Time-regularised and Periodic Event-triggered Control for Linear Systems

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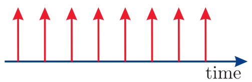
Where innovation starts

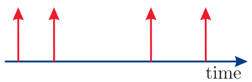
Outline

- Introduction event-triggered control
- Requirements for event-triggered control
- Challenges (disturbances, output-based, decentralised)
- Discussion different ETC schemes
  - Relative, absolute and mixed event generators
  - Periodic event-triggered control
  - ETC with time regularisation (waiting times)
  - Dynamic event generators
- New design methods
- Conclusions

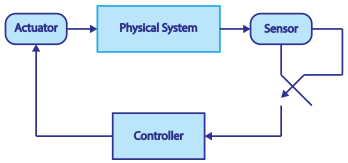
Introduction

Paradigm shift: Periodic control $\longrightarrow$ Aperiodic control





- Event-triggered control:

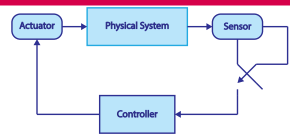


$$\begin{aligned} u(t) &= \mathcal{K}(x(t_k)), \text{ when } t \in [t_k, t_{k+1}) \\ t_{k+1} &= \inf\{t > t_k \mid \|x(t) - x(t_k)\| \geqslant \sigma\|x(t)\|\} \end{aligned}$$

[1] Hendricks *et al*, ACC'94
[2] Astrom & Bernhardsson, IFAC WC'99
[3] Arzen, IFAC WC'99
[4] Heemels *et al*, CEP'99

Requirements

$$\begin{aligned} u(t) &= \mathcal{K}(x(t_k)), \text{ when } t \in [t_k, t_{k+1}) \\ t_{k+1} &= \inf\{t > t_k \mid \|x(t) - x(t_k)\| \geqslant \sigma\|x(t)\|\} \end{aligned}$$



- Quality of Control (QoC): Stability, Performance, Robustness

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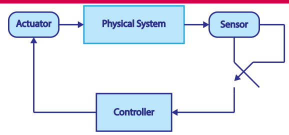

- Quality of Control (QoC): Stability, Performance, Robustness
  - Stability to equilibrium (under vanishing disturbances) or to a set
  - Performance/robustness : $\mathcal{L}_2$—gain from disturbance w to output z

$$\|z\|_{\mathcal{L}_2} \leqslant \beta(|x(0)|) + \gamma\|w\|_{\mathcal{L}_2} \text{ with } \|z\|^2_{\mathcal{L}_2} = \int_0^\infty \|z(t)\|^2 dt$$

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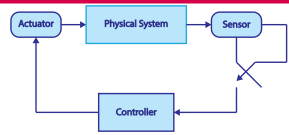

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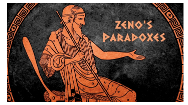

- Quality of Service (QoS) (required)
  - Zeno-freeness: Absence of infinite number of events in finite time
    - ▶ There is $\tau_{\text{MIET}} > 0$ s.t. $t_{k+1} - t_k \geqslant \tau_{\text{MIET}}$ for all $k \in \mathbb{N}$ (pos. MIET)

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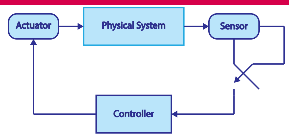

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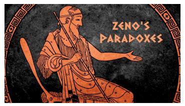

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  - Average inter-event times

Illustrative Example

- Consider $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ and $u(t) = \begin{bmatrix} 1 & -4 \end{bmatrix} x(t_k)$
- TTC: $t_k = k \cdot 0.025$
- ETC: $t_k = t \iff \| \underbrace{e(t)}_{=x(t)-\hat{x}(t)} \| \geqslant 0.05\|x(t)\|$ (relative triggering)
- Properties established in [1]:
  - Global exponential stability (GES)
  - Global positive lower bound on minimal inter-event time (MIET)

$$\inf\{t_{k+1} - t_k \mid k \in \mathbb{N}\} \geqslant \tau_{\text{MIET}} > 0$$

[1] Tabuada, *Event-triggered real-time scheduling of stabilizing control tasks*, TAC 2007

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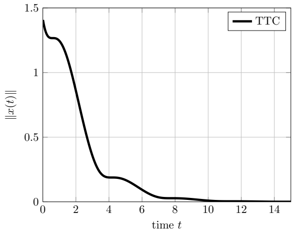

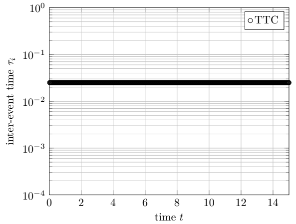

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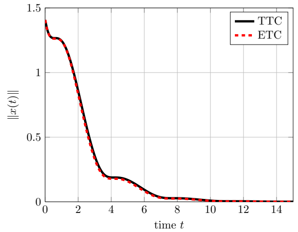

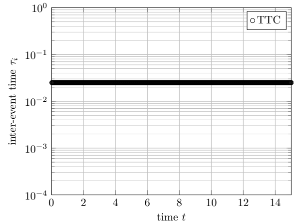

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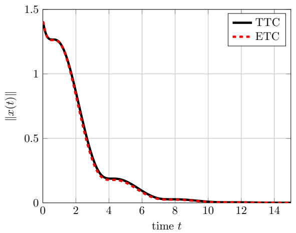

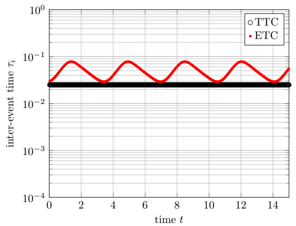

Illustrative example

- Consider $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w$ and $u(t) = \begin{bmatrix} 1 & -4 \end{bmatrix} x(t_k)$
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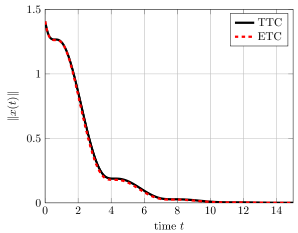

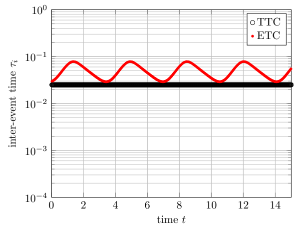

Borgers, Heemels, *Event-Separation Properties of Event-Triggered Control Systems*, TAC 2014

Illustrative example

- Consider $\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + {\color{red} w}$ and $u(t) = \begin{bmatrix} 1 & -4 \end{bmatrix} x(t_k)$
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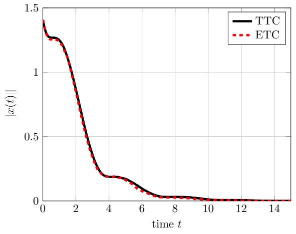

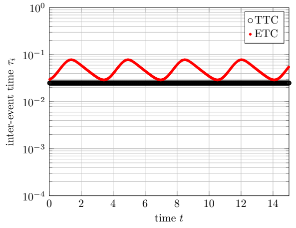

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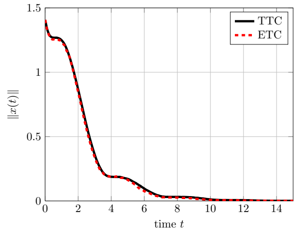

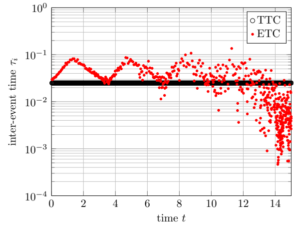

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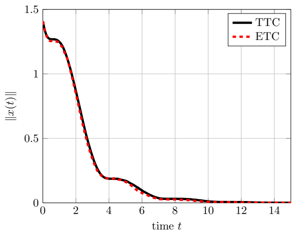

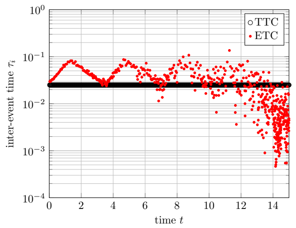

Borgers, Heemels, *Event-Separation Properties of Event-Triggered Control Systems*, TAC 2014

$\longrightarrow$ similar Zeno issues with output-based and distributed event-generators

Donkers, Heemels, *Output-Based Event-Triggered Control with Guaranteed L_∞-gain* ..., TAC 2012

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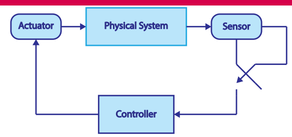

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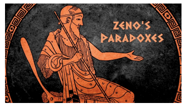

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  - Average inter-event times

Event-triggered control schemes

$$t_{k+1} = \inf\{t > t_k \mid \|\underbrace{y(t_k)}_{=\hat{y}(t)} - y(t)\| \geqslant \sigma\|y(t)\| + \delta\}$$

- Relative: $\|\hat{y} - y\| \geqslant \sigma\|y\|$ [1]
- Absolute: $\|\hat{y} - y\| \geqslant \delta$ [2-4]
- Mixed: $\|\hat{y} - y\| \geqslant \sigma\|y\| + \delta$ [5]

[1] Tabuada, *Event-triggered real-time scheduling of stabilizing control tasks*, TAC 2007
[2] Yook, Tilbury, Soparkar, *Trading computation for bandwidth: Reducing communication in distributed control systems using state estimators*, TCST 2002
[3] Miskowicz, *Send-on-delta concept: An event-based data-reporting strategy*, Sensors, 2006
[4] Lunze and Lehmann, *A state-feedback approach to event-based control*, Automatica, 2010
[5] Donkers, Heemels, *Output-Based Event-Triggered Control with Guaranteed L_∞-gain ...*, TAC 2012

Overview

State-feedback case

ETM	robust global	global	robust semi-global	semi-global	robust local	local
relative	×	✓	×	✓	×	✓
absolute	×	×	✓	✓	✓	✓
mixed	✓	✓	✓	✓	✓	✓

Output-feedback case

ETM	robust global	global	robust semi-global	semi-global	robust local	local
relative	×	×	×	×	×	×
absolute	×	×	✓	✓	✓	✓
mixed	×	×	✓	✓	✓	✓

- Robust semi-global at best!
- However, only practical stability / ultimate boundedness (no GAS)

Borgers, Heemels, *Event-Separation Properties of Event-Triggered Control Systems*, TAC 2014

Event-triggered control schemes

Time regularisation:

- Periodic Event-Triggered Control (PETC) [6-9]

$$t_{k+1} = \inf\{t > t_k \mid \|y(t) - \hat{y}(t)\| > \sigma\|y(t)\| \wedge t = kh,\ k \in \mathbb{N}\}$$

- Enforcing minimal inter-event/waiting time [7,9-11]

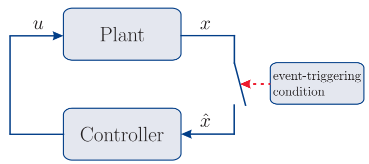
$$t_{k+1} = \inf\{t > t_k + T \mid \|y(t) - \hat{y}(t)\| > \sigma\|y(t)\|\}$$

Next: Analysis and design tools for these event generators

[6] Arzen, *A simple event-based PID controller*, IFAC 1999
[7] Heemels, Sandee, van den Bosch, *Analysis of event-driven controllers for linear systems*, IJC 2008
[8] Heemels, Donkers, Teel, *Periodic Event-Triggered Control for Linear Systems*, TAC 2013
[9] Henningsson, Johannesson, Cervin, *Sporadic event-based control of first-order linear stochastic ...*, Aut. 2008
[10] Tallapragada, Chopra, *Event-triggered dynamic output feedback control for LTI systems*, CDC 2012
[11] Tallapragada, Chopra, *Event-triggered decentralized dynamic output .. LTI systems*, NECSYS 2012

PETC

Description



$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = A^p x + B^p u + B^w w$$

$$u(t) = K\hat{x}(t)$$

$$\hat{x}(t) = \begin{cases} x(t_k), & \text{when } \xi^\top(t_k)Q\xi(t_k) > 0, \\ \hat{x}(t_k), & \text{when } \xi^\top(t_k)Q\xi(t_k) \leqslant 0 \end{cases} \quad \text{for } t \in (t_k, t_{k+1}]$$

with $t_k = kh$ and $h > 0$ fixed sampling period and $\xi = (x, \hat{x})$.

PETC

Hybrid system formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0, h],$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{cases}\begin{bmatrix}J_1\xi\\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi > 0,\ \tau = h\\ \begin{bmatrix}J_2\xi\\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi \leqslant 0,\ \tau = h\end{cases}$$

with $\xi = (x, \hat{x})$ and

$$A := \begin{bmatrix}A^p & B^pK\\ 0 & 0\end{bmatrix}, \quad B := \begin{bmatrix}B^w\\ 0\end{bmatrix}, \quad J_1 := \begin{bmatrix}I & 0\\ I & 0\end{bmatrix}, \quad J_2 := \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}$$

[1] Goebel, Sanfelice, Teel, *Hybrid dynamical systems: Modeling, Stability and Robustness*, 2012

PETC

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Problem formulation:

- Globally exponentially stable $(w = 0)$: $\|\xi(t)\| \leqslant ce^{-\rho t}\|\xi(0)\|$
- $\mathcal{L}_2$ gain smaller than or equal to γ with $z = C\xi + Dw$

$$\sqrt{\int_0^\infty \|z(t)\|^2 dt} \leqslant \beta(\xi_0) + \gamma\sqrt{\int_0^\infty \|w(t)\|^2 dt}$$

$\mathcal{L}_2$-gain: intersample behaviour

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0, h],$$

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Main idea $\mathcal{L}_2$ gain analysis: $z = C\xi + Dw$

- Timer-dependent quadratic Lyapunov function $V(\xi, \tau) = \xi^\top P(\tau)\xi$
- $\frac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}z^\top z + w^\top w$
- During jumps non-increase of LF:

$$V(J_1\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi > 0,$$
$$V(J_2\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi \leqslant 0$$

$\mathcal{L}_2$-gain: intersample behaviour

Main idea $\mathcal{L}_2$ gain analysis $V(\xi, \tau) = \xi^\top P(\tau)\xi$

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Nice idea ... but how?

$\mathcal{L}_2$-gain: intersample behaviour

Main idea $\mathcal{L}_2$ gain analysis $V(\xi,\tau)=\xi^\top P(\tau)\xi$

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- During jumps non-increase:

$$V(J_1\xi,0) \leqslant V(\xi,h), \text{ for all } \xi \text{ with } \xi^\top Q\xi > 0,$$
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Nice idea ... but how?

- Riccati differential equation

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}P = -A^\top P - PA - 2\rho P - \gamma^{-2}C^\top C - (PB + \gamma^{-2}C^\top D)M(B^\top P + \gamma^{-2}D^\top C)$$

- Hamiltonian $H := \begin{bmatrix} A+\rho I+\gamma^{-2}BMD^\top C & BMB^\top \\ -C^\top LC & -(A+\rho I+\gamma^{-2}BMD^\top C)^\top \end{bmatrix}$
- $F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}$
- $P_0 = (F_{21}(h) + F_{22}(h)P_h)\big(F_{11}(h) + F_{12}(h)P_h\big)^{-1}$

$\mathcal{L}_2$-gain: intersample behaviour

Theorem [1]: Suppose that there exist matrix $P_h \succ 0$, and scalars $\mu_i \geqslant 0$, such that for $i \in \{1,2\}$

$$\begin{bmatrix} P_h + (-1)^i\mu_i Q & J_i^\top \bar{F}_{11}^{-\top} P_h \bar{S} & J_i^\top(\bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1}) \\ \star & I - \bar{S}^\top P_h \bar{S} & 0 \\ \star & \star & \bar{F}_{11}^{-\top} P_h \bar{F}_{11}^{-1} + \bar{F}_{21}\bar{F}_{11}^{-1} \end{bmatrix} \succ 0$$

Then, the PETC system is GES with convergence rate ρ (when $w=0$) and has an $\mathcal{L}_2$-gain smaller than or equal to γ.

[1] Heemels, Donkers, Teel, *Periodic Event-Triggered Control for Linear Systems*, TAC 2013

The years 2011-2013

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, \text{ when } \tau \in [0,h],$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{cases} \begin{bmatrix} J_1\xi \\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi > 0,\ \tau = h \\ \begin{bmatrix} J_2\xi \\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi \leqslant 0,\ \tau = h \end{cases}$$

- Observation:
  - The LMIs can be re-interpreted as a conservative ℓ_2-gain test for a discrete-time PWL system (not being the discretisation) using CQLF

.... food for thought ....

PETC

Alternative approach: lifting ...

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix} A\xi + Bw \\ 1 \end{bmatrix}, \text{ when } \tau \in [0,h],$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{cases} \begin{bmatrix} J_1\xi \\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi > 0,\ \tau = h \\ \begin{bmatrix} J_2\xi \\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi \leqslant 0,\ \tau = h \end{cases}$$

$$z = C\xi + Dw$$

PETC

Hybrid systems formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0,h]$$
$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{bmatrix}\phi(\xi)\\ 0\end{bmatrix}, \quad \text{when } \tau = h$$
$$z = C\xi + Dw$$

PETC

Hybrid systems formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0,h]$$
$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{bmatrix}\phi(\xi)\\ 0\end{bmatrix}, \quad \text{when } \tau = h$$
$$z = C\xi + Dw$$

Other applications for this framework:

- Reset controllers with testing for reset only at kh, $k \in \mathbb{N}$ [1]
- Linear systems/controllers with one sensor/actuator node transmitting at kh, $k \in \mathbb{N}$ determined by quadratic protocol [1]
- Linear systems controlled by arbitrarily switching sampled-data controllers (in this case ϕ setvalued) [2]
- Linear systems controlled by saturating sampled-data controllers [1]

[1] Heemels, Dullerud, Teel, *$\mathcal{L}_2$-gain Analysis for a Class of Hybrid Systems with Applications to Reset and Event-triggered Control: A Lifting Approach*, TAC'16
[2] CDC'15 version of above

PETC

Hybrid systems formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0,h]$$
$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{bmatrix}\phi(\xi)\\ 0\end{bmatrix}, \quad \text{when } \tau = h$$
$$z = C\xi + Dw$$

- Internally stable[a]: $\|\xi\|_{\mathcal{L}_2} \leqslant \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ for $\mathcal{K}$-function β

[a] implies $\lim_{t\to\infty} \xi(t) = 0$ and Lyapunov stability $\|\xi\|_{\mathcal{L}_\infty} \leqslant \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$

PETC

Hybrid systems formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \text{ when } \tau \in [0,h]$$
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$$z = C\xi + Dw$$

- Internally stable[a]: $\|\xi\|_{\mathcal{L}_2} \leqslant \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$ for $\mathcal{K}$-function β
- $\mathcal{L}_2$-contractive: There is $\gamma_0 \in [0,1)$ and a $\mathcal{K}$-function β s.t.

$$\|z\|_{\mathcal{L}_2} \leqslant \beta(|\xi_0|) + \gamma_0\|w\|_{\mathcal{L}_2} \text{ with } \|z\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \|z(t)\|^2 dt}$$

[a] implies $\lim_{t\to\infty} \xi(t) = 0$ and Lyapunov stability $\|\xi\|_{\mathcal{L}_\infty} \leqslant \beta(\max(|\xi_0|, \|w\|_{\mathcal{L}_2}))$

Lifting-based approach

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \quad \text{when } \tau \in [0,h]$$
$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{bmatrix}\phi(\xi)\\ 0\end{bmatrix}, \quad \text{when } \tau = h$$
$$z = C\xi + Dw$$

$$\begin{aligned}\bar{\xi}_{k+1} &= A_d\phi(\bar{\xi}_k) + B_d v_k\\ r_k &= C_d\phi(\bar{\xi}_k)\end{aligned}$$

Main result: The hybrid system is internally stable and $\mathcal{L}_2$-contractive iff the discrete-time system is internally ℓ_2-stable and ℓ_2-contractive.

[1] Heemels, Dullerud, Teel, *$\mathcal{L}_2$-gain Analysis for a Class of Hybrid Systems with Applications to Reset and Event-triggered Control: A Lifting Approach*, TAC'16

Lifting-based approach

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \quad \text{when } \tau \in [0,h]$$
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Main result: The hybrid system is internally stable and $\mathcal{L}_2$-contractive iff the discrete-time system is internally ℓ_2-stable and ℓ_2-contractive.

- ℓ_2-contractive: there is $\gamma_0 \in [0,1)$ s.t.

$$\|r\|_{\ell_2} \leqslant \beta(|\bar{\xi}_0|) + \gamma_0\|v\|_{\ell_2} \text{ with } \|r\|_{\ell_2}^2 = \sum_{k=0}^{\infty}|r_k|^2$$

- internally ℓ_2-stable: $\|\bar{\xi}\|_{\ell_2} \leqslant \beta(\max(|\bar{\xi}_0|, \|v\|_{\ell_2})$

[1] Heemels, Dullerud, Teel, *$\mathcal{L}_2$-gain Analysis for a Class of Hybrid Systems with Applications to Reset and Event-triggered Control: A Lifting Approach*, TAC'16

Lifting-based approach

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \quad \text{when } \tau \in [0,h]$$
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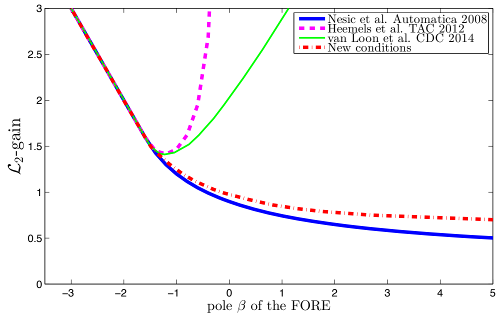
Main result: The hybrid system is internally stable and $\mathcal{L}_2$-contractive iff the discrete-time system is internally ℓ_2-stable and ℓ_2-contractive.

- Lifting with verifiable conditions without linearity
- For PETC *piecewise linear* system $\longrightarrow$ contractivity/stability analysis via LMIs using piecewise quadratic Lyapunov functions

[1] Heemels, Dullerud, Teel, *$\mathcal{L}_2$-gain Analysis for a Class of Hybrid Systems with Applications to Reset and Event-triggered Control: A Lifting Approach*, TAC'16

Reset control example

- Reset control example taken from [1]. ϕ also piecewise linear.



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \quad \text{when } \tau \in [0,h],$$
$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{cases}\begin{bmatrix}J_1\xi\\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi > 0,\ \tau = h\\ \begin{bmatrix}J_2\xi\\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi \leqslant 0,\ \tau = h\end{cases}$$

- LMI-based ℓ_2-gain analysis using piecewise quadratic storage functions for discrete-time PWL system
- Earlier results [2]: Sufficient LMI-based results with $V(\xi) = \xi^\top P(\tau)\xi$

[1] Nešić, Zaccarian, Teel, *Stability properties of reset systems*, Automatica 2008
[2] Heemels, Donkers, Teel, *Periodic event-triggered control for linear systems*, TAC 2013
[3] Heemels, Dullerud, Teel, *$\mathcal{L}_2$-gain Analysis for a Class of Hybrid Systems with Applications to Reset and Event-triggered Control: A Lifting Approach*, TAC 2016

Conclusions PETC

- Analysis/design tools for PETC
  - Riccati differential equation approach (2011-2013)
  - Lifting approach giving tight characterisation based on nonlinear/PWL discrete-time system

$\longrightarrow$ What about CETC with time-regularisation ?

Conclusions PETC

- Analysis/design tools for PETC
  - Riccati differential equation approach (2011-2013)
  - Lifting approach giving tight characterisation based on nonlinear/PWL discrete-time system

$\longrightarrow$ What about CETC with time-regularisation ?

$\longrightarrow$ Where the years 2011-2013 lost?

CETC with time regularisation

Event times:

$$t_{k+1} = \inf\{t > t_k + h \mid \xi^\top Q \xi > 0\}$$

Waiting time h

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \qquad \text{when } \tau \in [0,h] \text{ or } \xi^\top Q\xi \leqslant 0$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{bmatrix}J_1\xi\\ 0\end{bmatrix}, \qquad \text{when } \tau \in [h,\infty) \text{ and } \xi^\top Q\xi \geqslant 0$$

$$z = C\xi + Dw$$

Recall: PETC

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix} = \begin{bmatrix}A\xi + Bw\\ 1\end{bmatrix}, \quad \text{when } \tau \in [0,h],$$

$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix} = \begin{cases}\begin{bmatrix}J_1\xi\\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi > 0,\ \tau = h\\[2ex] \begin{bmatrix}J_2\xi\\ 0\end{bmatrix}, & \text{when } \xi^\top Q\xi \leqslant 0,\ \tau = h\end{cases}$$

Main idea $\mathcal{L}_2$ gain analysis: $z = C\xi + Dw$

- Timer-dependent quadratic Lyapunov function $V(\xi,\tau) = \xi^\top P(\tau)\xi$
- $\frac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2} z^\top z + w^\top w$
- During jumps non-increase:

$$V(J_1\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi > 0,$$
$$V(J_2\xi, 0) \leqslant V(\xi, h), \text{ for all } \xi \text{ with } \xi^\top Q\xi \leqslant 0$$

Recall: PETC

Main idea $\mathcal{L}_2$ gain analysis $V(\xi,\tau)=\xi^\top P(\tau)\xi$

- $\frac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}z^\top z + w^\top w$
- During jumps non-increase:

$$V(J_1\xi,0)\leqslant V(\xi,h),\ \text{for all } \xi \text{ with } \xi^\top Q\xi>0,$$
$$V(J_2\xi,0)\leqslant V(\xi,h),\ \text{for all } \xi \text{ with } \xi^\top Q\xi\leqslant 0$$

Nice idea ... but how?

- Riccati differential equation

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}P=-A^\top P-PA-2\rho P-\gamma^{-2}C^\top C-(PB+\gamma^{-2}C^\top D)M(B^\top P+\gamma^{-2}D^\top C)$$

- Hamiltonian $H := \begin{bmatrix} A+\rho I+\gamma^{-2}BMD^\top C & BMB^\top \\ -C^\top LC & -(A+\rho I+\gamma^{-2}BMD^\top C)^\top \end{bmatrix}$
- $F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau)\end{bmatrix}$
- $P_0 = (F_{21}(h)+F_{22}(h)P_h)\big(F_{11}(h)+F_{12}(h)P_h\big)^{-1}$

CETC with time regularisation

Event times:

$$t_{k+1}=\inf\{t>t_k+h \mid \xi^\top Q\xi>0\}$$

Waiting time h

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \tau\end{bmatrix}=\begin{bmatrix}A\xi+Bw\\ 1\end{bmatrix},\qquad \text{when } \tau\in[0,h] \text{ or } \xi^\top Q\xi\leqslant 0$$
$$\begin{bmatrix}\xi^+\\ \tau^+\end{bmatrix}=\begin{bmatrix}J_1\xi\\ 0\end{bmatrix},\qquad \text{when } \tau\in[h,\infty) \text{ and } \xi^\top Q\xi\geqslant 0$$
$$z=C\xi+Dw$$

$\mathcal{L}_2$-gain analysis using timer-dependent quadratic Lyapunov function:

$$V(\xi,\tau)=\begin{cases}\xi^\top P(\tau)\xi, & \text{when } \tau\in[0,h)\\ \xi^\top P(h)\xi, & \text{when } \tau\in[h,\infty)\end{cases}$$

CETC with time regularisation

- Ricatti differential equation

$$\rightarrow \tfrac{\mathrm{d}}{\mathrm{d}t}V\leqslant -2\rho V-\gamma^{-2}z^\top z+w^\top w \text{ when } \tau\in[0,h]$$

CETC with time regularisation

- Ricatti differential equation

$$\rightarrow \tfrac{\mathrm{d}}{\mathrm{d}t}V\leqslant -2\rho V-\gamma^{-2}z^\top z+w^\top w \text{ when } \tau\in[0,h]$$

- LMI guaranteeing (only for J_1) - similar as for PETC

$$\rightarrow V(J_1\xi,0)\leqslant V(\xi,h),\ \text{for all } \xi \text{ with } \xi^\top Q\xi\geqslant 0$$

CETC with time regularisation

- Ricatti differential equation

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- LMI guaranteeing (only for J_1) - similar as for PETC

$$\rightarrow V(J_1\xi,0) \leqslant V(\xi,h), \text{ for all } \xi \text{ with } \xi^\top Q\xi \geqslant 0$$

- $\begin{bmatrix} A^\top P_h + P_hA - \beta Q + 2\rho P_h + \gamma^{-2}C^\top C & \star \\ B^\top P_h + \gamma^{-2}D^\top C & \gamma^{-2}D^\top D - I \end{bmatrix} \preceq 0$

$$\rightarrow \tfrac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}z^\top z + w^\top w \text{ when } \tau \in [h,\infty) \text{ and } \xi^\top Q\xi \leqslant 0$$

CETC with time regularisation

30/37

- Ricatti differential equation

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- LMI guaranteeing (only for J_1) - similar as for PETC

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- $\begin{bmatrix} A^\top P_h + P_hA - \beta Q + 2\rho P_h + \gamma^{-2}C^\top C & \star \\ B^\top P_h + \gamma^{-2}D^\top C & \gamma^{-2}D^\top D - I \end{bmatrix} \preceq 0$

$$\rightarrow \tfrac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}z^\top z + w^\top w \text{ when } \tau \in [h,\infty) \text{ and } \xi^\top Q\xi \leqslant 0$$

$\rightarrow$ Then, the CETC with TR system is GES with convergence rate ρ (when $w = 0$) and has an $\mathcal{L}_2$-gain smaller than or equal to γ

CETC with time regularisation

Observations:

- Extra decrease along jumps with $\xi^\top Q\xi \geqslant 0$:

$$V(J_1\xi,0) \leqslant V(\xi,h) - \mu_1\xi^\top Q\xi$$

- Extra decrease during flow with $\tau \in [h,\infty)$ and $\xi^\top Q\xi \leqslant 0$:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V \leqslant -2\rho V - \gamma^{-2}z^\top z + w^\top w + \beta\xi^\top Q\xi$$

Idea: store extra decrease in buffer η

$$\dot\eta = \begin{cases} -2\rho\eta & \text{when } \tau \in [0,h] \\ -2\rho\eta - \beta\xi^\top Q\xi & \text{when } \tau \in [h,\infty) \end{cases}$$
$$\eta^+ = \eta + \mu_1\xi^\top Q\xi$$

$\rightarrow$ new Lyapunov/storage function: $U = V + \eta$

Dynamic CETC + TR

New storage function $U(\xi,\tau,\eta) = V(\xi,\tau) + \eta$

- $\tfrac{\mathrm{d}}{\mathrm{d}t}U \leqslant -2\rho U - \gamma^{-2}z^\top z + w^\top w$ for all $\tau \in [0,\infty)$
- $U(J_1\xi,0) \leqslant U(\xi,h)$ for all $\xi \in \mathbb{R}^{n_\xi}$
- U proper storage function as long as $\eta \geqslant 0$:

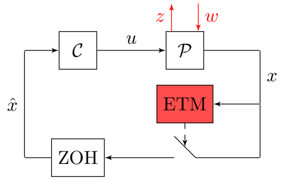
$$t_{k+1} = \inf\{t > t_k + h \mid \eta < 0\}$$

Benefits dynamic ETC vs. static ETC (with TR):

- Extended inter-event times compared to static CETC with TR
- Identical control performance guarantees!

[1] Postoyan et al., "Event-triggered and self-triggered stabilization ...," CDC 2011
[2] Girard, "Dynamic triggering mechanisms for event-triggered control," TAC 2015
[3] Dolk, Borgers, Heemels, "Dynamic Event-triggered Control...," CDC 2014, TAC 2017
[4] Borgers, Dolk, Heemels, "Dynamic event-triggered control with time regularization for linear systems," CDC 2016

Dynamic ETC: Example

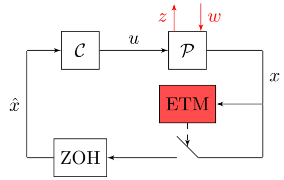


$$\mathcal{P}: \dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w$$

$$\mathcal{C}: u = \begin{bmatrix} 1 & -4 \end{bmatrix} x$$

Case study: $\mathcal{L}_2$-gain $\theta = 4$ from input w to state x: $h = 9.1 \cdot 10^{-3}$

Dynamic ETC: Example



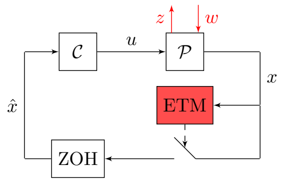
$$\mathcal{P}: \dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w$$

$$\mathcal{C}: u = \begin{bmatrix} 1 & -4 \end{bmatrix} x$$

Case study: $\mathcal{L}_2$-gain $\theta = 4$ from input w to state x: $h = 9.1 \cdot 10^{-3}$

- *Dynamic* event generator $t_{k+1} := \inf\{t > t_k + h \mid \eta(t) < 0\}$
- *Static* event generator: $t_{k+1} := \inf\{t > t_k + h \mid \Psi(x, e, \tau, \eta) < 0\}$

Dynamic ETC: Example



$$\mathcal{P}: \dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + w$$

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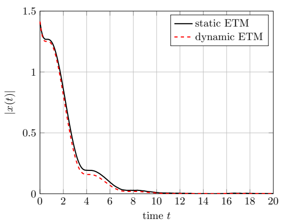
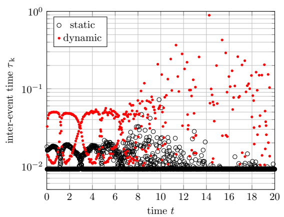
Case study: $\mathcal{L}_2$-gain $\theta = 4$ from input w to state x: $h = 9.1 \cdot 10^{-3}$

- *Dynamic* event generator $t_{k+1} := \inf\{t > t_k + h \mid \eta(t) < 0\}$
- *Static* event generator: $t_{k+1} := \inf\{t > t_k + h \mid \Psi(x, e, \tau, \eta) < 0\}$





- New dynamic ETM does not converge to TTC

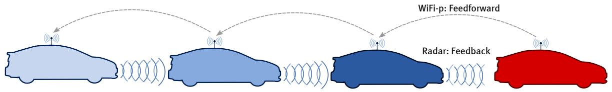
Conclusions

- Event-triggered control: A resource-aware control paradigm
- Challenges:
  - Disturbances & Output-based / distributed event generators
- Relative/absolute/mixed at best: practical stability and semi-global MIET
- $\Rightarrow$ CETC with time-regularisation or PETC
- (Tight) analysis/design tools for linear systems
  - PETC: Riccati differential equation approach (2011-2013)
  - PETC: Lifting approach
  - CETC + TR: Riccati differential equation approach + dynamic CETC with TR
- Extensions using PWQ LF and dynamic PETC design (HSCC 2017?)
- Many interesting and practical and theoretical issues open

More info: http://www.heemels.tue.nl

Motivation

Cooperative Adaptive Cruise Control



- String stability: disturbance attenuation along the vehicle string
  - $\mathcal{L}_p$-gain$\leqslant 1$
- Communication resources limited $\longrightarrow$ event-triggered communication

TU/e Technische Universiteit Eindhoven University of Technology

Presentation based on:

- D.P. Borgers and W.P.M.H. Heemels, *Event-separation properties of event-triggered control systems*, IEEE Transactions on Automatic Control, 59(10):2644-2656, 2014.
- D. P. Borgers, V. S. Dolk, and W. P. M. H. Heemels. *Dynamic event-triggered control with time-regularization for linear systems*. In 55th IEEE Conf. Decision and Control, 2016.
- V. S. Dolk, D. P. Borgers, and W. P. M. H. Heemels. *Output-based and decentralized dynamic event-triggered control with guaranteed Lp-gain performance and zeno-freeness*. IEEE Trans. Automat. Contr., 2017.
- M.C.F. Donkers and W.P.M.H. Heemels, *Output-Based Event-Triggered Control with Guaranteed L_∞-gain and Improved and Decentralised Event-Triggering*, IEEE Transactions on Automatic Control, 57(6), p. 1362-1376, 2012.
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