Realization Theory of Hybrid Systems and Its Application to Control, System Identification and Model Reduction

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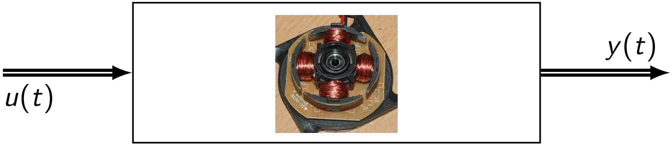
What is realization theory ?

We observe the input-output behavior (black-box)

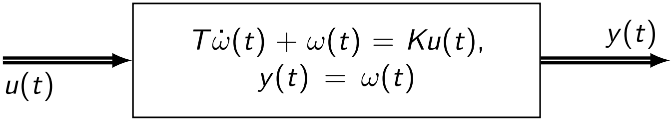
of a physical process



What is realization theory ?

We observe the input-output behavior (black-box)

Which mathematical models (fixed structure)



can describe the observed behavior of the black-box ?

Realization theory: what is it good for ?

- Existence of a solution depends only on

  input-output behavior

  $$u \mapsto y$$

- Input output behavior is what we really know.
- Input-output behavior is the fundamental object of study.

But, for actually computing a controller, we need a

state-space representation.

This raises several problems.

Realization theory: what is it good for ?

- State-space models are artifacts.
- All models are wrong, some are useful...

  Models are like sausages: best not to see how they are made.
- Bad model $\Longrightarrow$ bad controller (garbage in garbage out)

Good state-space model:

1. approximates sufficiently well the input-output behavior,
2. suitable algebraic structure for computing a controller.

Realization theory: what is it good for ?

Non-uniqueness of models and computation of a controller:

- Many state-space models for the same input-output behavior

  (even if we fix the model structure).

- Choice of the model influences computability of controllers ?

  If we fail with one model, do we try another one ? Which ?

- Unique class of good (minimal) models ?

System identification: from experiments to models

- How to find from experiments a good model for control ?
- Design experiments to validate models.

Model reduction: 'As simple as possible but not simpler ..... '
How to compute the simplest but still adequate model ?

Realization theory: what is it good for ?

Systems identification

- realization algorithms yield identification algorithms
- minimality helps to characterize identifiability

Model reduction

- Minimization algorithms yield model reduction algorithms.
- Partial realization = model reduction by moment matching.
- Helps to formulate the notion of distance and error bounds for model reduction algorithms.

Realization theory: what is it good for ? cont.

Control design

- Realization theory helps to derive conditions for observability and controllability. The latter are necessary for existence of observers and controllers.
- The existence of a solution to many control problems depends only on the input-output behavior, not on the state-space representation.
- Observer design, filtering = almost realization theory.

Example: linear systems

$$\Sigma : \begin{cases} \left.\begin{cases} \dot{x}(t) \\ x(t+1) \end{cases}\right\} = Ax(t) + Bu(t),\ x(0) = 0 \\ y(t) = Cx(t) \end{cases}$$

A potential input-output map Y maps input $u(.)$ to output $y(.)$. Since Y is linear in $u(.)$ it is determined by its impulse response

Impulse response $G(t) = Y(\delta_0)(t)$

- δ_0 is the Dirac-delta for continuous-time
- $\delta_0(0) = 1$, $\delta_0(t) = 0, t > 0$ for discrete-time

Σ is a realization of Y, iff

$$G(t) = Ce^{At}B(\text{cont.time})$$
$$G(t) = CA^{t}B(\text{disc.-time})$$

Example: linear systems

Realization problem

For the specified input-output map Y find a (preferably minimal) linear system Σ such that Σ realizes Y.

Markov parameters

$$M_k = \begin{cases} \frac{d^k}{dt^k} G(t)|_{t=0} & \text{continuous time, or} \\ G(k+1) & \text{discrete time} \end{cases}$$

Classical step. Σ is a realization of $Y \iff M_k = CA^k B$

Hankel matrix of Y $\quad H_Y = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots \\ M_1 & M_2 & M_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

Example: linear systems

Theorem (Kalman 1964)

1. *Y has a linear system realization* $\iff \mathrm{rank}\, H_Y < +\infty$.
2. *A linear system Σ is a minimal realization of Y* $\iff$

$$\mathrm{rank}\, \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$$

i.e. Σ is reachable, and

$$\mathrm{rank}\, \begin{bmatrix} C^T & A^T C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix} = n$$

i.e. Σ is observable. All minimal realizations are isomorphic.

3. *Let H_N be the upper $pmN \times Npm$ block of H_Y. If* $\mathrm{rank}\, H_N = \mathrm{rank}\, H_Y$, *then we can compute by a numerical algorithm a minimal realization of Y from H_N.*

Realization theory for hybrid systems

Questions

- What is a ‘good’ notion for dimension and minimality of hybrid systems ?
- Find necessary and sufficient conditions for an input-output map to admit a realization by a hybrid system.
- Find a characterization of minimality and its uniqueness.
- Find minimization algorithms, i.e. algorithms for transforming a hybrid system to a minimal one
- Find realization algorithms, i.e. algorithms for computing a hybrid system representation from input-output data.

Hybrid systems with complete realization theory

- Linear switched systems
- Bilinear switched systems
- Linear hybrid systems without guards (but with state jumps)
- Bilinear hybrid systems without guards (but with state jumps)
- Stochastic Jump-Markov Systems
- Affine LPV systems

Definition of linear switched systems

$$\Sigma \begin{cases} \left.\begin{cases} \dot{x}(t) \\ x(t+1) \end{cases}\right\} = A_{q(t)}x(t) + B_{q(t)}u(t),\ x(0) = 0 \\ y(t) = C_{q(t)}x(t) \end{cases}$$

Inputs

$q(t) \in Q$ – switching sequence, $u(t)$ – continuous input

Outputs

$y(t)$ – continuous output

Dimension – n, the dimension of the state $x(t)$.

Motivation for linear switched systems

- the simplest type of hybrid systems
- occur in practice: quantized systems, chemical processes, high-tech machines.

Impulse response of linear switched systems

- Potential nput-output map Y of a linear switched system
  1. Maps switching $q(.)$ and input $u(.)$ to outputs $y(.)$.
  2. Linear in continuous input $u()$.

- Y is completely described by its impulse response

  Impulse response for switching $q(.)$

  Switching $q()$: stay in discrete mode $q_1, \ldots, q_k$ for times $t_1, \ldots, t_k$.

$$G_{q_1 \ldots q_k}(t_1, \ldots, t_k) = Y(q(.), \delta_0)$$

  - δ_0 is the Dirac-delta for continuous-time
  - $\delta_0(0) = 1$, $\delta_0(t) = 0, t > 0$ for discrete-time

Markov parameters for linear switched systems

Markov parameters, $q_0, q \in Q$ – discrete modes, $j = 1, 2, \ldots, m$

$$S_{q,q_0}(q_1 q_2 \cdots q_k) = \left\{ \begin{array}{l} G_{q_1 \cdots q_k}(1, 1, \ldots, 1) \\ \frac{d}{dt_1} \cdots \frac{d}{dt_k} G_{q_0 q_1 \cdots q_k q}(0, t_1, \ldots, t_k, 0)|_{t_1 = \cdots = t_k = 0} \end{array} \right\}$$

Markov parameters are indexed by sequences of discrete modes Q^*

Σ is a realization of Y $\iff$

$$S_{q,q_0}(q_1 q_2 \cdots q_k) = C_q A_{q_k} \cdots A_{q_1} B_{q_0}$$

Hankel matrix for linear switched systems

$Q = \{1, 2, \ldots, D\}$

$v_1 \prec \ldots \prec v_k, \ldots$ lexicographic ordering of all sequences.

$$M(v) = \begin{bmatrix} S_{1,1}(v) & \ldots & S_{1,D}(v) \\ \vdots & \ldots & \vdots \\ S_{D,1}(v) & \ldots & S_{D,D}(v) \end{bmatrix}$$

Hankel matrix: H_Y

$$H_Y = \begin{bmatrix} M(v_1 v_1) & M(v_2 v_1) & \cdots & M(v_k v_1) & \cdots \\ M(v_1 v_2) & M(v_2 v_2) & \cdots & M(v_k v_2) & \cdots \\ M(v_1 v_3) & M(v_2 v_3) & \cdots & M(v_k v_3) & \cdots \\ \vdots & \vdots & \ldots & \vdots & \ldots \end{bmatrix},$$

Realization theorem for linear switched systems

Theorem

- *Y has a realization* $\iff \mathrm{rank}\ \ H_Y < +\infty$,
- *Σ is a minimal realization of Y* $\iff$ *Σ is reachable, i.e.*

$$n = \dim \mathrm{rank}\ [A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} \mid q_0, q_1, \ldots, q_k \in Q, k < n]$$

and observable

$$n = \dim \mathrm{rank}\ [(C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1})^T \mid q, q_1, \ldots, q_k \in Q, k < n]$$

Minimal realizations are unique up to isomorphisms.
Any realization can be transformed to a minimal one.

There is a minization algorithm.

Example

$$A_{q_1} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} C_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

This system is neither observable nor reachable, hence it is not minimal.

Example: cont

After minimization, we obtain

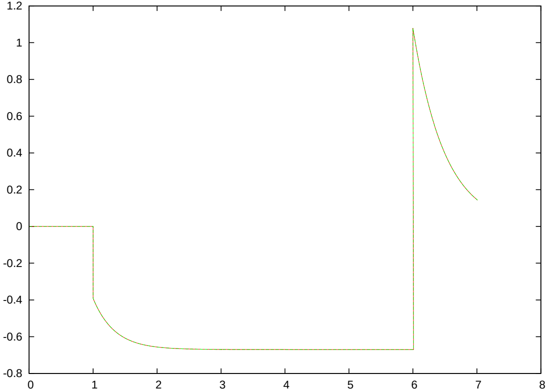
$$A_{q_1} = \begin{bmatrix} -3 & 0 & -0.02 \\ 0 & -3 & 0 \\ 0.98 & 0 & 0.006 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; C_{q_1} = \begin{bmatrix} 0.95 \\ 0 \\ -0.31 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -4 & 0 & -0.02 \\ 0 & -2 & 0 \\ 0.98 & 0 & -0.99 \end{bmatrix} B_{q_2} = \begin{bmatrix} 0.31 \\ 0 \\ 0.95 \end{bmatrix} \; C_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

The system above is minimal, but none of the subsystems is minimal

Example: cont

If we simulate the two systems for white noise input and switching sequence $(q_2, 1)(q_1, 2)(q_1, 3)(q_2, 1)$.



Realization algorithm

$$H_{Y,N+1,N} = \begin{bmatrix} M(v_1 v_1) & \cdots & M(v_M v_1) \\ \vdots & \cdots & \vdots \\ M(v_1 v_N) & \cdots & M(v_N v_N) \\ M(v_1 v_{N+1}) & \cdots & M(v_N v_{N+1}) \end{bmatrix}$$

1: $H_{f,N+1,N} = OR$

2: $B_q = m(q-1)+1, \ldots, mq$th columns of R.

3: $C_q = p(q-1)+1, \ldots, pq$th rows of O.

4: $A_q = \bar{O}^{+} O_q$

- $\bar{O}$ – the block rows of O which are indexed by $v_1, \ldots, v_N$.
- $\bar{O}^{+}$–pseudo-inverse of $\bar{O}$.
- O_q – shifted $\bar{O}$: the row of O_q indexed by sequence v is the row of O indexed by sequence qv.

Partial realization theorem for linear switched systems

$$H_{Y,N,N} = \begin{bmatrix} M(v_1 v_1) & \cdots & M(v_N v_1) \\ \vdots & \dots & \vdots \\ M(v_1 v_N) & \cdots & M(v_N v_N) \end{bmatrix}$$

$$H_{Y,N,N+1} = \begin{bmatrix} M(v_1 v_1) & \cdots & M(v_N v_1) & M(v_{N+1} v_1) \\ \vdots & \dots & \vdots & \vdots \\ M(v_1 v_N) & \cdots & M(v_N v_N) & M(v_{N+1} v_N) \end{bmatrix},$$

Theorem

1. *If* $\mathrm{rank}\ H_{Y,N,N} = \mathrm{rank}\ H_{Y,N,N+1} = \mathrm{rank}\ H_{Y,N+1,N}$ *then the result of the algorithm recreates the Markov-parameters* $M(v_1), \dots M(v_{2N+1})$.
2. *If N is bigger than the dimension of a realization of Y, then the algorithm returns a minimal realization of Y.*

Example

Consider the switched system from the previous example and Y the input-output map of that system.

$$H_{Y,2,1} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & -3 & 0 & -2 & 0 \\ 0 & -1 & 0 & 3 & 0 & 4 \\ -3 & 0 & 9 & 0 & 6 & 0 \\ 0 & -1 & 0 & 4 & 0 & 5 \\ -2 & 0 & 6 & 0 & 4 & 0 \\ 0 & 3 & 0 & -9 & 0 & -12 \\ 9 & 0 & -27 & 0 & -18 & 0 \\ 0 & 4 & 0 & -12 & 0 & -16 \\ 6 & 0 & -18 & 0 & -12 & 0 \\ 0 & 4 & 0 & -12 & 0 & -16 \\ 6 & 0 & -18 & 0 & -12 & 0 \\ 0 & 5 & 0 & -16 & 0 & -21 \\ 4 & 0 & -12 & 0 & -8 & 0 \end{bmatrix}$$

Example: cont.

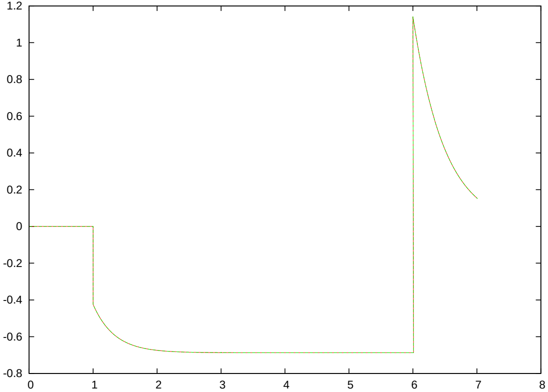
Applying the realization algorithm to $H_{Y,2,1}$ yields.

$$A_{q_1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3.02 & 0.17 \\ 0 & -0.32 & 0.018 \end{bmatrix}, B_{q_1} = \begin{bmatrix} -1.9 \\ 0 \\ 0 \end{bmatrix}, C_{q_1} = \begin{bmatrix} 0 \\ 0.21 \\ 0.46 \end{bmatrix}^T$$

$$A_{q_2} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4.02 & 0.17 \\ 0 & -0.32 & -0.98 \end{bmatrix}, B_{q_2} = \begin{bmatrix} 0 \\ 1.25 \\ -0.57 \end{bmatrix}, C_{q_2} = \begin{bmatrix} -0.53 \\ 0 \\ 0 \end{bmatrix}^T$$

Example: cont

If we simulate the two systems for white noise input and switching sequence $(q_2, 1)(q_1, 2)(q_1, 3)(q_2, 1)$.



Example of minimization

4 discrete modes, total number of continuous states: $3+2+3+3=11$.

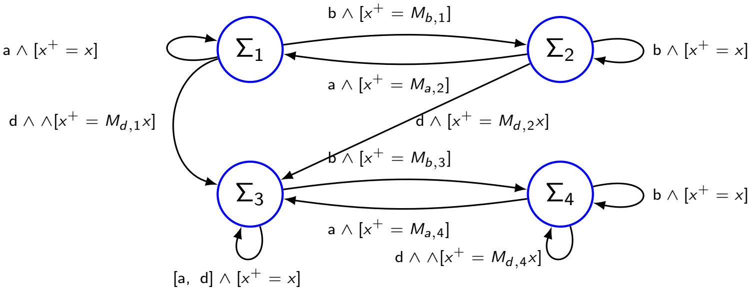
$$\Sigma_1 \begin{cases} \dot{x}_1 &= -x_1 + u_1 + u_2 \\ \dot{x}_2 &= x_1 - 3x_2 \\ \dot{x}_3 &= 2x_1 + 5x_2 - 4x_2 + 2.6u_2 \\ y &= x_1 + x_2 + x_3 \end{cases}$$

$$\Sigma_2 \begin{cases} \dot{x}_1 &= -2x_1 + u_2 \\ \dot{x}_2 &= -1x_1 + u_1 - u_2 \\ y &= x_1 + x_2 \end{cases}$$

$$\Sigma_3 \begin{cases} \dot{x}_1 &= -5x_1 + u_1 + 4u_2 \\ \dot{x}_2 &= -9x_1 - 10x_2 - 2u_1 + 5u_2 \\ \dot{x}_3 &= -3.14x_3 \\ y &= 0.01x_1 + 1.2x_2 \end{cases}$$

$$\Sigma_4 \begin{cases} \dot{x}_1 &= -11x_1 + 4x_2 + 5x_3 \\ \dot{x}_2 &= -5x_2 + 6x_3 + u_1 + 4u_2 \\ \dot{x}_3 &= -10x_3 - 2u_1 + 5u_2 \\ y &= 0.01x_2 + 1.2x_3 \end{cases}$$

Example of minimization:continued



$$M_{b,1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, M_{d,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} M_{a,2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_{d,2} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 0 & 0 \end{bmatrix} M_{b,4} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} M_{d,4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of minimization:continued

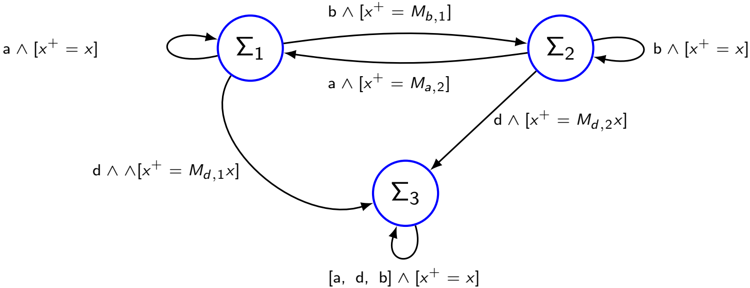
Result of minimization: 3 discrete states, number of continous states: $2+2+3=7$.

$$\Sigma_1 \begin{cases} \dot{x} & = \begin{bmatrix} -5.46 & 3.58 & 2.6 \\ -0.78 & -2.01 & 1.23 \\ -0.4 & -0.19 & -0.53 \end{bmatrix} x + \begin{bmatrix} 0.03 & -2.43 \\ 0.29 & -0.47 \\ -0.95 & -1.28 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y & = \begin{bmatrix} -0.59 & -0.91 & -1.35 \end{bmatrix} x \end{cases}$$

$$\Sigma_2 \begin{cases} \dot{x} & = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y & = \begin{bmatrix} -1 & 1 \end{bmatrix} x \end{cases}$$

$$\Sigma_3 \begin{cases} \dot{x} & = \begin{bmatrix} -12.23 & -8.6 \\ 1.87 & -2.77 \end{bmatrix} x + \begin{bmatrix} 1.72 & -5.71 \\ -1.33 & -2.07 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ y & = \begin{bmatrix} -1.16 & 0.29 \end{bmatrix} x \end{cases}$$

Example of minimization:continued



$$M_{b,1} = \begin{bmatrix} -0.32 & 0.91 & 0.27 \\ 0.03 & 0.29 & -0.95 \end{bmatrix}, M_{d,1} = \begin{bmatrix} 0.92 & 0.29 & 0.12 \\ -0.24 & -0.07 & -0.03 \end{bmatrix}$$

$$M_{d,2} = \begin{bmatrix} 1.94 & -3.12 \\ -0.5 & -0.08 \end{bmatrix}, M_{b,2} = \begin{bmatrix} -0.32 & 0.03 \\ 0.91 & 0.29 \\ 0.27 & -0.95 \end{bmatrix}$$

Consequence of realization theory: counter-examples

- If all the continuous subsystems are minimal, then the switched system is minimal.
- A switched system can be minimal (resp. observable, reachable), without any of the subsystems being minimal (resp. observable, reachable).
- Certain linear switched systems can never be brought to a form where all the continuous subsystems are minimal.
- A minimal switched system can be computed from finite input-output data (finitely many impulse responses).

Application of realization theory

- Identifiability [with Laurent Bako, Jan van Schuppen]
- Persistence of excitation [with Laurent Bako]
- Identification algorithms [with Laurent Bako, Roland Toth, Pepijn Cox]
- Manifold structure of systems, distances, canonical forms [with René Vidal and Ralf Peeters]
- Model reduction [with Mert Bastug, Rafael Wisniewski, John Leth]

Identifiability of hybrid systems

Identifiability

A parameterization of hybrid systems is identifiable, if no two distinct parameter values yield the same input-output behavior.

Example:

The parameterization below is not identifiable

$$\begin{aligned} \dot{x} &= (a+b)x + u, \\ y &= x \end{aligned}$$

$a, b \in \mathbb{R}$ are parameters.

Motivation:

it is impossible to identify non-identifiable parameterizations.

Identifiability of hybrid systems: continued

Uniqueness of minimal realization $\Longrightarrow$
A parameterization is identifiable $\Longleftrightarrow$ for every parameter the corresponding system is minimal and no two parameter values lead to isomorphic systems.

Identifiability can be checked numerically.

Counter-example A liner switched system can be identifiable, without the linear subsystems being identifiable.

Naive approach: identify the linear subsystems separately will not work in general.

Identifiability of hybrid systems: continued

Specific counter-example

$$\begin{aligned} x(t+1) &= A_{q_t}(\theta)x(t) + B_{q_t}(\theta)u_t \\ y(t) &= C_{q_t}(\theta)x(t), \quad q_t \in \{1,2\} \end{aligned}$$

$\theta = (a_{1,11}, a_{1,21}, a_{1,33}) \in \mathbb{R}^3$ – parameter

$$A_1(\theta) = \begin{bmatrix} a_{1,11} & 1 & 0 \\ a_{1,21} & 0 & 0 \\ 0 & 0 & a_{1,33} \end{bmatrix}, \; B_1(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \;\; C_1(\theta) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$$

$$A_2(\theta) = \begin{bmatrix} a_{2,11} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a_{2,21} & a_{2,22} \end{bmatrix}, \; B_2(\theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_2(\theta) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Persistence of excitation for hybrid systems

Definition
An input signal is persistently exciting, if the system parameters can be reconstructed from the response to this input signal.

Realization algorithm $\Longrightarrow$
finitely many Markov-parameters $\{M(v_i)\}_{i=1}^{2n-1}$ are sufficient to compute a linear switched system realization.

The $\{u(t), q(t)\}_{t=0}^{\infty}$ is persistently exciting,
if $\{M(v_i)\}_{i=1}^{2n-1}$ can be computed from $\{u(t), q(t), y(t)\}_{t=0}^{\infty}$.

Persistence of excitation is a property of the inputs, not of the outputs.

Persistence of excitation for hybrid systems

Conditions for persistence of excitation

1. The switched system is stable.
2. $u(.)$ and $q(.)$ are independent processes and
   2.1 $u(.)$ and $q(.)$ are ergodic.
   2.2 $u(.)$ is a coloured noise.
   2.3 $q(.)$ contains any sequence of length $2n-1$ with non-zero probability

$u(.)$ white noise, $q(.)$ binary noise $\implies$ persistence of excitation.

Identification algorithm

Realization algorithm + persistently exciting input = identification algorithm

Suppose $\{u(t), q(t)\}_{t=0}^{\infty}$ are persistently exciting.

1. Compute the Markov parameters $\{M(v_i)\}_{i=1}^{2n-1}$ from the input/output time series $\{u(t), q(t), y(t)\}_{t=0}^{\infty}$
2. Apply the realization algorithm $\{M(v_i)\}_{i=1}^{2n-1}$.

The above procedure is a black-box subspace-like method.

Improved version for LPV [LPVS'15 with P. Cox, R.Toth]

Model reduction

Model reduction by moment matching [Mert Bastug, John Leth, Rafael Wisniewski:] Consider the Markov parameters

$$M(q_1 \cdots q_k) = \begin{bmatrix} C_1 \\ \vdots \\ C_D \end{bmatrix} A_{q_k} \cdots A_{q_1} \begin{bmatrix} B_1 & \cdots & B_D \end{bmatrix}$$

Compute a smaller system $\hat{\Sigma} = \{\hat{A}_q, \hat{B}_q, \hat{C}_q\}_{q=1}^{D}$ s.t.

$$M(q_1 \cdots q_k) = \begin{bmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_D \end{bmatrix} \hat{A}_{q_k} \cdots \hat{A}_{q_1} \begin{bmatrix} \hat{B}_1 & \cdots & \hat{B}_D \end{bmatrix}$$

for all sequences $q_1, \ldots, q_k$ from some set $\mathcal{L}$.

$\forall$ set S of switching signals, $\exists$, $\mathcal{L}$ s.t. Σ and $\hat{\Sigma}$ have the same input-output behavior along any $q(.) \in S$.

Model reduction: continued

Balanced truncation: with J. Leth, R. Wisniewski]

1. Find grammians $\mathcal{Q}, \mathcal{P} > 0$ s.t.

$$\forall q \in Q : A_q^T \mathcal{Q} + \mathcal{Q} A_q + C_q^T C_q \leq 0$$
$$\forall q \in Q : A_q \mathcal{P} + \mathcal{P} A_q^T + B_q B_q^T \leq 0$$

2. Apply a linear basis transformation $\mathcal{S}$ s.t. in the new basis

$$\mathcal{P} = \mathcal{Q} = \Lambda = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$

   $\sigma_1, \ldots, \sigma_n$ singular values of the grammians.

3. Choose $r < n$ an let $\hat{A}_q \in \mathbb{R}^{r \times r}$, $\hat{B}_q \in \mathbb{R}^{r \times \cdot}$ and $\hat{C}_q \in \mathbb{R}^{\cdot \times r}$

$$A_q = \begin{bmatrix} \hat{A}_q & \star \\ \star & \star \end{bmatrix}, \quad B_q = \begin{bmatrix} \hat{B}_q \\ \star \end{bmatrix}, \quad C_q^T = \begin{bmatrix} \hat{C}_q^T \\ \star \end{bmatrix}$$

4. $\hat{\Sigma} = \{\hat{A}_q, \hat{B}_q, \hat{C}_q)\}_{q \in Q}$.

Model reduction: continued

- $||\Sigma - \hat{\Sigma}||_2 \leq 2(\sigma_{r+1} + \cdots + \sigma_n)$
- Grammians $\mathcal{P}, \mathcal{Q}$ and hence $\sigma_1, \ldots, \sigma_n$ are not unique. From realization theory:
  1. Minimization preserves the existence of grammians and does not increase their singular values.
  2. All minimal and i/o equivalent systems yield the same collection of singular values.

Realization theory: some consequences for control

- Quadratic stability is the property of input-output behavior: minimization preserves quadratic stability, and if one minimal system is quadratically stable, then so are all the other minimal systems with the same i/o behavior.

  Is this true for non-quadratic Lyapunov functions ?

- Quadratic L_2 gain is a propert of i/o behavior: if a L_2 gain can be achieved by quadratic storage function (LMI) for some switched system representing a i/o behavior, then the same gain can be achieved by a qudaratic storage function for any minimal switched system representation of the i/o behavior.

- (TAC'17, Chitour & Mason & Sigalotti): the same is true for general L_2 gain (non-quadratic storage function), minimality is used