Certifying Non-negativity with Lasserre's Hierarchy and Semidefinite Programming

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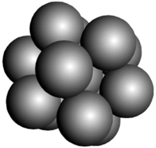
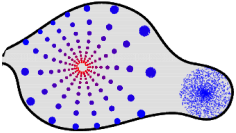


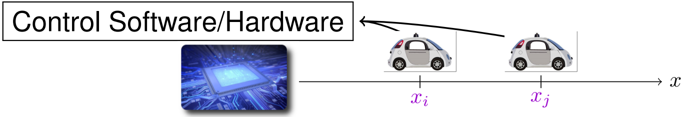


Introduction

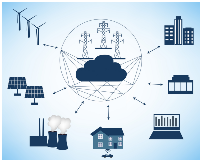
VERIFICATION OF NONLINEAR SYSTEMS . . .

SAFETY of critical parts for **computing** $\oplus$ **physical** devices

Cars

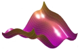


Smart Grids



Space Systems

. . . CAST AS CERTIFIED OPTIMIZATION $\rightsquigarrow$ SOLVE **OFFLINE**

Input: linear  semidefinite  polynomial 

Output: value + numerical/symbolic/formal **certificate**

SDP for Polynomial Optimization

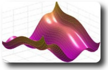
NP-hard NON CONVEX Problem $p^\star = \inf p(\mathbf{x})$

Theory

(Primal)

$$\inf \int p \, d\mu$$

with μ proba $\Rightarrow$



INFINITE LP

(Dual)

$$\sup \lambda$$

$\Leftarrow$ with $p - \lambda \geqslant 0$

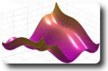
SDP for Polynomial Optimization

NP-hard NON CONVEX Problem $p^\star = \inf p(\mathbf{x})$

Practice

(Primal **Relaxation**)

moments $\int \mathbf{x}^\alpha \, d\mu$



(Dual **Strengthening**)

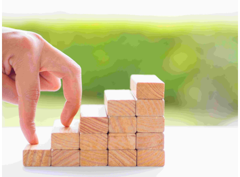
$p - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow p^*$
[Lasserre/Parrilo 01]

degree d
n vars $\Longrightarrow \binom{n+d}{n}$ **SDP** VARIABLES

Numeric Solvers $\Longrightarrow$ **Approx** Certificate

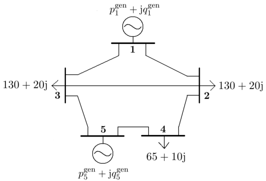


Success Stories: Lasserre's Hierarchy

MODELING POWER: **Cast** as ∞-*dimensional* LP over measures

STATIC **Polynomial Optimization**

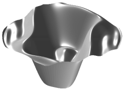
Optimal Powerflow $n \simeq \mathbf{10^3}$ [Josz et al 16]



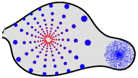
Roundoff Error $n \simeq \mathbf{10^2}$ [Magron et al 17]

DYNAMICAL **Polynomial Optimization**

Regions of attraction [Henrion et al 14]



Reachable sets [Magron et al 17]

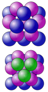


APPROXIMATE OPTIMIZATION BOUNDS!

Success Stories: Certified Optimization

Kepler's Conjecture(1611)

The max density of sphere packings is $\pi/\sqrt{18}$



Flyspeck : **F**ormalizing the **p**roof of **K**epler by T.Hales (1994)
Verification of thousands of "tight" nonlinear inequalities

Seminal Paper:

Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017 $\sim$ **120 citations**

MY CONTRIBUTION:

(Non)-Polynomial optimization to verify **Flyspeck** inequalities

Certification Challenges

Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$
$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$?

Certification Challenges

Scalability

[Joswig et al 16] **20000+** terms, $d = 39$, $n = 6$

$$2d_1^2d_2^2k_1^{12}k_5^4x_1^6x_9^{13} + 46d_1^2d_2^2k_1^{12}k_5^4x_1^5x_9^{14} + 2d_1^2d_2^2k_1^{11}k_5^5x_1^5x_9^{14} + 46d_1^2d_2^2k_1^{11}k_5^5x_1^4x_9^{15} + 11d_1d_2k_1^{13}k_5^4x_1^6x_9^{14} +$$
$$297d_1d_2k_1^{13}k_5^4x_1^5x_9^{15} + 11d_1d_2k_1^{12}k_5^5x_1^5x_9^{15} + 297d_1d_2k_1^{12}k_5^5x_1^4x_9^{16} + 242k_1^{14}k_5^4x_1^5x_9^{16} + 242k_1^{13}k_5^5x_1^4x_9^{17} +$$
$$2d_1^3d_2^3k_1^{11}k_5^4x_1^6x_9^{11} + 46d_1^3d_2^3k_1^{11}k_5^4x_1^5x_9^{12} + 2d_1^3d_2^3k_1^{10}k_5^5x_1^5x_9^{12} + 46d_1^3d_2^3k_1^{10}k_5^5x_1^4x_9^{13} + 6d_1^3d_2^2k_1^{11}k_5^4x_1^5x_9^{13} +$$
$$138d_1^3d_2^2k_1^{11}k_5^4x_1^4x_9^{14} + 4d_1^3d_2^2k_1^{10}k_5^5x_1^4x_9^{14} + 92d_1^3d_2^2k_1^{10}k_5^5x_1^3x_9^{15} + 8d_1^2d_2^3k_1^{11}k_5^4x_1^6x_9^{12} + 184d_1^2d_2^3k_1^{11}k_5^4x_1^5x_9^{13} +$$
$$6d_1^2d_2^3k_1^{10}k_5^5x_1^5x_9^{13} + 138d_1^2d_2^3k_1^{10}k_5^5x_1^4x_9^{14} + 2d_1^2d_2^2k_1^{12}k_5^4x_1^7x_9^{11} + 73d_1^2d_2^2k_1^{12}k_5^4x_1^6x_9^{12} + 617d_1^2d_2^2k_1^{12}k_5^4x_1^5x_9^{13} +$$
$$2d_1^2d_2^2k_1^{12}k_5^3x_1^6x_9^{13} + 46d_1^2d_2^2k_1^{12}k_5^3x_1^5x_9^{14} + 2d_1^2d_2^2k_1^{11}k_5^5x_1^6x_9^{12} + 73d_1^2d_2^2k_1^{11}k_5^5x_1^5x_9^{13} + 617d_1^2d_2^2k_1^{11}k_5^5x_1^4x_9^{14} +$$
$$4d_1^2d_2^2k_1^{11}k_5^4x_1^5x_9^{14} + 92d_1^2d_2^2k_1^{11}k_5^4x_1^4x_9^{15} + 2d_1^2d_2^2k_1^{10}k_5^5x_1^4x_9^{15} + 46d_1^2d_2^2k_1^{10}k_5^5x_1^3x_9^{16} + 45d_1^2d_2k_1^{12}k_5^4x_1^5x_9^{14} +$$
$$1215d_1^2d_2k_1^{12}k_5^4x_1^4x_9^{15} + 34d_1^2d_2k_1^{11}k_5^5x_1^4x_9^{15} + 918d_1^2d_2k_1^{11}k_5^5x_1^3x_9^{16} + d_1d_2^2k_1^{12}k_5^4x_1^7x_9^{12} + 91d_1d_2^2k_1^{12}k_5^4x_1^6x_9^{13} +$$
$$1760d_1d_2^2k_1^{12}k_5^4x_1^5x_9^{14} + d_1d_2^2k_1^{11}k_5^5x_1^6x_9^{13} + 80d_1d_2^2k_1^{11}k_5^5x_1^5x_9^{14} + 1463d_1d_2^2k_1^{11}k_5^5x_1^4x_9^{15} + 12d_1d_2k_1^{13}k_5^4x_1^7x_9^{12} +$$
$$467d_1d_2k_1^{13}k_5^4x_1^6x_9^{13} + 3575d_1d_2k_1^{13}k_5^4x_1^5x_9^{14} + 11d_1d_2k_1^{13}k_5^3x_1^6x_9^{14} + 297d_1d_2k_1^{13}k_5^3x_1^5x_9^{15} + 12d_1d_2k_1^{12}k_5^5x_1^6x_9^{13} +$$
$$467d_1d_2k_1^{12}k_5^5x_1^5x_9^{14} + 3575d_1d_2k_1^{12}k_5^5x_1^4x_9^{15} + 22d_1d_2k_1^{12}k_5^4x_1^5x_9^{15} + 594d_1d_2k_1^{12}k_5^4x_1^4x_9^{16} + 11d_1d_2k_1^{11}k_5^5x_1^4x_9^{16} +$$
$$297d_1d_2k_1^{11}k_5^5x_1^3x_9^{17} + 1254d_1k_1^{13}k_5^4x_1^4x_9^{16} + 1012d_1k_1^{12}k_5^5x_1^3x_9^{17} + +43d_2k_1^{13}k_5^4x_1^6x_9^{14} + 1834d_2k_1^{13}k_5^4x_1^5x_9^{15} +$$
$$43d_2k_1^{12}k_5^5x_1^5x_9^{15} + 1592d_2k_1^{12}k_5^5x_1^4x_9^{16} + 286k_1^{14}k_5^4x_1^6x_9^{14} + 2904k_1^{14}k_5^4x_1^5x_9^{15} + 242k_1^{14}k_5^3x_1^5x_9^{16} + 286k_1^{13}k_5^5x_1^5x_9^{15} + \dots$$

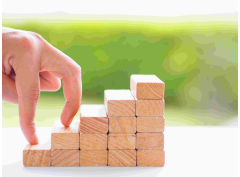
Certification Challenges

"In theory, theory and practice are the same. In practice, they are different." - *A. Einstein*

CONVERGENCE RATE

THEORY $\frac{1}{\sqrt[c]{\log \frac{\text{STAIRS}}{c}}}$

[Nie-Schweighofer 07]

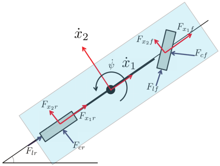


$\uparrow$ PRACTICE ?

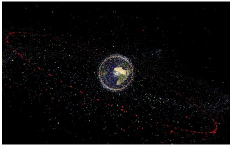
Scientific challenge: bridge THEORY & PRACTICE gap

Modeling Challenges Cyber-Physical

Control systems



Vehicules



Collisions

Fluid mechanics

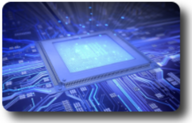
Partial Differential Equations

Mixing Discrete/Continuous Equations

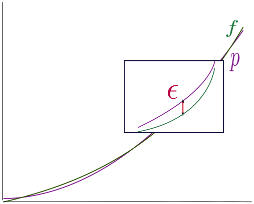
Discrete	$\mathbf{x}_{t+1} = f(\mathbf{x}_t)$	$\Longrightarrow \mu_T = \mu_0 + f_{\#}\, \mu$	Liouville Transport Equation
Continuous	$\dot{\mathbf{x}} = f(\mathbf{x})$	$\Longrightarrow \mu_T = \mu_0 + \operatorname{div} f\, \mu$	

Modeling Challenges Cyber-Physical

FINITE-PRECISION SOFTWARE/HARDWARE



Tuned Precision FPGAs



Approx Math Functions

$$a + (b + c) \neq (a + b) + c$$

Optimize Programs

PERFORMANCE

VS

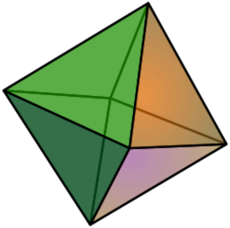
ACCURACY

MIXED PRECISION $\bigoplus$ Scalability $\bigoplus$ Loops

What is Semidefinite Optimization?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d} \,.$$



- Linear cost $\mathbf{c}$
- Linear inequalities "$\sum_i A_{ij} z_j \geqslant d_i$"

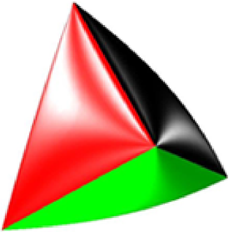
Polyhedron

What is Semidefinite Optimization?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \; .$$

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

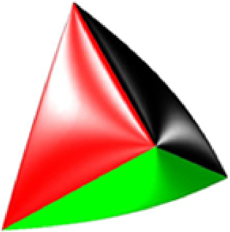


Spectrahedron

What is Semidefinite Optimization?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \,, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \,.$$



Spectrahedron

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)

Applications of SDP

- Combinatorial optimization
- Control theory
- Matrix completion
- Unique Games Conjecture (Khot '02) :
  "A *single concrete algorithm* provides **optimal guarantees** among all efficient algorithms for a large class of computational problems."
  (Barak and Steurer survey at ICM'14)
- Solving polynomial optimization (Lasserre '01)

Lasserre's Hierarchy

- Prove **polynomial inequalities** with SDP:

$$f(a,b) := a^2 - 2ab + b^2 \geqslant 0 \; .$$

- Find $\mathbf{z}$ s.t. $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} \; .$

- Find $\mathbf{z}$ s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Lasserre's Hierarchy

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve:

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \ . \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2 \, .$

- Solving **SDP** $\implies$ Finding **Sums of Squares** certificates

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

- Semialgebraic set
  $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$

Lasserre's Hierarchy

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- $:= [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geqslant 0, \quad x_2(1 - x_2) \geqslant 0\}$

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

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$$\overbrace{x_1 x_2}^{f} + \frac{1}{8} = \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

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$$\overbrace{x_1 x_2}^{f} + \frac{1}{8} = \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

- Sums of squares (SOS) σ_i

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

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  $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$

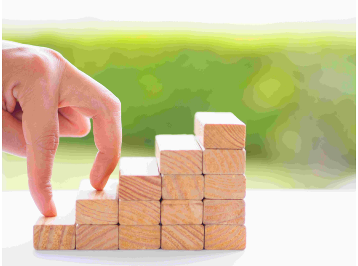
- $:= [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geqslant 0, \quad x_2(1 - x_2) \geqslant 0\}$

$$\overbrace{x_1 x_2}^{f} + \frac{1}{8} = \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

- Sums of squares (SOS) σ_i

- Bounded degree:
  $\mathcal{Q}_d(\mathbf{K}) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2d \right\}$

Lasserre's Hierarchy



- **Hierarchy of SDP relaxations**:
  $\lambda_d := \sup_{\lambda} \Big\{ \lambda : f - \lambda \in \mathcal{Q}_d(\mathbf{K}) \Big\}$

- Convergence guarantees $\lambda_d \uparrow f^*$ [Lasserre 01]

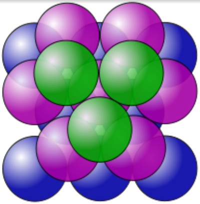
- Can be computed with SDP solvers (CSDP, SDPA)

- **"No Free Lunch" Rule**: $\binom{n+2d}{n}$ SDP variables

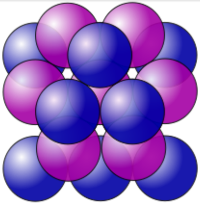
Oranges Stack

Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is $\frac{\pi}{\sqrt{18}}$



Face-centered cubic Packing



Hexagonal Compact Packing

A “Simple” Example

In the computational part:

- Multivariate Polynomials:
  $\Delta\mathbf{x} := x_1 x_4(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2 x_5(x_1 - x_2 + x_3 + x_4 - x_5 + x_6) + x_3 x_6(x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2(x_3 x_4 + x_1 x_6) - x_5(x_1 x_3 + x_4 x_6)$

A "Simple" Example

In the computational part:

- Semialgebraic functions: composition of polynomials with $|\cdot|, \sqrt{}, +, -, \times, /, \sup, \inf, \ldots$

$$p(\mathbf{x}) := \partial_4 \Delta \mathbf{x} \qquad q(\mathbf{x}) := 4x_1 \Delta \mathbf{x}$$
$$r(\mathbf{x}) := p(\mathbf{x}) / \sqrt{q(\mathbf{x})}$$

$$l(\mathbf{x}) := -\frac{\pi}{2} + 1.6294 - 0.2213 \left(\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0\right) + 0.913 \left(\sqrt{x_4} - 2.52\right) + 0.728 \left(\sqrt{x_1} - 2.0\right)$$

A "Simple" Example

In the computational part:

- Transcendental functions $\mathcal{T}$: composition of semialgebraic functions with $\arctan, \exp, \sin, +, -, \times, \dots$

A “Simple” Example

In the computational part:

- Feasible set $\mathbf{K} := [4, 6.3504]^3 \times [6.3504, 8] \times [4, 6.3504]^2$

Lemma$_{9922699028}$ from Flyspeck:

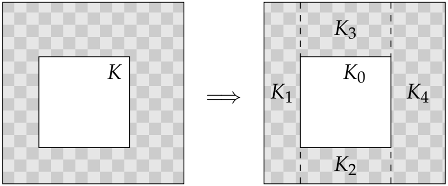
$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{p(\mathbf{x})}{\sqrt{q(\mathbf{x})}}\right) + l(\mathbf{x}) \geqslant 0$$

Existing Certified Frameworks

Lemma$_{9922699028}$ from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{\partial_4 \Delta \mathbf{x}}{\sqrt{4x_1 \Delta \mathbf{x}}}\right) + l(\mathbf{x}) \geqslant 0$$

- Dependency issue using Interval Calculus:
  - One can bound $\partial_4 \Delta \mathbf{x} / \sqrt{4x_1 \Delta \mathbf{x}}$ and $l(\mathbf{x})$ separately
  - Too coarse lower bound: -0.87
  - Subdivide $\mathbf{K}$ to prove the inequality



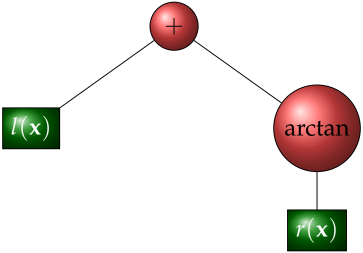
Nonlinear Function Representation

Tree representations of multivariate functions:

- leaves are semialgebraic functions
- nodes are univariate functions or binary operations

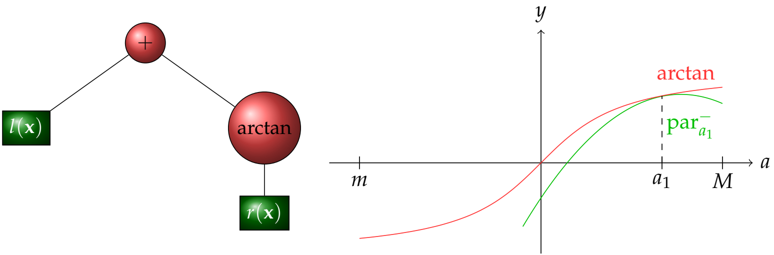
Nonlinear Function Representation

- For the “Simple” Example from Flyspeck:



Maxplus Optimization Algorithm

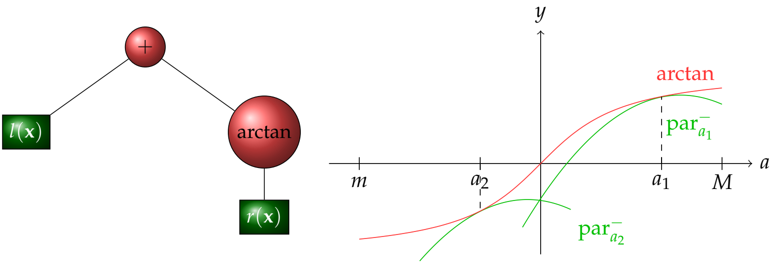
First iteration:



1. control point $\{a_1\}$: $m_1 = -4.7 \times 10^{-3} < 0$

Maxplus Optimization Algorithm

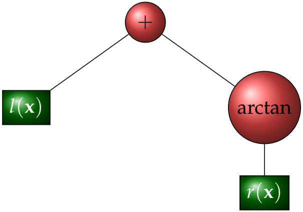
Second iteration:

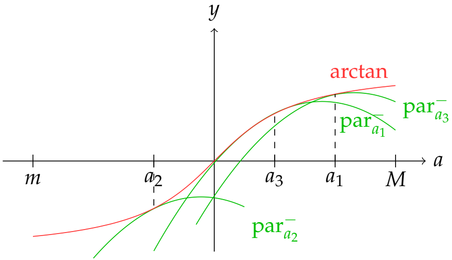


2 control points $\{a_1, a_2\}$: $m_2 = -6.1 \times 10^{-5} < 0$

Maxplus Optimization Algorithm

Third iteration:





- 3 control points $\{a_1, a_2, a_3\}$: $m_3 = 4.1 \times 10^{-6} > 0$

OK!

Contribution: Publications and Software

M., Allamigeon, Gaubert, Werner.
Formal Proofs for Nonlinear Optimization,
Journal of Formalized Reasoning 8(1):1–24, 2015.

Hales, Adams, Bauer, Dang, Harrison, Hoang, Kaliszyk, M., Mclaughlin, Nguyen, Nguyen, Nipkow, Obua, Pleso, Rute, Solovyev, Ta, Tran, Trieu, Urban, Vu & Zumkeller, *Forum of Mathematics, Pi*, 5 2017

Software Implementation `NLCertify`:

15 000 lines of OCAML code

4000 lines of COQ code

M. NLCertify: A Tool for Formal Nonlinear Optimization, *ICMS*, 2014.

Roundoff Error Bounds

- Exact:

$$f(\mathbf{x}) := x_1 x_2 + x_3 x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \mathbf{e}) := [x_1 x_2 (1 + e_1) + x_3 x_4 (1 + e_2)](1 + e_3)$$

- $\mathbf{x} \in \mathbf{X}\,, \quad \mid e_i \mid \leqslant 2^{-p} \quad p =$ 24 (single) or 53 (double)

Roundoff Error Bounds

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \mathbf{e})$
Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \mathbf{e}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \mathbf{e}) = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}}(\mathbf{e}) \mathbf{x}^{\boldsymbol{\alpha}}$

2: Decompose $r(\mathbf{x}, \mathbf{e}) = l(\mathbf{x}, \mathbf{e}) + h(\mathbf{x}, \mathbf{e})$, l **linear** in $\mathbf{e}$

3: Bound $h(\mathbf{x}, \mathbf{e})$ with interval arithmetic

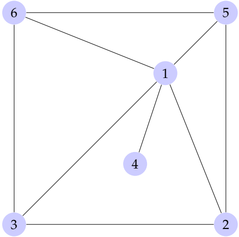
4: Bound $l(\mathbf{x}, \mathbf{e})$ with **SPARSE SUMS OF SQUARES**

Roundoff Error Bounds

Sparse SDP Optimization [Waki, Lasserre 06]

- Correlative sparsity pattern (csp) of vars

$$x_2x_5 + x_3x_6 - x_2x_3 - x_5x_6 + x_1(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6)$$



Roundoff Error Bounds

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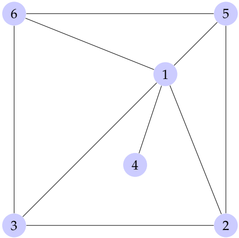


1. Maximal cliques $C_1, \ldots, C_l$
2. Average size $\kappa \rightsquigarrow \binom{\kappa+2k}{\kappa}$ vars

$C_1 := \{1, 4\}$

$C_2 := \{1, 2, 3, 5\}$

$C_3 := \{1, 3, 5, 6\}$

Dense SDP: 210 vars

Sparse SDP: 115 vars

Contributions

$l(\mathbf{x}, \mathbf{e}) = \sum_{i=1}^{m} s_i(\mathbf{x}) e_i$

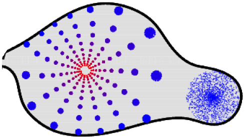
Maximal cliques correspond to $\{\mathbf{x}, e_1\}, \ldots, \{\mathbf{x}, e_m\}$

M., Constantinides, Donaldson. Certified Roundoff Error Bounds Using Semidefinite Programming, *Trans. Math. Soft.*, 2016

Reachable Sets of Polynomial Systems

Iterations $\mathbf{x}_{t+1} = f(\mathbf{x}_t)$
Uncertain $\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u})$

- **Converging** SDP hierarchies
- Image measure
- Liouville equation (conservation)



$$\mu_t + \mu = f_{\#}\,\mu + \mu_0$$

Reachable Sets of Polynomial Systems

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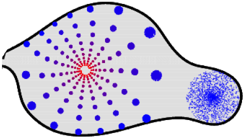


- **Converging** SDP hierarchies
- Image measure
- Liouville equation (conservation)

$$\mu_t + \mu = f_\# \mu + \mu_0$$

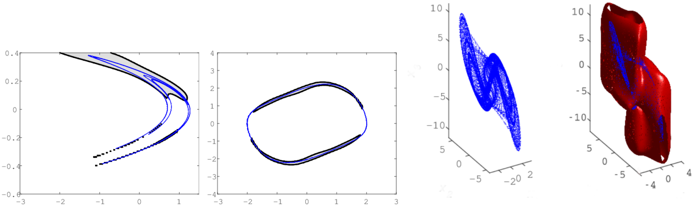
M., Garoche, Henrion, Thirioux. Semidefinite Approximations of Reachable Sets for Discrete-time Polynomial Systems, 2017.

Invariant Measures of Polynomial Systems

Discrete $\mathbf{x}_{t+1} = f(\mathbf{x}_t) \implies f_{\#}\mu - \mu = 0$

Continuous $\dot{\mathbf{x}} = f(\mathbf{x}) \implies \operatorname{div} f \mu = 0$

- **Converging** SDP hierarchies
- measures with density in L_p
- singular measures $\implies$ chaotic attractors



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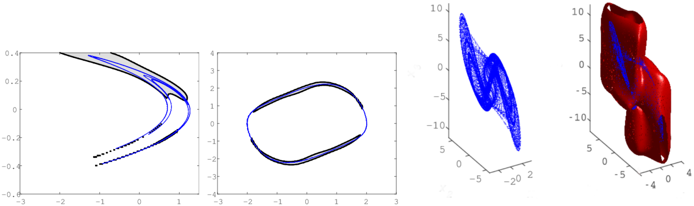

M., Forets, Henrion. Semidefinite Characterization of Invariant Measures for Polynomial Systems. 2018.

SDP for Polynomial Optimization

Success Stories

Challenges

SDP for Nonlinear Optimization

`RealCertify`: Certify Non-negativity

RealCertify: Certify Non-negativity

$X = (X_1, \dots, X_n)$ **co-NP hard problem: check $f \geqslant 0$ on $\mathbf{K}$**

$f \in \mathbb{Q}[X]$

RealCertify: Certify Non-negativity

$X = (X_1, \ldots, X_n)$ **co-NP hard problem: check $f \geqslant 0$ on $\mathbf{K}$**

$f \in \mathbb{Q}[X]$

1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

$$\boxed{n = 1} \quad f = 1 + X + X^2 + X^3 + X^4$$

$$\boxed{n > 1} \quad f = 4X_1^4 + 4X_1^3X_2 - 7X_1^2X_2^2 - 2X_1X_2^3 + 10X_2^4$$

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2. Constrained $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$

   $g_j \in \mathbb{Q}[X]$

   $f = -X_1^2 - 2X_1X_2 - 2X_2^2 + 6 \quad \mathbf{K} = \{1 - X_1^2 \geqslant 0, 1 - X_2^2 \geqslant 0\}$

RealCertify: Certify Non-negativity

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   $f = -X_1^2 - 2X_1X_2 - 2X_2^2 + 6$ $\mathbf{K} = \{1 - X_1^2 \geqslant 0, 1 - X_2^2 \geqslant 0\}$

1. $f \in \Sigma$ = sums of squares (SOS)

   $f = \sigma = h_1{}^2 + \cdots + h_p{}^2 \geqslant 0$

2. Weighted SOS $f = \sigma_0 + \sigma_1\, g_1 + \cdots + \sigma_m\, g_m \geqslant 0$ on $\mathbf{K}$

From Approximate to Exact Solutions

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

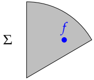
$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$?

From Approximate to Exact Solutions

Win TWO-PLAYER GAME

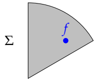


sum of squares of f?

≃ Output!

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



Hybrid Symbolic/Numeric Algorithms

sum of squares of $f + \varepsilon$?

Error Compensation

$\simeq \quad \rightarrow \quad =$

$\simeq$ Output!

Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geqslant 0$ on $\mathbb{R}$ ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = {f_1}^2 + {f_2}^2$.

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Proof.

$f = h^2(q + ir)(q - ir)$ □

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

Rational SOS Decompositions

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 = ???$$

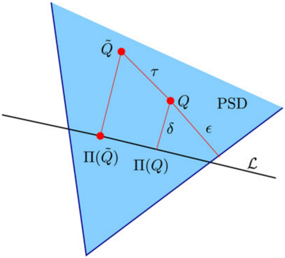
Existing Frameworks

- project & round [Peyrl-Parrilo 08] [Kaltofen-Yang-Zhi 08]
  $f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

Find $\tilde{\mathbf{Q}}$ with semidefinite programming



$f(X) = \mathbf{v}_D{}^T(X) \prod(\mathbf{Q})\, \mathbf{v}_D(X)$

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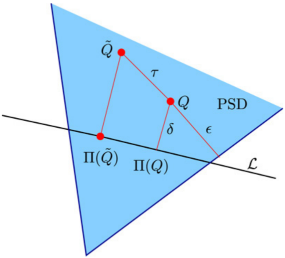


- RAGLib (critical points) [Safey El Din]

- SamplePoints (CAD) [Moreno Maza-Alvandi et al.]

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1\, g_1 + \cdots + \tilde{\sigma}_m\, g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1\, g_1 + \cdots + \tilde{\sigma}_m\, g_m$$

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$

Compact $\mathbf{K} \subseteq [0,1]^n$

$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$

$\boxed{\simeq \quad \to \quad =}$

$\forall \mathbf{x} \in [0,1]^n, u(\mathbf{x}) \leqslant -\varepsilon$

$\min_{\mathbf{K}} f \geqslant \varepsilon$ when $\varepsilon \to 0$

COMPLEXITY?

Modules & Install

`gricad-gitlab:RealCertify`

Depends on Maple &

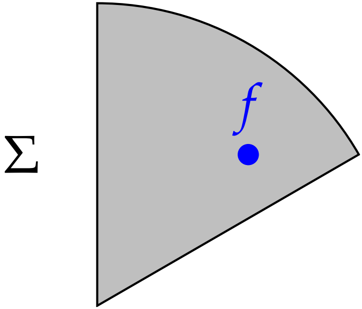
`univsos` $\boxed{n = 1}$

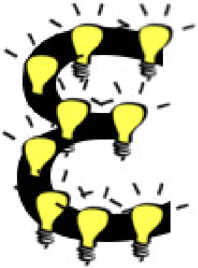
- Square free decomposition with `sqrfree`
- PARI/GP for complex zero isolation

`multivsos` $\boxed{n > 1}$

- arbitrary precision SDP solver SDPA-GMP [Nakata 10]
- Newton Polytope with `convex` package [Franz 99]
- Cholesky's decomposition with `LUDecomposition`

intsos with $n \geqslant 1$: Perturbation





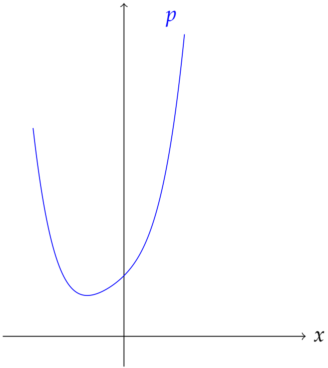
PERTURBATION idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$



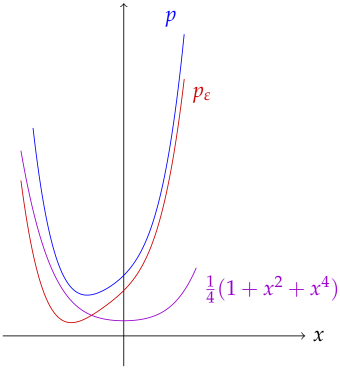
$p = 1 + X + X^2 + X^3 + X^4$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$

💡 **Perturb**: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, deg $p = d = 2k$, $p > 0$

PERTURB: find $\varepsilon \in \mathbb{Q}$ s.t.

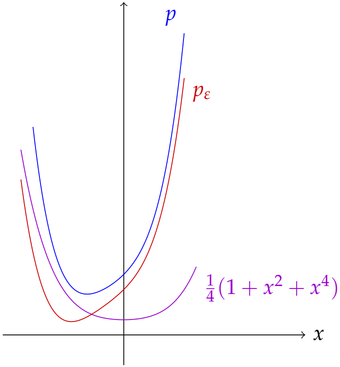
$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$

SDP Approximation:

$$p - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

ABSORB: small enough u_i

$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS



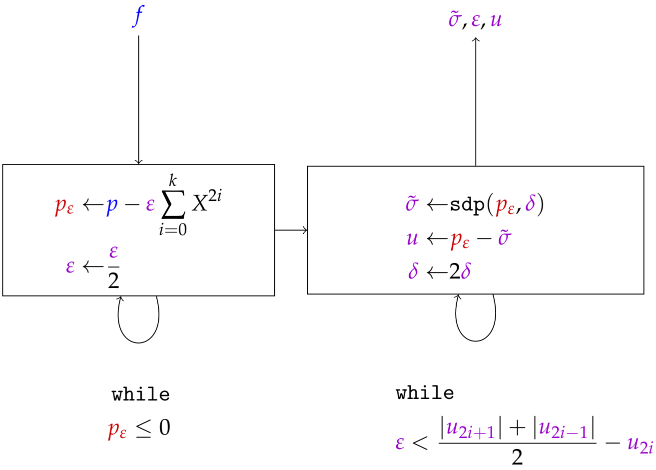
$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ and SDP Approximation

- **Input**: $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



intsos with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

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$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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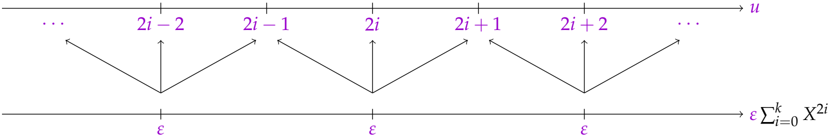

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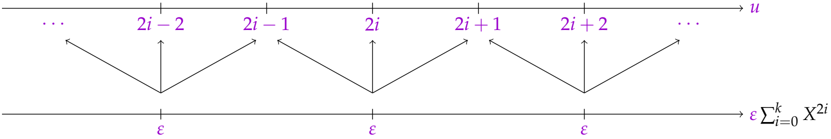
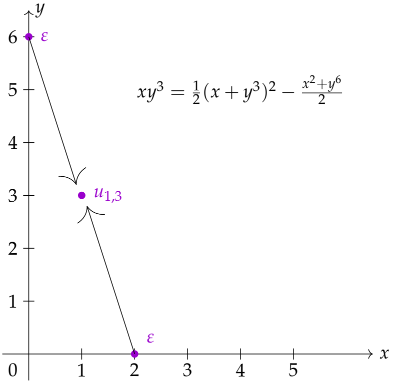


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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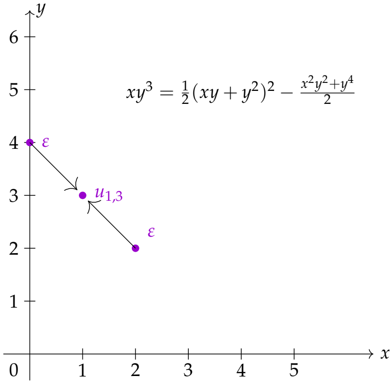

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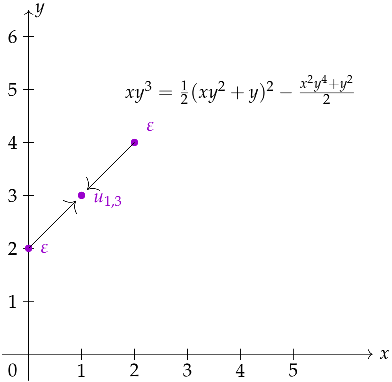

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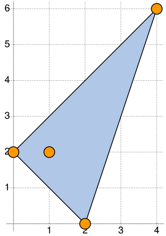
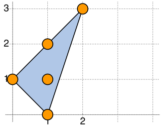
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$



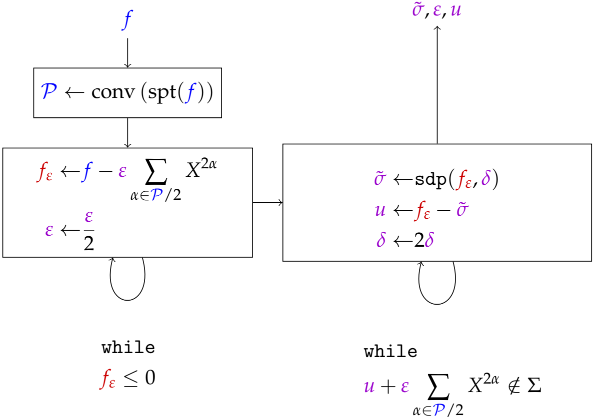
Newton Polytope $\mathcal{P} = \text{conv}\,(\text{spt}(f))$

Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm `intsos`

- **Input**: $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



Algorithm `Putinarsos`

Assumption: $\exists i$ s.t. $g_i = 1 - \|X\|_2^2$

$f > 0$ on $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$ has **Putinar**'s representation:

$$f = \sigma_0 + \sum_j \sigma_j \, g_j \quad \text{with } \sigma_j \in \Sigma[X]\,, \deg \sigma_j \leqslant 2D$$

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ABSORBTION as in Algorithm `intsos`: $u = f_\varepsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j \, g_j$

Unconstrained Benchmarks

Id	n	d	multivsos		RoundProject		RAGLib	CAD
			τ_1 (bits)	t_1 (s)	τ_2 (bits)	t_2 (s)	t_3 (s)	t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	598.	–
f_{10}	10	4	344 347	2.45	8 374 082	4.59	–	–

Constrained Benchmarks

Id	n	d	multivsos D	τ_1 (bits)	t_1 (s)	RAGLib t_2 (s)	CAD t_3 (s)
f_{260}	6	3	2	114 642	2.72	4.19	–
f_{491}	6	3	2	108 359	9.65	0.01	0.05
f_{752}	6	2	2	10 204	0.26	0.07	–
f_{859}	6	7	4	6 355 724	303.	0.05	–
f_{863}	4	2	1	5 492	0.14	0.01	0.01
f_{884}	4	4	3	300 784	25.1	113.	–
butcher	6	3	2	247 623	1.32	231.	–
heart	8	4	2	618 847	2.94	24.7	–

Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	**Output Bit Size**
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau\, d^{d^{\mathcal{O}(n)}}$

💡 How to handle degenerate situations?

Conclusion and Perspectives

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Algo	Input	$\mathbf{K}$	Output Bit Size
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau \, d^{d^{\mathcal{O}(n)}}$

- How to handle degenerate situations?
- Better arbitrary-precision SDP solvers
- Extension to other relaxations, sums of hermitian squares

Crucial need for polynomial systems certification
Available PhD/Postdoc Positions

End

Thank you for your attention!

gricad-gitlab:RealCertify

https://homepages.laas.fr/vmagron

Magron, Safey El Din & Schweighofer. Algorithms for Weighted Sums of Squares Decomposition of Non-negative Univariate Polynomials, *JSC*. arxiv:1706.03941

Magron & Safey El Din. On Exact Polya and Putinar's Representations, *ISSAC'18*. arxiv:1802.10339

Magron & Safey El Din. RealCertify: a Maple package for certifying non-negativity, *ISSAC'18*. arxiv:1805.02201