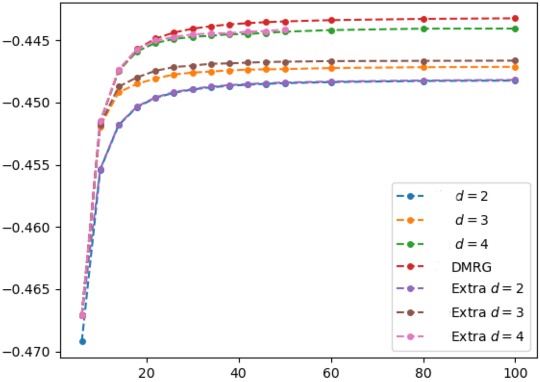


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**Sparse** solved within 1 hour on PFCALCUL at LAAS

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Eigenvalue optimization

Trace optimization

SDP hierarchies

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$$\begin{aligned} = \sup \quad & m \\ \text{s.t.} \quad & \mathrm{tr}(a(\underline{X}) - m) \geqslant 0\,, \quad \forall \underline{X} \in \mathcal{D}_S \end{aligned}$$

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$\mathrm{tr}_{\min}^{\mathrm{II}_1}$ = minimal trace over the union of type $- \mathrm{II}_1$ vN algebras

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Converging hierarchy with cyclic quadratic modules:

💡 replace "$\mathrm{tr}(a - m) \geqslant 0$ on $\mathcal{D}_S^{\mathrm{II}_1}$" by $a - m\mathbf{1} \in \mathcal{M}^{\mathrm{cyc}}(S)_r$

$\mathcal{M}^{\mathrm{cyc}}(S)_r$ = polynomials with same trace as some from $\mathcal{M}(S)_r$

# Kadison-Dubois representation theorem

$$\chi_{\mathcal{M}} := \{\varphi : \mathsf{T} \to \mathbb{R} \mid \varphi \text{ homomorphism}, \varphi(\mathcal{M}) \subseteq \mathbb{R}_{\geqslant 0}, \varphi(1) = 1\}$$

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**Theorem: Kadison-Dubois [Marshall 08]**

Given an Archimedean quadratic module $\mathcal{M} \subseteq \mathrm{T}$ & $a \in \mathrm{T}$:

$$\forall \varphi \in \chi_{\mathcal{M}} \quad \varphi(a) \geqslant 0 \qquad \Leftrightarrow \qquad \forall \varepsilon > 0 \quad a + \varepsilon \in \mathcal{M}$$

# Non-cyclic representation for T

For $S \subseteq \mathsf{T}$, “augment” $S$ with traces of hermitian squares:

$$S(N) = S \cup \{\mathrm{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \mathrm{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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**Lemma [Klep-M.-Volcic 20]**

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**Proof**

By induction: $\forall w \in \langle \underline{x} \rangle$, $m \pm \mathrm{tr}(w) \in \mathcal{M}(S(N))$ for some $m > 0$

$$w = x_j^{2k} \implies N^k + 1 + 2\,\mathrm{tr}(x_j^k) = \left(N^k - \mathrm{tr}(x_j^{2k})\right) + \mathrm{tr}\left((x_j^k + 1)^2\right)$$

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$$S[N] = S \cup \{N - x_j^2\} \subset \mathbb{T}$$

**Lemma [Klep-M.-Volcic 20]**

$\mathcal{M}(S(N))$ is archimedean

**Proof**

By induction: $\forall w \in \langle \underline{x} \rangle$, $m \pm \mathrm{tr}(w) \in \mathcal{M}(S(N))$ for some $m > 0$

$$w = x_j^{2k} \implies N^k + 1 + 2\,\mathrm{tr}(x_j^k) = \left(N^k - \mathrm{tr}(x_j^{2k})\right) + \mathrm{tr}\left((x_j^k + 1)^2\right)$$

**Theorem: Non-cyclic representation for T [Klep-M.-Volcic 20]**

$a \geqslant 0$ on $\mathcal{D}_{S[N]}^{\mathrm{II}_1}$ $\Leftrightarrow$ $a + \varepsilon \in \mathcal{M}(S(N))$ for all $\varepsilon > 0$

# Cyclic quadratic modules in $\mathbb{T}$

$\mathcal{M}^{\text{cyc}} \subseteq \operatorname{Sym} \mathbb{T}$ is a cyclic quadratic module if

$1 \in \mathcal{M}^{\text{cyc}}$, $\mathcal{M}^{\text{cyc}} + \mathcal{M}^{\text{cyc}} \subseteq \mathcal{M}^{\text{cyc}}$, $a^\star \mathcal{M}^{\text{cyc}} a \subseteq \mathcal{M}^{\text{cyc}} \ \forall a \in \mathbb{T}$

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**Proposition [Klep-M.-Volcic 20]**

$\mathcal{M}^{\text{cyc}}$ is archimedean $\Leftrightarrow N - \sum_{i=1}^n x_j^2 \in \mathcal{M}^{\text{cyc}}$ for some $N > 0$

# Positivity of elements in T

**Theorem [Klep-M.-Volcic 20]**

Let $\mathcal{M}^{\rm cyc} \subseteq \operatorname{Sym} \mathbb{T}$ & $a \in$ T.

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# Positivity of elements in $\mathrm{Sym}\,\mathbb{T}$

**Proposition [Klep-M.-Volcic 20]**

Let $(\mathcal{F}, \tau)$ be a tracial pair and $X = X^* \in \mathcal{F}$. Tfae:

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(i) $X \succcurlyeq 0$

(ii) $\tau(XY) \geq 0$ for all positive semidefinite contractions $Y \in \mathcal{F}$

(iii) $\tau(Xp(X)^2) \geq 0$ for all $p \in \mathbb{R}[t]$

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Let $\mathcal{M}^{\text{cyc}} \subseteq \operatorname{Sym}\mathbb{T}$ & $a \in \operatorname{Sym}\mathbb{T}$. The following are equivalent:

(i) $a \succcurlyeq 0$ on $\mathcal{D}^{\text{II}_1}_{\mathcal{M}^{\text{cyc}}}$

(ii) $\forall \varepsilon > 0$, there exist SOS $s_1, s_2 \in \mathbb{R}[t]$, $q \in \mathcal{M}^{\text{cyc}}$ such that

$$\operatorname{tr}(ay) + \varepsilon = \operatorname{tr}(s_1(a)y + s_2(a)(1-y)) + q$$

where $y$ is an auxiliary symmetric free variable.

💡 $\operatorname{tr}(ay) + \varepsilon$ is in the module generated by $\mathcal{M}^{\text{cyc}}, y, 1-y$

What is noncommutative optimization?

Eigenvalue optimization

Trace optimization

SDP hierarchies

# Tracial words & moment matrices

$\mathbb{T}$-words = $\{\prod_i \mathrm{tr}(u_i)v \mid u_i, v \in \langle \underline{x} \rangle\}$ and T-words

$\mathrm{tr}(x_1)^2$ is a T-word, $\mathrm{tr}(x_1)x_1$ is a $\mathbb{T}$-word

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Tracial degree = up to cyclic equivalence

$\mathbf{W}_r^{\mathbb{T}}$ = vector of $\mathbb{T}$-words of with tracial degree $\leqslant r$

$n = 1$: $\mathbf{W}_2^{\mathbb{T}}$ contains $1, x_1, x_1^2, \mathrm{tr}(x_1), \mathrm{tr}(x_1^2), \mathrm{tr}(x_1)x_1$

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Tracial moment matrix $\mathbf{M}_r^{\mathbb{T}}(L)$ for a linear functional $L$ : T $\to \mathbb{R}$:

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Trace localizing matrix $\mathbf{M}_r^{\mathbb{T}}(sL)$ for $s \in \mathbb{T}$:

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Reminder:

$$S(N) = S \cup \{\mathrm{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \mathrm{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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Elements of $\mathcal{M}(S(N))_r$ are

$$a_1^2 s \quad a_2^2 (N^k - \mathrm{tr}(x_j^{2k})) \quad \mathrm{tr}(ff^\star)$$

for $s \in S$, $a_i \in \mathsf{T}$, $f \in \mathbb{T}$

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**Theorem [Klep-M.-Volcic 20]**

There is no duality gap and $a_r \to a_{\min}^{\mathrm{II}_1}$ as $r \to \infty$

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Reduction from the general trace setting to the pure trace

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# Polynomial Bell inequalities

CLASSICAL WORLD

$$\psi^*(A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2)\psi \leqslant 2$$

for separable states $\psi \in \mathbb{C}^k \otimes \mathbb{C}^k$ and matrices $A_j, B_j$ satisfying $A_j^* = A_j$, $A_j^2 = I$, $B_j^* = B_j$, $B_j^2 = I$

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COVARIANCES OF QUANTUM CORRELATIONS

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2nd sparse SDP gives also 5 . . . 10 times faster

# Conclusion and perspectives

CONVERGING HIERARCHIES to minimize pure trace polynomials

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FREE PROBABILITIES for minimizer approximation: noncommutative Christoffel-Darboux kernels and the Siciak function [Beckermann et al. 20, Belinschi et al. 22]

# Advertisement

Two summer schools in QMATH departement, Copenhagen mid-end August

- Quantum entanglement via nonlocal games
  qmath.ku.dk/events/conferences/quantum-entanglement
- Entropy inequalities in quantum information
  indico.nbi.ku.dk/event/1317

# Thank you for your attention!


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# Thank you for your attention!


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