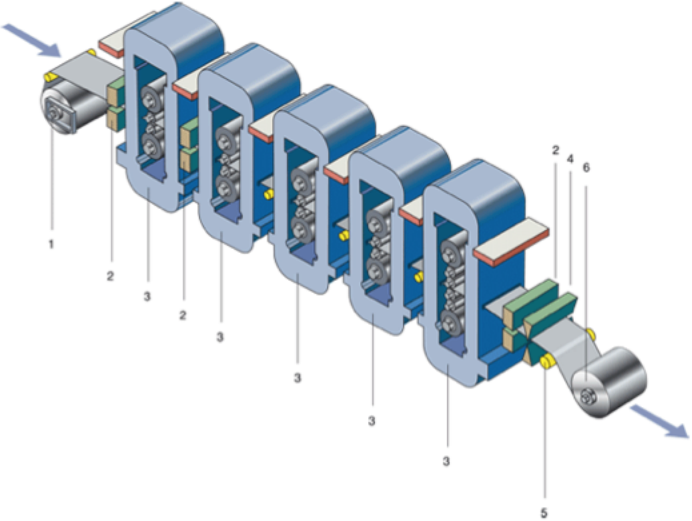
Stability analysis of a general class of hybrid singularly perturbed systems

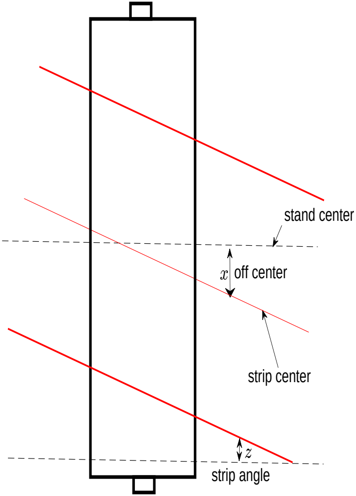
J. Ben Rejeb, I.-C. Morărescu, A. Girard, J. Daafouz

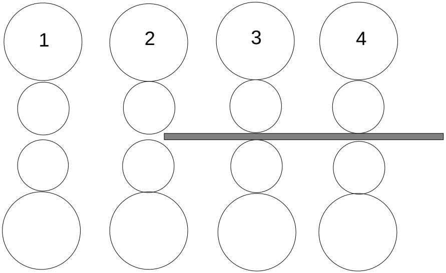
CRAN UMR 7039 CNRS - Université de Lorraine
constantin.morarescu@univ-lorraine.fr

Rolling in finishing mill



Tail end phase





Low dimensional system behavior

Consider the following dynamics :

$$\begin{cases} \dot{u}(t) = a_1 u(t) + b_1 v(t) \\ \varepsilon \dot{v}(t) = c_1 u(t) + d_1 v(t) \end{cases} \quad t \in [t_{2k}, t_{2k+1}), k \in \mathbb{N}$$

and

$$\begin{cases} \varepsilon \dot{u}(t) = a_2 u(t) + b_2 v(t) \\ \dot{v}(t) = c_2 u(t) + d_2 v(t) \end{cases} \quad t \in [t_{2k+1}, t_{2k+2}), k \in \mathbb{N}$$

with slow manifolds

$$c_1 u(t) + d_1 v(t) = 0 \text{ and } a_2 u(t) + b_2 v(t) = 0.$$

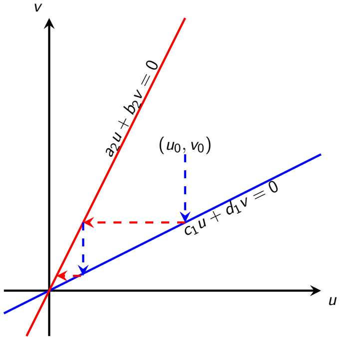


FIGURE – In blue the slow manifold associated with first mode and in red the slow manifold associated with the second mode. The dashed lines represent the asymptotic behavior of the overall system with initial state (u_0, v_0) when $\varepsilon \to 0$ and no dwell-time (or $\mathcal{O}(\varepsilon)$ dwell-time) is imposed.

Objective

Analyze and better understand the behavior of the general class of singularly perturbed switched impulsive systems with switch dependent nature of the state variable.

- rewrite the general class under study as a traditional system with switch independent nature of the state variable
- appropriate methodology for stability analysis in this framework
- characterization of the minimal dwell-time between two events that ensures the stability

Dynamics under consideration

$$\mathbf{D}^{\sigma_k}\dot{\mathbf{X}}(t) = \mathbf{A}^{\sigma_k}\mathbf{X}(t),\ \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}$$

with impulsive dynamics :

$$\mathbf{X}(t_k) = \mathbf{J}^{\nu_k}\mathbf{X}(t_k^-), \quad \forall k \geq 1$$

$0 = t_0 < t_1 < \ldots$ are the times of switches and/or impulses

$\sigma_k \in \mathcal{I}$ and $\nu_k \in \mathcal{J}$ with $\mathcal{I}$ and $\mathcal{J}$ finite sets of indices

$\mathbf{D}^i$ diagonal matrices whose diagonal elements belong $\{\varepsilon, 1\}$

Dynamics under consideration

$$\begin{cases} \dot{u}(t) = a_1 u(t) + b_1 v(t) \\ \varepsilon \dot{v}(t) = c_1 u(t) + d_1 v(t) \end{cases} \quad t \in [t_k, t_{k+1}),$$

$\Leftrightarrow$

$$\mathbf{D}\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t), \ \forall t \in [t_k, t_{k+1}),$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \mathbf{X}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

Remarks

$$\begin{cases} \mathbf{D}^{\sigma_k}\dot{\mathbf{X}}(t) = \mathbf{A}^{\sigma_k}\mathbf{X}(t), \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N} \\ \mathbf{X}(t_k) = \mathbf{J}^{\nu_k}\mathbf{X}(t_k^-), \quad \forall k \geq 1 \end{cases}$$

includes

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includes

- singularly perturbed switched impulsive linear systems ($\mathbf{D}^i = \mathbf{D}, \forall i \in \mathcal{I}$)

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- singularly perturbed switched impulsive linear systems ($\mathbf{D}^i = \mathbf{D}, \forall i \in \mathcal{I}$)
- singularly perturbed switched linear systems ($\mathcal{J} = \{1\},\ \mathbf{J}^1 = \mathbf{I}$)

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Variable reordering - Traditional hybrid framework

Introduce the permutation matrix S_i such that

$$S_i \mathbf{D}^i S_i^\top = \begin{pmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x,n_z} \\ \mathbf{0}_{n_z,n_x} & \varepsilon \mathbf{I}_{n_z} \end{pmatrix}, \quad \forall i \in \mathcal{I}$$

and define the change of variable

$$\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = S_{\sigma_k} \mathbf{X}(t), \ \forall t \in [t_k, t_{k+1}).$$

Let also

$$A^{\sigma_k} \triangleq S_{\sigma_k} \mathbf{A}^{\sigma_k} S_{\sigma_k}^\top, \ J^{\nu_k} \triangleq S_{\sigma_k} \mathbf{J}^{\nu_k} S_{\sigma_{k-1}}^\top,$$

leading to

$$\begin{cases} \begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{z}(t) \end{pmatrix} = A^{\sigma_k} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \quad \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N} \\ \begin{pmatrix} x(t_k) \\ z(t_k) \end{pmatrix} = J^{\nu_k} \begin{pmatrix} x(t_k^-) \\ z(t_k^-) \end{pmatrix}, \ k \in \mathbb{N} \end{cases}$$

Classical singular perturbation analysis

Let

$$(S)\begin{cases}\dot{x}(t) = A_{11}x(t) + A_{12}z(t)\\ \varepsilon\dot{z}(t) = A_{21}x(t) + A_{22}z(t)\end{cases}$$

with $(x(0), z(0)) = (x_0, z_0)$ and A_{22} non-singular.
The associated reduced order & boundary layer systems are

$$\begin{cases}\dot{x}_s(t) = A_0 x_s(t)\\ z_s(t) = A_{22}^{-1}A_{21}x_s(t)\end{cases}, \quad \& \quad \dot{z}_f(t) = A_{22}z_f(t)$$

with $x_s(0) = x_0$ and $z_f(0) = z_0 - A_{22}^{-1}A_{21}x_0$
Then, A_0 and A_{22} Hurwitz guarantee that (S) is stable. Moreover $x - x_s = \mathcal{O}(\varepsilon)$ and $z - z_f - z_s = \mathcal{O}(\varepsilon)$.

Change of variable

For $i \in \mathcal{I}$, $j \in \mathcal{J}$, let $A^i = \begin{pmatrix} A^i_{11} & A^i_{12} \\ A^i_{21} & A^i_{22} \end{pmatrix}$, $J^j = \begin{pmatrix} J^j_{11} & J^j_{12} \\ J^j_{21} & J^j_{22} \end{pmatrix}$,

Assumption

A^i_{22} is non-singular for all $i \in \mathcal{I}$.

For all $i \in \mathcal{I}$ we define

$$P_i = \begin{pmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x,n_z} \\ (A^i_{22})^{-1} A^i_{21} & \mathbf{I}_{n_z} \end{pmatrix},\ P_i^{-1} = \begin{pmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x,n_z} \\ -(A^i_{22})^{-1} A^i_{21} & \mathbf{I}_{n_z} \end{pmatrix}.$$

and perform the following time dependent change of variable :

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P_{\sigma_k} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix},\ \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}$$

Continuous dynamics in x, y variables

$$\left(\begin{array}{c} \dot{x}(t) \\ \varepsilon \dot{y}(t) \end{array} \right) = \left(\begin{array}{cc} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \varepsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \varepsilon B_3^{\sigma_k} \end{array} \right) \left(\begin{array}{c} x(t) \\ y(t) \end{array} \right),$$
$$\forall t \in [t_k, t_{k+1}), \; k \in \mathbb{N}$$

where for all $i \in \mathcal{I}$ one has

$$\begin{aligned} A_0^i &= A_{11}^i - A_{12}^i (A_{22}^i)^{-1} A_{21}^i, \\ B_1^i &= A_{12}^i, \\ B_2^i &= (A_{22}^i)^{-1} A_{21}^i A_0^i, \\ B_3^i &= (A_{22}^i)^{-1} A_{21}^i A_{12}^i. \end{aligned}$$

Jump map in x, y variables

$$\begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} = R^{\sigma_{k-1} \xrightarrow{\nu_k} \sigma_k} \begin{pmatrix} x(t_k^-) \\ y(t_k^-) \end{pmatrix}, \quad \forall k \geq 1$$

where for all $i, i' \in \mathcal{I}$, $j \in \mathcal{J}$,

$$R^{i \xrightarrow{j} i'} = P_{i'} J^j P_i^{-1} = \begin{pmatrix} R_{11}^{i \xrightarrow{j} i'} & R_{12}^{i \xrightarrow{j} i'} \\ R_{21}^{i \xrightarrow{j} i'} & R_{22}^{i \xrightarrow{j} i'} \end{pmatrix}$$

Main assumptions

For all $i \in \mathcal{I}$ one defines $A_0^i = A_{11}^i - A_{12}^i (A_{22}^i)^{-1} A_{21}^i$ and impose the following :

Assumption

A_0^i and A_{22}^i are Hurwitz for all $i \in \mathcal{I}$.

Consequently, $\exists Q_s^i \geq \mathbf{I}_{n_x},\ Q_f^i \geq \mathbf{I}_{n_z},\ i \in \mathcal{I}$ and $\lambda_s > 0,\ \lambda_f > 0$ such that :

$$A_0^{i\top} Q_s^i + Q_s^i A_0^i \leq -2\lambda_s Q_s^i$$
$$A_{22}^{i\top} Q_f^i + Q_f^i A_{22}^i \leq -2\lambda_f Q_f^i$$

Stability parameters

Let γ_{11}, γ_{12}, γ_{21}, γ_{22} be defined as :

$$
\begin{aligned}
\gamma_{11} &= \max_{i,i'\in\mathcal{I},j\in\mathcal{J}} \left\|(Q_s^{i'})^{\frac{1}{2}} R_{11}^{i\xrightarrow{j}i'} (Q_s^{i})^{-\frac{1}{2}}\right\|, \\
\gamma_{12} &= \max_{i,i'\in\mathcal{I},j\in\mathcal{J}} \left\|(Q_s^{i'})^{\frac{1}{2}} R_{12}^{i\xrightarrow{j}i'} (Q_f^{i})^{-\frac{1}{2}}\right\|, \\
\gamma_{21} &= \max_{i,i'\in\mathcal{I},j\in\mathcal{J}} \left\|(Q_f^{i'})^{\frac{1}{2}} R_{21}^{i\xrightarrow{j}i'} (Q_s^{i})^{-\frac{1}{2}}\right\|, \\
\gamma_{22} &= \max_{i,i'\in\mathcal{I},j\in\mathcal{J}} \left\|(Q_f^{i'})^{\frac{1}{2}} R_{22}^{i\xrightarrow{j}i'} (Q_f^{i})^{-\frac{1}{2}}\right\|.
\end{aligned}
\tag{1}
$$

Reduced order model

Original system :

$$\begin{cases} \begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{y}(t) \end{pmatrix} = \begin{pmatrix} A_0^{\sigma_k} & B_1^{\sigma_k} \\ \varepsilon B_2^{\sigma_k} & A_{22}^{\sigma_k} + \varepsilon B_3^{\sigma_k} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \\ \begin{pmatrix} x(t_k) \\ y(t_k) \end{pmatrix} = R^{\sigma_{k-1} \overset{\nu_k}{\to} \sigma_k} \begin{pmatrix} x(t_k^-) \\ y(t_k^-) \end{pmatrix} \end{cases}$$

Corresponding reduced order model :

$$\begin{cases} \dot{x}(t) = A_0^{\sigma_k} x(t), \ \forall t \in [t_k, t_{k+1}), \ k \in \mathbb{N} \\ x(t_k) = R_{11}^{\sigma_{k-1} \overset{\nu_k}{\to} \sigma_k} x(t_k^-), \quad \forall k \geq 1. \end{cases}$$

Analysis ideas

Consider the slow and fast Lyapunov functions :

$$\begin{cases} W_s(t) = \sqrt{x(t)^\top Q_s^{\sigma_k} x(t)} \\ W_f(t) = \sqrt{y(t)^\top Q_f^{\sigma_k} y(t)} \end{cases}, \ \forall t \in [t_k, t_{k+1}),\ k \in \mathbb{N}.$$

Introduce

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \quad M_\tau = \begin{pmatrix} e^{-\lambda_s \tau} + \varepsilon\beta_3 & \varepsilon(\beta_2 + \beta_3) \\ \varepsilon\beta_1 & e^{-\frac{\lambda_f}{\varepsilon}\tau} + \varepsilon\beta_1 \end{pmatrix}.$$

Let $\tau_k = t_{k+1} - t_k$. We show that :

$$\begin{pmatrix} W_s(t_{k+1}^-) \\ W_f(t_{k+1}^-) \end{pmatrix} \leq M_{\tau_k} \begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix}, \text{ for } \epsilon \text{ small}$$

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \Gamma \begin{pmatrix} W_s(t_k^-) \\ W_f(t_k^-) \end{pmatrix}.$$

Analysis ideas

Choose $\tau^* \geq 0$ such that the positive matrix ΓM_{τ^*} is Schur.
For ϵ sufficiently small and $(t_k)_{k\geq 0}$ sequence of event times satisfying the dwell-time property $\tau_k \geq \tau^*$, for all $k \in \mathbb{N}$ one has

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \Gamma M_{\tau_{k-1}} \dots \Gamma M_{\tau_0} \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix}$$

leading to

$$\begin{pmatrix} W_s(t_k) \\ W_f(t_k) \end{pmatrix} \leq \left(\Gamma M_{\tau^*}\right)^k \begin{pmatrix} W_s(t_0) \\ W_f(t_0) \end{pmatrix}.$$

Therefore we have to characterize τ^* that renders ΓM_{τ^*} is Schur.

Stability analysis results

Table – Summary of the main results establishing dwell-time conditions for the stability of the original system.

γ_{11}	$\gamma_{12},\ \gamma_{21},\ \gamma_{22}$	dwell-time condition
$\gamma_{11} > 1$	–	$\tau^* > \frac{\ln(\gamma_{11})}{\lambda_s} + \mathcal{O}(\varepsilon)$
$\gamma_{11} = 1$	–	$\tau^* > -\frac{\varepsilon}{\lambda_f}\ln(\varepsilon) + \mathcal{O}(\varepsilon)$
	$\gamma_{12} = 0$	$\tau^* > \mathcal{O}(\varepsilon)$
$\gamma_{11} < 1$	–	
	$\gamma_{22} < 1,\ \frac{\gamma_{12}\gamma_{21}}{(1-\gamma_{11})(1-\gamma_{22})} < 1$	$\tau^* \geq 0$

Stability analysis results

TABLE – Summary of the main results establishing dwell-time conditions for the reduced order system .

γ_{11}	dwell-time condition
$\gamma_{11} > 1$	$\tau^* > \frac{\ln(\gamma_{11})}{\lambda_s}$
$\gamma_{11} \leq 1$	$\tau^* \geq 0$

Illustration on scalar fast and slow dynamics

Consider again the system with the following two modes

$$\begin{cases} \dot{u}(t) = a_1 u(t) + b_1 v(t) \\ \varepsilon \dot{v}(t) = c_1 u(t) + d_1 v(t) \end{cases} \quad t \in [t_{2k}, t_{2k+1}), k \in \mathbb{N}$$

and

$$\begin{cases} \varepsilon \dot{u}(t) = a_2 u(t) + b_2 v(t) \\ \dot{v}(t) = c_2 u(t) + d_2 v(t) \end{cases} \quad t \in [t_{2k+1}, t_{2k+2}), k \in \mathbb{N}$$

Notice that

$$R_{11}^{1\to 2} = -\frac{c_1}{d_1}, \quad R_{11}^{2\to 1} = -\frac{b_2}{a_2}.$$

and $\gamma_{11} < 1$ if and only if

$$\left|\frac{b_2}{a_2}\right| < q < \left|\frac{d_1}{c_1}\right|,$$

where $q = \sqrt{\dfrac{Q_s^2}{Q_s^1}}$.

Illustration on scalar fast and slow dynamics

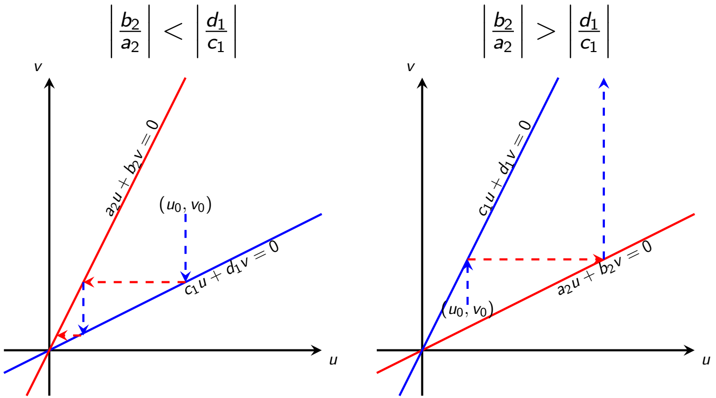


FIGURE – In blue the slow manifold associated with first mode and in red the slow manifold associated with the second mode.

Numerical examples

Consider :

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} -1 & 0.5 \\ -3 & -2 \end{pmatrix}, \quad \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} -2.5 & -4 \\ 1 & 0.5 \end{pmatrix}.$$

The two slow manifolds are :

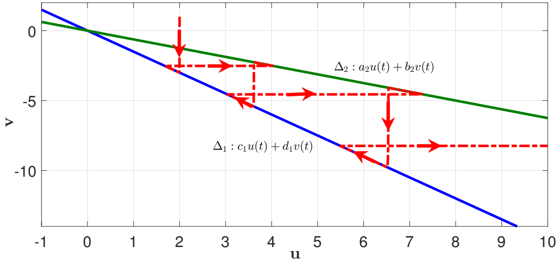
$$\begin{cases} -3u(t) - 2v(t) = 0 \\ -2.5u(t) - 4v(t) = 0. \end{cases}$$

and

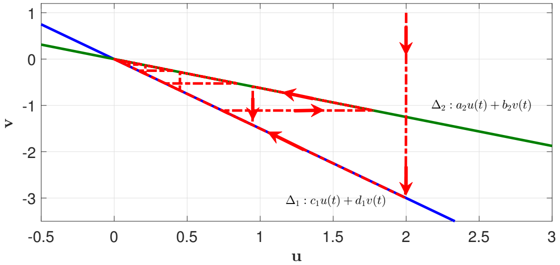
$$R_{11}^{1\to 2} = -1.5, \quad R_{11}^{2\to 1} = -1.6 \Rightarrow \gamma_{11} = 1.6 > 1.$$

We also consider $\mathbf{X}_0 = (2,\ 1)^\top$, $\varepsilon = 10^{-3}$.
The required dwell-time equals 0.57 *sec*.

Numerical examples



$\tau = 0.2$ *sec*



$\tau = 0.57$ *sec*

Conclusions and perspectives

Conclusions :

- Stability analysis for hybrid singularly perturbed systems with switch-dependent nature of the variable
- A characterization of the dwell-time ensuring stability is presented.
- Time-varying dimension of the state can be considered provided that we add some artificial state-variables with stable dynamics.

Perspectives :

- Apply the results to Arcelor-Mital problem.
- Design automatons ensuring stability under pre-defined dwell-time constraints.