Certified and efficient polynomial optimization via conic programming

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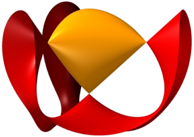
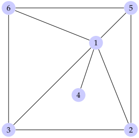
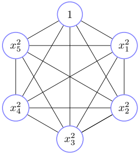


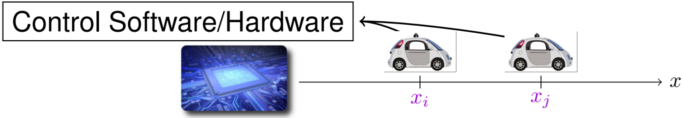




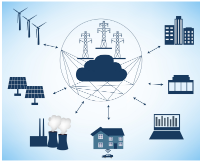
Motivation: verification of nonlinear systems

SAFETY of critical parts for **computing** $\oplus$ **physical** devices

Cars

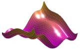


Smart Grids



Space Systems

... CAST AS CERTIFIED OPTIMIZATION $\rightsquigarrow$ SOLVE **OFFLINE**

Input: linear  semidefinite  polynomial 

Output: value + numerical/symbolic/formal **certificate**

The Moment-Sums of Squares Hierarchy

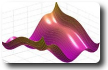
NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Theory

(Primal)

$\inf \int f \, d\mu$

with μ proba $\Rightarrow$



INFINITE LP

(Dual)

$\sup \lambda$

$\Leftarrow$ with $f - \lambda \geqslant 0$

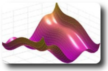
The Moment-Sums of Squares Hierarchy

NP-hard NON CONVEX Problem $f^\star = \inf f(\mathbf{x})$

Practice

(Primal **Relaxation**) | (Dual **Strengthening**)

moments $\int \mathbf{x}^\alpha \, d\mu$

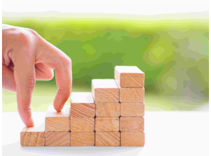


$f - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f^*$
[Lasserre/Parrilo 01]

degree d & n vars $\Longrightarrow \binom{n+2d}{n}$ **SDP** VARIABLES
Numeric solvers $\Longrightarrow$ **Approx** Certificate

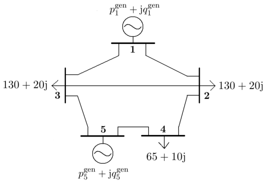


Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as ∞-*dimensional* LP over measures

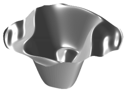
STATIC **Polynomial Optimization**

Optimal Powerflow $n \simeq \mathbf{10^3}$ [Josz et al 16]

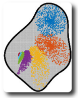


DYNAMICAL **Polynomial Optimization**

Regions of attraction [Henrion-Korda 14]



Reachable sets [Magron et al 17]



SDP for Polynomial Optimization

SDP Characterization of Reachable Sets

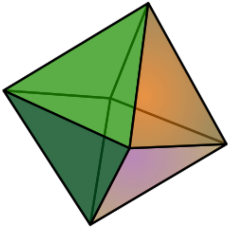
Sparse Polynomial Optimization

Conic Programming: LP

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d}$$

  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij}\, z_j \geqslant d_i$"

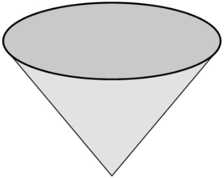


Polyhedron

Conic Programming: SOCP

- Second-order Cone Programming (SOCP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad z_1 \geq \sqrt{{z_2}^2 + \cdots + {z_n}^2}$$



Second-order Cone

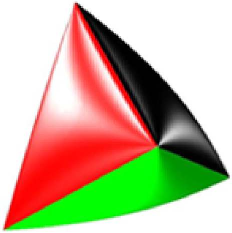
  - Linear cost $\mathbf{c}$
  - convex set defined by the linear/quadratic inequalities

Conic Programming: SDP

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0$$

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

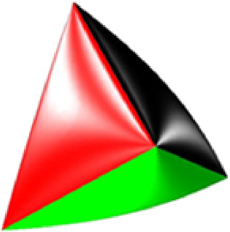


Spectrahedron

Conic Programming: SDP

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \ .$$



Spectrahedron

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

SDP for Polynomial Optimization

Prove **polynomial inequalities** with SDP:

$$f = a^2 - 2ab + b^2 \geqslant 0$$

Find $\mathbf{z}$ s.t. $f = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$

$$a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$$

$$\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$$

SDP for Polynomial Optimization

Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

$$a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a-b)^2$$

Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

Semialgebraic set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

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$\mathbf{X} = [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geqslant 0, \quad x_2(1 - x_2) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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Sums of squares (SOS) σ_j

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Sums of squares (SOS) σ_j

Fixed degree: $\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^{l} \sigma_j g_j, \text{ with } \deg \sigma_j\, g_j \leqslant 2r \right\}$

SDP for Polynomial Optimization

Hierarchy of SDP relaxations: $\lambda_r := \sup_{\lambda} \left\{ \lambda : f - \lambda \in \mathcal{Q}_r(\mathbf{X}) \right\}$

Convergence guarantees $\lambda_r \uparrow f^*$ [Lasserre 01]

Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

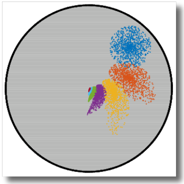
"No Free Lunch" Rule: $\binom{n+2d}{n}$ SDP variables

SDP for Polynomial Optimization

SDP Characterization of Reachable Sets

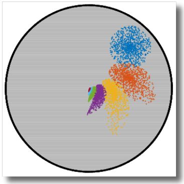
Sparse Polynomial Optimization

SDP Characterization of Reachable Sets



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$

SDP Characterization of Reachable Sets



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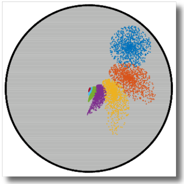

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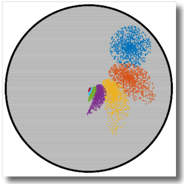
Reachable Set (RS) of admissible trajectories
$\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t\in\mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \mathbf{x}_0 \in \mathbf{X}_0\}$

SDP Characterization of Reachable Sets



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$\mathbf{X}^\infty = \mathbf{X}_0 \bigcup f(\mathbf{X}_0) \bigcup f^2(\mathbf{X}_0) \bigcup \cdots \subseteq \mathbf{X}$ (box or ball)

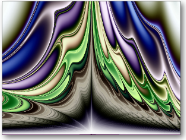
Tractable approximations of RS $\mathbf{X}^\infty$?

SDP Characterization of Reachable Sets

- Occurs in several contexts :
  1. program analysis: fixpoint computation

     toyprogram (x_1, x_2)
     requires $(0.25 \leqslant x_1 \leqslant 0.75$ && $0.25 \leqslant x_2 \leqslant 0.75)$
     ;
     while $(x_1^2 + x_2^2 \leqslant 1)$ {
     $x_1 = x_1 + 2x_1x_2$;
     $x_2 = 0.5(x_2 - 2x_1^3)$;
     }

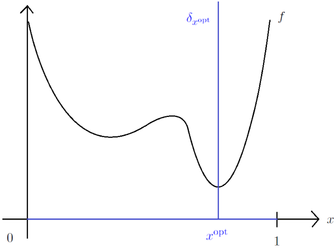
  2. hybrid systems, biology: Neuron Model, Growth Model
  3. control: integrator, Hénon map



Characterizing the RS

CHARACTERIZE A VALUE

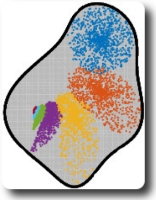
$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$



Dirac measure $\mu^\star = \delta_{\mathbf{x}^{\mathrm{opt}}}$

CHARACTERIZE A SET

?

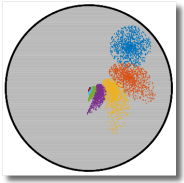


Lebesgue measure $\mu^\star = \lambda_{\mathbf{X}^\infty}$

Occupation Measures and Liouville's Equation

$\mathbf{x}_{t+1} = f(\mathbf{x}_t) \quad \mathbf{x}_0 \in \mathbf{X}_0$

$\mathbf{x}_1 = f(\mathbf{x}_0) \ldots \mathbf{x}_t = f(\mathbf{x}_{t-1})$



- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$

- **Pushforward** $f_\# : \mathcal{M}_+(\mathbf{X}_0) \to \mathcal{M}_+(\mathbf{X})$

$$\mu_1(\mathbf{A}) = f_\# \mu_0(\mathbf{A}) := \mu_0(f^{-1}(\mathbf{A}))$$

- $f_\# \mu_0$ is the **image measure** of μ_0 under f

Occupation Measures and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ and

$$\mu_1 = f_\# \mu_0$$
$$\dots$$
$$\mu_t = f_\# \mu_{t-1}$$
$$\nu_t = \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} f_\#^i \mu_0$$

- The measures μ_t, ν_t, μ_0 satisfy **Liouville's Equation**:

$$\mu_t + \nu_t = f_\# \nu_t + \mu_0$$

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\lambda_{\mathbf{X}_t} = f^t_{\#}\, \mu_0$
  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**

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- Lebesgue measure $\lambda_{\mathbf{X}^T}$ on $\mathbf{X}^T := \bigcup_{t=0}^T \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
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  $\Longrightarrow \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

$$\lambda_{\mathbf{X}^T} + \nu^T = f_{\#}\, \nu^T + \mu_0^T$$

average **occupation measure** ν^T: volume occupied in $\mathbf{X}^T$

Volume Assumption

Discrete Time

Define $\mathbf{Y}^0 = \mathbf{X}^0$ and $\mathbf{Y}^t = \mathbf{X}_t \backslash \mathbf{X}^{t-1}$

$$\lim_{T \to \infty} \frac{1}{\operatorname{vol}(\mathbf{X})} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t < \infty$$

Volume Assumption

Discrete Time

$\mathbf{Y}^0 = \mathbf{X}^0$ and $\mathbf{Y}^t = \mathbf{X}_t \backslash \mathbf{X}^{t-1}$

$$\lim_{T \to \infty} \underbrace{\frac{1}{\operatorname{vol}(\mathbf{X})} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t}_{\text{average minimal time to reach } \mathbf{X}^T} < \infty$$

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Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Infinite Primal LP for Discrete RS

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \nu = f_{\#} \nu + \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,, \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Infinite Primal LP for Discrete RS

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& \mu + \nu = f_{\#} \nu + \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) , \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Lemma

Volume Assumption $\Longrightarrow$ optimal solution $\mu^\star = \lambda_{\mathbf{X}^\infty}$

Primal-dual LP in Discrete Time

Primal LP

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \nu = f_{\#} \nu + \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \\
& \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^T := \inf_{u,v,w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{C}_+(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{C}_+(\mathbf{X}) \\
& w \in \mathcal{C}_+(\mathbf{X}) \\
& u + v \circ f - v \in \mathcal{C}_+(\mathbf{X}) \\
& u \geqslant 0 \\
& u \in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X})
\end{aligned}
$$

SDP Strengthening of the Dual LP

Discrete Time

$$
\begin{aligned}
d_r^T := \inf_{u,v,w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + Tu) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{Q}_r(\mathbf{X}_0) \\
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\end{aligned}
$$

Theorem [Magron-Garoche-Henrion-Thirioux 19]

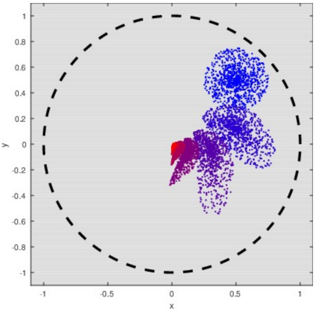
$\mathbf{X}_r^T = \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.

Volume Assumption $\Rightarrow \lim_{r \to \infty} \mathrm{vol}(\mathbf{X}_r^\infty \backslash \mathbf{X}^\infty) = 0$.

Toy Example

Trajectories from $\mathbf{X}_0 = \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ = \frac{1}{2}(x_1 + 2x_1x_2)$$
$$x_2^+ = \frac{1}{2}(x_2 - 2x_1^3)$$



$\mathbf{X}_2^{\infty}$

Toy Example

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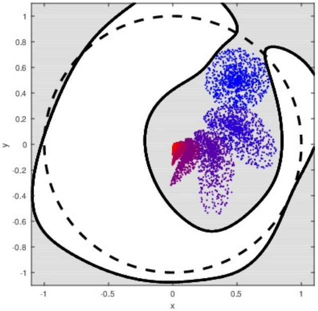

$\mathbf{X}_3^\infty$

Toy Example

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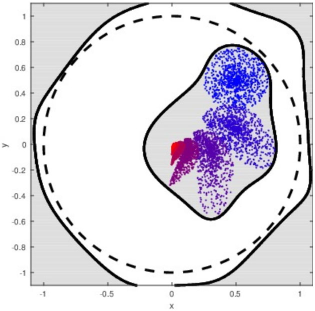

$\mathbf{X}_4^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 = \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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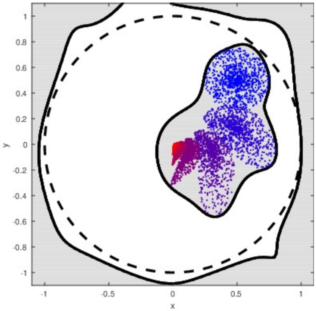

$\mathbf{X}_5^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 = \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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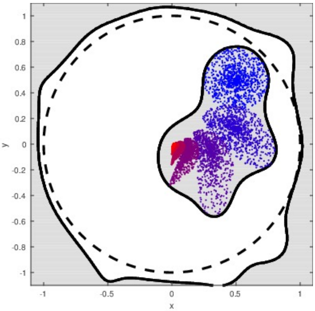

$\mathbf{X}_6^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 = \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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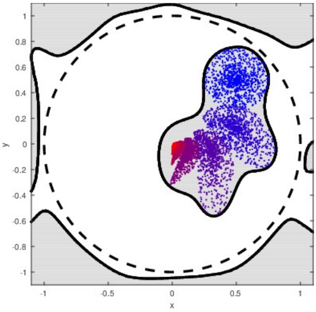
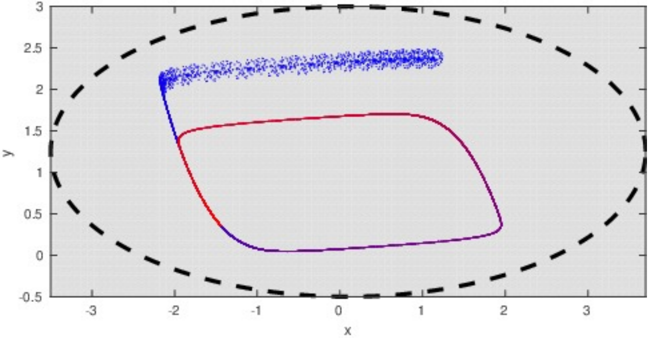


$\mathbf{X}_7^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 = [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ = x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)$$
$$x_2^+ = x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))$$



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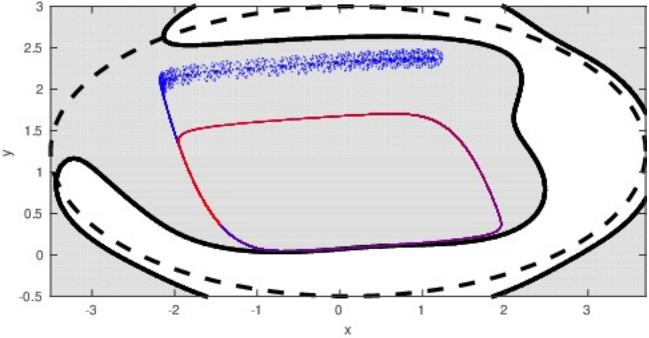

$\mathbf{X}_3^{\infty}$

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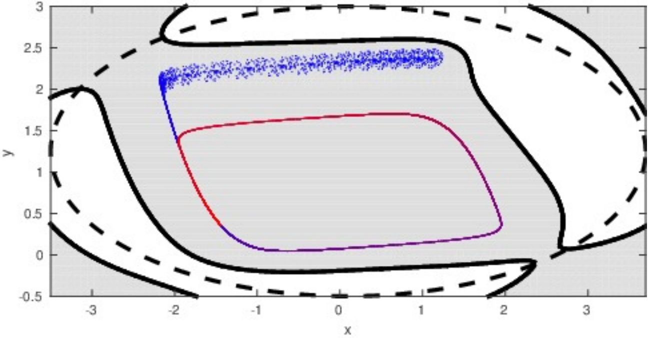

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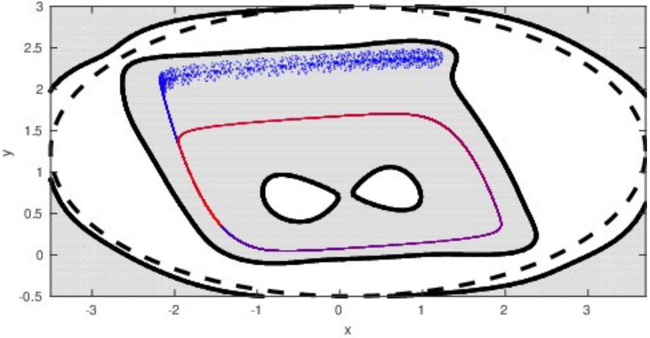

$\mathbf{X}_5^\infty$

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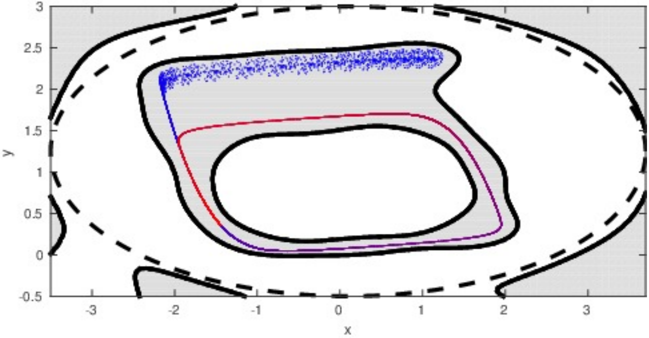

$\mathbf{X}_6^\infty$

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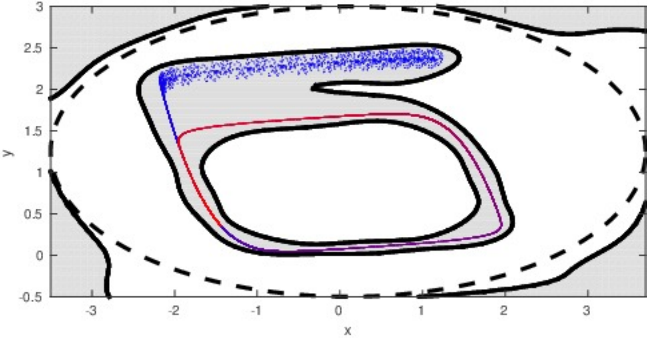


$\mathbf{X}_7^\infty$

SDP for Polynomial Optimization

SDP Characterization of Reachable Sets

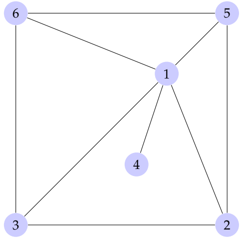
Sparse Polynomial Optimization

Sparse Polynomial Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$f = x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$

Chordal graph G



Sparse Polynomial Optimization [Waki, Lasserre 06]

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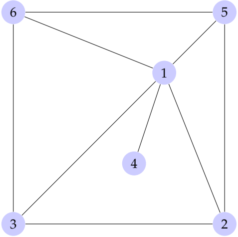


$C_1 = \{1, 4\}$
$C_2 = \{1, 2, 3, 5\}$
$C_3 = \{1, 3, 5, 6\}$

Dense SDP: 210 vars

Sparse SDP: 115 vars

1. Maximal cliques $C_1, \ldots, C_p$
2. Average size $\kappa \rightsquigarrow \binom{\kappa+2d}{\kappa}$ vars

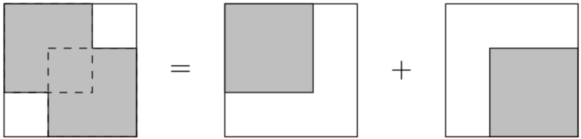
Sparse Polynomial Optimization [Waki, Lasserre 06]

Theorem [Griewank-Toint 84]

Chordal graph G with maximal cliques $C_1, \dots, C_p$

$Q_G \succcurlyeq 0$ with nonzero entries at edges of G

$\implies Q_G = \sum_k {P_{C_k}}^T Q_k P_{C_k}$ with $Q_k \succcurlyeq 0$ indexed by C_k



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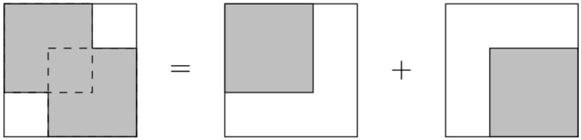

Sparse $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{R}[\mathbf{x}, C_k]$

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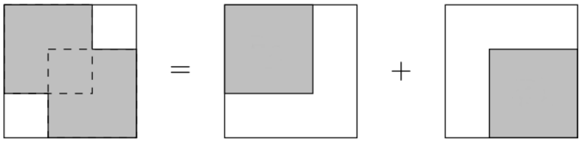

Sparse $f = f_1 + \cdots + f_p$ with $f_k \in \mathbb{R}[\mathbf{x}, C_k]$

Theorem: Sparse SOS representation [Waki 06]

f SOS + Chordal Graph G with cliques C_k

$\implies$ $\boxed{f = \sigma_1 + \cdots + \sigma_p}$ with σ_k SOS involving vars in C_k

Sparse Examples

Chained singular function:

$$f_{\mathsf{cs}} = \sum_{i \in J} ((x_i + 10x_{i+1})^2 + 5(x_{i+2} - x_{i+3})^2 + (x_{i+1} - 2x_{i+2})^4 + 10(x_i - x_{i+3})^4)$$

$J = \{1, 3, 5, \ldots, n-3\}$ & $4 \mid n$

$C_k = \{k, k+1, k+2, k+3\}$

Sparse Examples

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$J = \{1, 3, 5, \dots, n-3\}$ & $4 \mid n$

💡 $C_k = \{k, k+1, k+2, k+3\}$

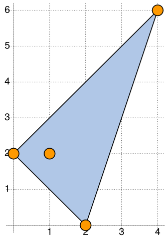
Generalized Rosenbrock function:

$$f_{\mathsf{gR}} = 1 + \sum_{i=1}^{n-1} \Big(100(x_{i+1} - x_i^2)^2 + (1 - x_{i+1})^2 \Big)$$

💡 $C_k = \{k, k+1\}$

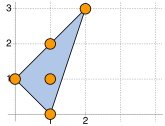
Exploiting sparsity via Newton polytope

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$



Newton polytope $\mathcal{P} = \mathrm{conv}\,(\text{spt}(f))$

Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



$$f = \begin{pmatrix} x & y & xy & xy^2 & x^2y^3 \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} x \\ y \\ xy \\ xy^2 \\ x^2y^3 \end{pmatrix}$$

Exploiting term sparsity [Wang-Magron-Lasserre 19]

$$f = x^2 - 2xy + 3y^2 - 2x^2y + 2x^2y^2 - 2yz + 6z^2 + 18y^2z - 54yz^2 + 142y^2z^2$$

Newton polytope $\to f = \begin{pmatrix} 1 & x & y & z & xy & yz \end{pmatrix} \underbrace{Q}_{\succcurlyeq 0} \begin{pmatrix} 1 \\ x \\ y \\ z \\ xy \\ yz \end{pmatrix}$

$\rightsquigarrow \frac{6 \times 7}{2} = 28$ "unknown" entries in Q

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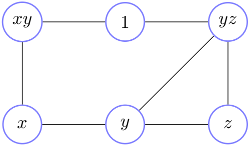
💡 **Term** sparsity pattern graph G



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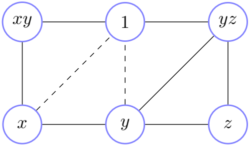
💡 **Term** sparsity pattern graph G

Chordal extension



Replace Q by Q_G with nonzero entries at edges of G

$\rightsquigarrow$ 6 + 9 = 15 "unknown" entries in Q_G

Lyapunov functions from networked systems

$$f = \sum_{i=1}^{N} a_i (x_i^2 + x_i^4) - \sum_{i,k=1}^{N} b_{ik} x_i^2 x_k^2 \quad a_i \in [1, 2] \quad b_{ik} \in [\frac{0.5}{N}, \frac{1.5}{N}]$$

$\rightsquigarrow \binom{N+2}{2}(\binom{N+2}{2}+1)/2$ "unknown" entries in $Q = 231$ for $N = 5$

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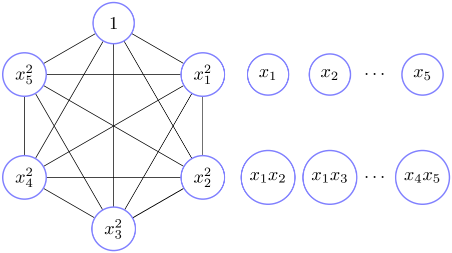
tsp graph G



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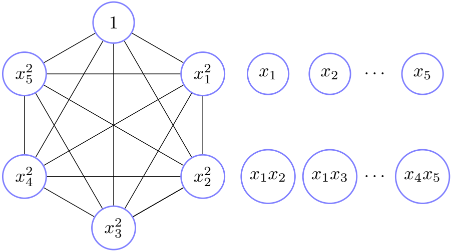

$\rightsquigarrow (N + 1)^2$ “unknown” entries in $Q_G = 36$ for $N = 5$

Proof that $f \geqslant 0$ for $N = 80$ in ~ 10 seconds!

Network of Duffing oscillators

Hamiltonian $V = \sum_{i=1}^{N} a_i (\frac{1}{2} x_i^2 - \frac{1}{4} x_i^4) + \frac{1}{8} \sum_{i,k=1}^{N} b_{ik} (x_i - x_k)^4$

On which domain $V > 0$?

$$V - \sum_{i=1}^{N} \underbrace{\lambda_i}_{>0} x_i^2 (g - x_i^2) \geqslant 0$$

$\implies$ $V > 0$ when $x_i^2 < g$

Network of Duffing oscillators

Hamiltonian $V = \sum_{i=1}^{N} a_i \left(\frac{1}{2}x_i^2 - \frac{1}{4}x_i^4\right) + \frac{1}{8}\sum_{i,k=1}^{N} b_{ik}(x_i - x_k)^4$

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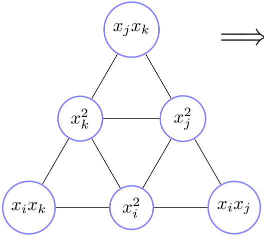

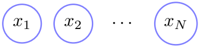

💡 **tsp** graph G

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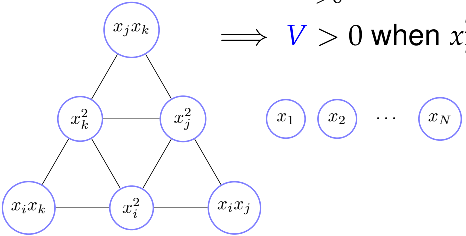
tsp graph G



$\rightsquigarrow \frac{N(N+1)}{2} + 6\binom{N}{2} + N$ "unknown" entries in $Q_G = 80$ for $N = 5$

Proof that $f \geqslant 0$ for $N = 50$ in ~ 1 seconds!

Circuit polynomials

$$f = b_{\boldsymbol{\alpha}_1} \mathbf{x}^{\boldsymbol{\alpha}_1} + \cdots + b_{\boldsymbol{\alpha}_r} \mathbf{x}^{\boldsymbol{\alpha}_r} - b_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$$

$b_{\boldsymbol{\alpha}_j} > 0 \quad \boldsymbol{\alpha}_j \in (2\mathbb{N})^n$

Circuit polynomials

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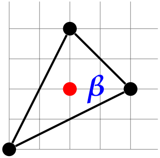
$b_{\boldsymbol{\alpha}_j} > 0 \quad \boldsymbol{\alpha}_j \in (2\mathbb{N})^n$

$f = 1 + x^2 y^4 + x^4 y^2 - 3x^2 y^2$



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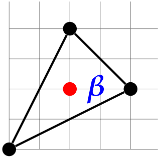

$\beta = \lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r \quad \lambda_j > 0$ and $\lambda_1 + \cdots + \lambda_r = 1$

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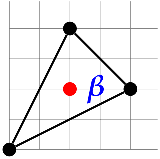

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Theorem [Illiman-de Wolff 16]

$f \geqslant 0 \Leftrightarrow |b_\beta| \leqslant \Theta_f$ or ($b_\beta \geqslant -\Theta_f$, β even)

SUMS OF CIRCUITS computed via Geometric programming

Rational mediated sets and SOCP

Averages of distinct rational points in M

$$A(M) := \{\frac{1}{2}(\mathbf{v} + \mathbf{w}) \mid \mathbf{v} \neq \mathbf{w}, \mathbf{v}, \mathbf{w} \in M\}$$

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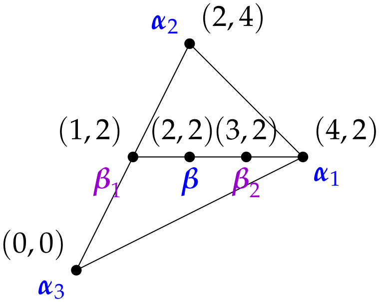

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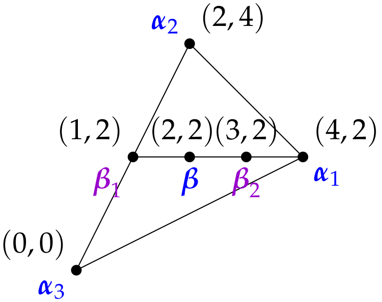

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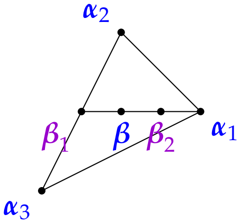
$$\mathbf{u} \in M \backslash \mathcal{A} \implies \mathbf{u} = \mathbf{v} + \mathbf{w}$$

Find a_i, b_i, c_i such that

$$f(\mathbf{x}) = \sum_i 2a_i \mathbf{x}^{\mathbf{v}_i} + b_i \mathbf{x}^{\mathbf{w}_i} - 2c_i \mathbf{x}^{\mathbf{u}_i}$$

(SOCP) $\quad 2a_i b_i \geqslant c_i^{\,2}, \quad a_i, b_i \geqslant 0$

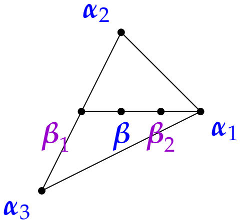
Circuits and sums of binomial squares



$$f = (1 - xy^2)^2 + 2(x^{\frac{1}{2}}y - x^{\frac{3}{2}}y)^2 + (xy - x^2y)^2$$

$\Longrightarrow$ sum of 3 binomial squares

Circuits and sums of binomial squares



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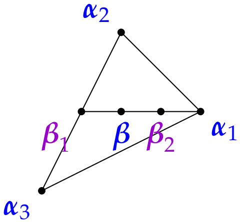
"Arbitrary support" $f = \sum_{\alpha \in \Lambda(f)} c_\alpha \mathbf{x}^\alpha - \sum_{\beta \in \Gamma(f)} d_\beta \mathbf{x}^\beta$

Monomial squares = $\Lambda(f)$ Complement = $\Gamma(f)$

Circuits and sums of binomial squares



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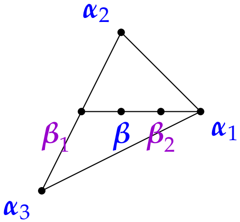
Simplex **Cover** of each $\beta \implies \mathcal{A}$ and mediated set $M_{\mathcal{A}\beta}$

Circuits and sums of binomial squares



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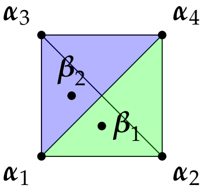
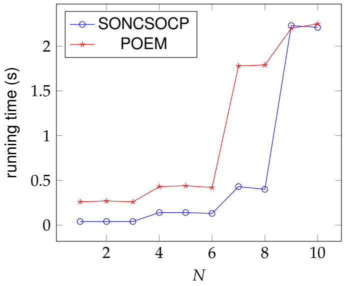
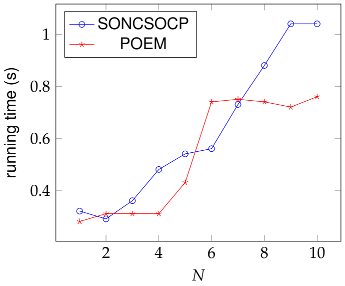
Simplex **Cover** of each $\beta \Longrightarrow \mathcal{A}$ and mediated set $M_{\mathcal{A}\beta}$

$$f = 50x^4y^4 + x^4 + 3y^4 + 800 - 100xy^2 - 100x^2y$$



Comparison with previous method (POEM)

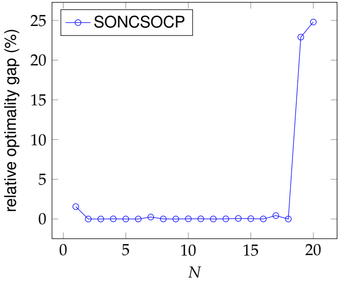




Arbitrary support

$n \sim 40$

$d \sim 60$



Conclusion and Perspectives

1. MODELING tool
   SDP approximation of the **whole reachable set** $\mathbf{X}^{\infty}$

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2. EFFICIENT tool, new applications of TSSOS/circuits?
   - control/stability analysis
   - optimal powerflow
   - quantum physics
   - robustness of deep networks

Thank you for your attention!

`https://homepages.laas.fr/vmagron`

- Magron, Garoche, Henrion & Thirioux. Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems, *SICON* arxiv:1703.05085
- Wang & Magron. A second order cone characterization for sums of nonnegative circuits `SONCSOCP`
- Wang, Magron & Lasserre. TSSOS: a moment-SOS hierarchy that exploits term sparsity. arxiv:1912.08899 `TSSOS`
- Klep, Magron & Povh. Sparse Noncommutative Polynomial Optimization, arxiv:1909.00569 `NCSOStools`
- Magron, Constantinides & Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming, *TOMS*. arxiv:1507.03331 `Real2Float`