Impact of access rates on the performance of networks

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Introduction

- Previous models, infinite access-rates: DPS, PS, $\alpha$-fair allocation
- Single bottleneck and Linear network
- Fluid and Diffusion scaling. Diffusion scaling allows to explicitly compute the covariance matrix of the steady-state number of flows
Single bottleneck

- DPS model. M classes. Poisson arrivals with rate $\lambda_i$ and job sizes exponentially distributed with mean $1/\mu_i$
- Each class-i user is access-link limited at $r_i$.
- Total capacity allocated to class-i user:

$$R_i(t) = \frac{1}{N_i(t)} \min\left\{ \frac{g_i N_i(t)}{\sum_{j=1}^{M} g_j N_j(t)}, r_i N_i(t) \right\}$$
Single bottleneck (cont.)

- The process

\[
\left( \tilde{N}(t) \right)_{t \geq 0} = (N_1(t), \ldots, N_M(t))_{t \geq 0}
\]

Is Markovian with transition rates

\[
\begin{align*}
\begin{cases}
\left( \tilde{N}(t) \right) \rightarrow \left( \tilde{N}(t) \right) + e_i : \lambda_i \\
\left( \tilde{N}(t) \right) \rightarrow \left( \tilde{N}(t) \right) - e_i : \mu_i N_i(t) R_i(t)
\end{cases}
\end{align*}
\]
Fluid limit

- Let \( \left( \overrightarrow{N}^{(L)}(t) \right)_{t \geq 0} \) denote the process where arrival rates are replaced by \( L \lambda_i \), and service rate by \( L \).
- Normalized process \( \left( L^{-1} \overrightarrow{N}^{(L)}(t) \right)_{t \geq 0} \) converges to a deterministic limit

\[
n'_i(t) = \lambda_i - \phi_i(\overrightarrow{n}(t))
\]

where \( \phi_i(\overrightarrow{n}) = \mu_i \min \left\{ \frac{g_i n_i(t)}{\sum_{j=1}^{M} g_j n_j(t)}, r_i n_i(t) \right\} \)
Equilibrium point

• Let $\gamma_i = g_i / r_i$. Relabel the classes such that $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_M$.

• Proposition: There exists a unique $s = \{1, \ldots, M\}$ such that $S = \{1, \ldots, s\}$. The classes belonging to $S$ are access-rate limited. For $i \in S$

$$n^*_i(s) = \frac{\lambda_i}{\mu_i r_i}$$

and for $i \in S^C$

$$n^*_i(s) = \frac{\rho_i}{g_i} \left( \frac{1}{1 - \sum_{j=s+1}^{M} \rho_j} \right) \left( \sum_{j=1}^{s} \frac{g_i \lambda_i}{\mu_i r_i} \right)$$
Numerical example: $r_1 = 0.1, r_2 = 0.8$
Linearized system

• Consider the M-dimensional vector

\[ \vec{m}(t) = \vec{n}(t) - \vec{n}^* \]

• We determine a matrix \( P \equiv (p_{ij})_{i,j=1}^M \) such that

\[ \vec{m}'(t) = -P\vec{m}(t) \]

• Proposition: All eigenvalues of \( P \) are positive
  – Stability of linearized system
Diffusion scaling

• Let us introduce the perturbation process

\[ \vec{Y}(t) = \frac{1}{\sqrt{L}} \left( \mathcal{N}^{(L)}(t) - L\vec{n}^* \right) \]

• It satisfies the stochastic differential equation

\[ d\vec{Y}(t) = -P\vec{Y}(t)dt + d\vec{W}(t) \]

where \( \vec{W}(t) = A\vec{B}(t) \) with \( \vec{B}(t) \) an M-dimensional vector of independent Brownian motions, and A a diagonal matrix with \( A_{ii} = \sqrt{2\lambda_i} \)
Covariance matrix

\[
E \left[ \vec{Y} \vec{Y}^T \right] = \Sigma = \int_0^\infty e^{-Pt} A A^T e^{-P^T t} \, dt
\]

- For \( M > 2 \) cumbersome. For \( M = 2 \)

\[
\Sigma = \begin{pmatrix}
\frac{\lambda_1}{r_1 \mu_1} & -\lambda_1 \frac{p_{21}}{p_{11}(p_{11} + p_{22})} & \lambda_1 \frac{p_{21}}{p_{11}p_{22}(p_{11} + p_{22})} + \lambda_1 \\
-\lambda_1 \frac{p_{21}}{p_{11}(p_{11} + p_{22})} & \frac{\lambda_1}{r_1} & \lambda_1 \frac{p_{21}^2}{p_{11}p_{22}(p_{11} + p_{22})} \\
\lambda_1 \frac{p_{21}}{p_{11}p_{22}(p_{11} + p_{22})} + \lambda_1 & \lambda_1 \frac{p_{21}^2}{p_{11}p_{22}(p_{11} + p_{22})} & \lambda_1
\end{pmatrix}
\]

- As expected positive correlation
Covariance: $C=1, r_2=0.8$
Linear network

• M nodes of capacity C and M+1 classes
• Poisson arrival processes with rates $\lambda_i$
• Exponentially distributed job sizes with mean $1/\mu_i$

• Assume stability condition $\rho_i + \rho_0 < 1$, $\forall i$, with $\rho_i = \frac{\lambda_i}{\mu_i C}$
Linear network with access-rate limitation

- $\alpha$-fair allocation
  $$\max_{R_i, \forall i} \{ \sum_{k=1}^{M} g_i N_i \frac{R_i^{1-\alpha}}{1-\alpha} \}$$

  subject to
  $$N_0 R_0 + N_i R_i \leq C$$

- Approximately the allocations are
  $$R_i(t) = \frac{1}{N_i(t)} \min\left\{ \frac{S_\alpha(N(t)) C}{g_0 N_0(t) + S_\alpha(N(t))}, r_0 N_i(t) \right\}$$
  $$R_0(t) = \frac{1}{N_0(t)} \min\left\{ \frac{g_0 N_0(t) C}{g_0 N_0(t) + S_\alpha(N(t))}, r_0 N_0(t) \right\}$$

- Give rise to an ergodic continuous time Markov chain
Fluid limit

- Equilibrium point is the solution of

$$\rho_0 = \min \left\{ \frac{g_0 n_0}{g_0 n_0 + S_\alpha(n)}, r_0 n_0 \right\}$$

$$\rho_i = \min \left\{ \frac{S_\alpha(n)}{g_0 n_0 + S_\alpha(n)}, \frac{r_i n_i}{C} \right\}$$

- Assume $\rho_i \neq \rho_j$, for all i,j. Either a crossing class ($S^C = \{i^*\}$) or the common class ($S^C = \{0\}$) is binding.
  - If $S^C = \{i^*\}$, then $i^* = \text{argmax} \{\rho_i\}$
Equilibrium point

• Proposition: Assume $\rho_i \neq \rho_j$, for all i,j.
  – Crossing class is binding: $S^C = \{i^*\}$. For all $i \in S$
    \[ n_i^* = \frac{\lambda_i}{\mu_i r_i} \]
    and
    \[ n_i^* = \left( \frac{\rho_i^*}{(1 - \rho_i^*) g_i^*} \left( \frac{g_0 \lambda_0}{r_0 \mu_0} \right)^\alpha - 1 \right) \sum_{i=1, i \neq i^*}^M \frac{g_i \lambda_i}{r_i \mu_i} \right)^{1/\alpha} \]
  – Common class is binding: $S^C = \{0\}$. Then for all $i=1,\ldots,M$
    \[ n_i^*(s) = \frac{\lambda_i}{\mu_i r_i} \]
    and
    \[ n_i^* = \frac{\rho_0}{(1 - \rho_0) g_0} \left( \sum_{i=1}^M g_i \left( \frac{\lambda_i}{r_i \mu_i} \right)^\alpha \right)^{1/\alpha} \]
Diffusion scaling

- Construct Linearized system (matrix P)
  - P positive eigenvalues
- Diffusion scaling
  \[
  \tilde{Y}(t) = \frac{1}{\sqrt{L}} \left( \tilde{N}(t)^{(L)} - L\tilde{n}^* \right)
  \]
- Steady state covariance matrix
  \[
  E\left[ \tilde{Y}\tilde{Y}^T \right] = \Sigma = \int_0^\infty e^{-Pt} AA^T e^{-P^Tt} \, dt
  \]
Steady-state covariance

• Proposition:
  – Crossing class is binding: $S^C = \{i^*\}$.
    For $i \neq 0$, $i^*$:
    $$\sum_{i^*i} = -\lambda_i \frac{p_{i^*i}}{p_{ii}} \frac{1}{p_{ii} + p_{i^*i^*}} < 0$$

    and
    $$\sum_{i^*0} = -\lambda_0 \frac{p_{i^*0}}{p_{00}} \frac{1}{p_{00} + p_{i^*i^*}} > 0$$

  – Common class is binding
    • All covariances are positive
Common class binding: $r_0 = 1, r_2 = 0.05$
NS simulations: cross class binding (only class 1 access rate limited)
Conclusions and future work

• Impact of access link: Analytical expressions for the steady-state covariance

• Other networks: Trees, star topology

• Load sharing in order to reduce the steady-state covariance