Stability analysis of discrete-time infinite-horizon optimal control with discounted cost

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ANR project COMPACS

Introduction

Discrete-time systems

$$x(k+1) = f(x(k), u(k))$$

Discounted cost function

$$J(x, \mathbf{u}) := \sum_{k=0}^{\infty} \gamma^k \ell(x(k), u(k))$$

- $x(k)$ solution to the system at step k starting from x with inputs $(u(0), u(1), \dots, u(k))$
- ℓ: stage cost, non-negative, example: $\ell(x, u) = x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u$
- $\gamma \in (0, 1)$: discount factor

Introduction: motivations

Fields

- **Control engineering**
- Artificial intelligence
- Operations research

Optimal control of systems subject to commmunication/computation constraints, see *e.g.*

- Radio-mode management event-based control of linear systems in [de Castro et al., ACC 2012]
- Event-triggered control of linear systems in [Antunes and Heemels, IEEE TAC 2014]
- Self-triggered control of linear systems in [Gommans et al., Automatica 2014]
- Time-triggered and self-triggered control of nonlinear systems using an optimal planning algorithm in [Busoniu et al., IEEE TAC 2016]

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Introduction: a simple example

Consider

$$x(k+1) = 2x(k) + u(k)$$

and

$$J(x,\mathbf{u}) = \sum_{k=0}^{\infty} \gamma^k \left(x(k)^2 + u(k)^2\right)$$

Solution [Bertsekas, 2012]:

$$u(k) = K(\gamma)x(k)$$

with

$$K(\gamma) = -2\left(1 + 2\left(5\gamma - 1 + \sqrt{(5\gamma-1)^2 + 4\gamma}\right)^{-1}\right)^{-1}$$

Hence

$$x(k+1) = (2 + K(\gamma))x(k)$$

and stability is guaranteed iff $2 + K(\gamma) \in (-1,1)$, i.e. $\gamma \in (\frac{1}{3}, 1]$

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Introduction: existing results

Stability results:

- Linear systems with discounted quadratic costs [Bertsekas, 2012]
- Local stability for continuous-time systems *e.g.*, [Rodriguez, JEDC 2004; Sorger, JOTA 1992]
- Analysis for specific algorithms
- Semiglobal practical stability for nonlinear systems affine in the input, quadratic stage cost [Boussios et al., ACC 2001]

No general conditions to guarantee stability for nonlinear systems

Introduction: objectives

- Stability guarantees for general nonlinear systems and stage costs (i.e. ℓ) when applying optimal inputs
- Robust stability
- Same guarantees when applying near-optimal inputs
- Relationship between the optimal cost function when $\gamma = 1$ and $\gamma \in (0, 1)$

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Plan

1. Introduction
2. Stability guarantees
3. Robustness
4. Near-optimal inputs
5. Miscellanies
6. Conclusions

Optimization problem

Consider the system

$$x(k+1) \quad = \quad f(x(k), u(k))$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U}(x) \subseteq \mathbb{R}^m$, $\mathcal{U}(x)$ non-empty set of admissible inputs

We define $\mathcal{W} := \{(x, u) \,:\, x \in \mathbb{R}^n \text{ and } u \in \mathcal{U}(x)\}$

Objective is to minimize the cost function

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Assumption: existence of optimal inputs

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For any $x \in \mathbb{R}^n$ and $\gamma \in (0,1)$, there exists an infinite-length input sequence $\mathbf{u}^*_\gamma(x)$, called *optimal solution*, such that

$$J(x, \mathbf{u}^*_\gamma(x)) \quad = \quad \inf_{\mathbf{u}} J_\gamma(x, \mathbf{u}) \quad =: \quad V_\gamma(x),$$

where V_γ is the *optimal value function*.

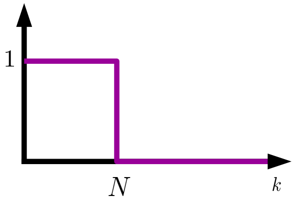
Conditions available in e.g., [Keerthi and Gilbert, IEEE TAC 1985]

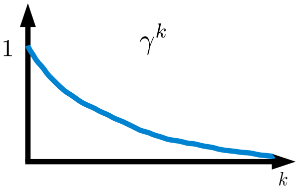
Optimization problem (c'd)

Assumptions and stability analysis inspired by

G. Grimm, M.J. Messina, S.E. Tuna, A.R. Teel, Model predictive control : for a want of a local control Lyapunov function, all is not lost, IEEE TAC 2005

$$J(x,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x(k),u(k)) + g(x(N)) \qquad J(x,\mathbf{u}) = \sum_{k=0}^{\infty} \gamma^k \ell(x(k),u(k))$$





Assumptions: controllability

Stability with respect to $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (continuous)

Examples of σ: $x \mapsto |x|$, $x \mapsto |x|^2$, $x \mapsto |x|_{\mathcal{A}}$ for some set $\mathcal{A}$

Controllability assumption

There exists $\overline{\alpha}_V \in \mathcal{K}_\infty$ such that for any $\gamma \in (0,1)$ and $x \in \mathbb{R}^n$,

$$V_\gamma(x) \leq \overline{\alpha}_V(\sigma(x)).$$

Recall: $V_\gamma(x) = \inf_{\mathbf{u}} J_\gamma(x, \mathbf{u})$

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Assumptions: controllability (c'd)

Lemma [ℓ is globally exponentially controllable to zero w.r.t. σ]

$\exists M, \lambda > 0$, $\forall x \in \mathbb{R}^n$, $\exists \mathbf{u}$, $\forall k \in \mathbb{Z}_{>0}$,

$$\ell(x(k), u(k)) \leq M\sigma(x)e^{-\lambda k}.$$

Then $\overline{V}_\gamma(x) \leq \overline{\alpha}(\sigma(x))$ for all $x \in \mathbb{R}^n$, with $\overline{\alpha} \in \mathcal{K}_\infty$.

Idea of the proof:

$$\begin{aligned} V_\gamma(x) &\leq J(x, \mathbf{u}) = \sum_{k=0}^{\infty} \gamma^k \ell(x(k), u(k)) \leq \sum_{k=0}^{\infty} \ell(x(k), u(k)) \leq \sum_{k=0}^{\infty} M\sigma(x)e^{-\lambda k} \\ &= \frac{M}{1 - e^{-\lambda}}\sigma(x) \end{aligned}$$

Assumptions: detectability

Detectability assumption

There exist a continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_W, \chi_W \in \mathcal{K}_\infty$ and $\overline{\alpha}_W : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous, nondecreasing and zero at zero, such that for any $(x, u) \in \mathcal{W}$

$$\begin{aligned} W(x) &\leq \overline{\alpha}_W(\sigma(x)) \\ W(f(x,u)) - W(x) &\leq -\alpha_W(\sigma(x)) + \chi_W(\ell(x,u)). \end{aligned}$$

Example: $\ell(x,u) = x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru$ where $Q = Q^{\mathrm{T}} > 0$ and $R = R^{\mathrm{T}} \geq 0$

$$\begin{aligned} W(x) = 0 & \\ W(f(x,u)) - W(x) = 0 &\leq -\lambda_{\min}(Q)|x|^2 + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru \end{aligned}$$

$\Rightarrow \sigma(x) = |x|^2$, $W = 0$, $\overline{\alpha}_W = 0$, $\chi_W(s) = s$ and $\alpha_W(s) = \lambda_{\min}(Q)s$ for $s \geq 0$

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System model: difference inclusion

For any $x \in \mathbb{R}^n$, in view of the Bellman equation

$$\mathcal{U}^*_\gamma(x) \quad := \quad \underset{u \in \mathcal{U}(x)}{\arg\min} \left[\ell(x,u) + \gamma V_\gamma(f(x,u))\right],$$

Hence,

$$x(k+1) \quad \in \quad f\left(x(k), \mathcal{U}^*_\gamma(x(k))\right) \quad =: \quad F^*_\gamma(x(k))$$

where $f(x, \mathcal{U}^*_\gamma(x))$ is the set $\{f(x,u) \,:\, u \in \mathcal{U}^*_\gamma(x)\}$ for $x \in \mathbb{R}^n$.

Lyapunov analysis: main idea

Consider

$$V_\gamma(x) = \inf_{\mathbf{u}} \sum_{k=0}^{\infty} \gamma^k \ell(x(k), u(k))$$

- Controllability assumption $\Rightarrow V_\gamma(x) \leq \overline{\alpha}_V(\sigma(x))$
- We have

$$V_\gamma(x) = \sum_{k=0}^{\infty} \gamma^k \ell(x(k), u^*(k)) = \ell(x, u^*(0)) + \gamma \ell(x(1), u^*(1)) + \ldots \geq \ell(x, u^*(0))$$

Suppose the detectability condition holds with $W = 0$,

$$W(f(x, u^*(0))) - W(x) = 0 \leq -\alpha_W(\sigma(x)) + \ell(x, u^*(0))$$

Hence

$$\alpha_W(\sigma(x)) \leq \ell(x, u^*(0)) \leq V_\gamma(\sigma(x))$$

$\Rightarrow V_\gamma$ is positive definite and radially unbounded

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Lyapunov analysis: main idea (c'd)

- In view of Bellman equation

$$V_\gamma(x(0)) = \ell(x(0), u^*(0)) + \gamma V_\gamma(x(1))$$

therefore

$$\begin{aligned} V_\gamma(x(1)) - V_\gamma(x(0)) &= V_\gamma(x(1)) - \ell(x(0), u(0)) - \gamma V_\gamma(x(1)) \\ &= -\ell(x(0), u(0)) + (1-\gamma) V_\gamma(x(1)) \\ &\leq -\alpha_W(\sigma(x(0))) + (1-\gamma)\overline{\alpha}_V(\sigma(x(1))) \end{aligned}$$

After some manipulations

$$V_\gamma(x(1)) - V_\gamma(x(0)) \leq -\alpha_W(\sigma(x(0))) + \frac{1-\gamma}{\gamma}\overline{\alpha}_V(\sigma(x(0))).$$

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Lyapunov analysis: theorem

Theorem

There exist $\underline{\alpha}_Y, \overline{\alpha}_Y, \alpha_Y \in \mathcal{K}_\infty$, $\Upsilon \in \mathcal{KK}$ and for any $\gamma \in (0,1)$ there exists $Y_\gamma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the following holds.

(a) For any $x \in \mathbb{R}^n$,

$$\underline{\alpha}_Y(\sigma(x)) \leq Y_\gamma(x) \leq \overline{\alpha}_Y(\sigma(x)).$$

(b) For any $x \in \mathbb{R}^n$, $\upsilon \in F^*_\gamma(x)$,

$$Y_\gamma(\upsilon) - Y_\gamma(x) \leq -\alpha_Y(\sigma(x)) + \Upsilon(\sigma(x), \frac{1-\gamma}{\gamma}).$$

Lyapunov function: $Y_\gamma = V + W$ or $Y_\gamma = \rho_V(V_\gamma) + \rho_W(W)$ with $\rho_V, \rho_W \in \mathcal{K}_\infty$

Main stability result

Theorem [Uniform semiglobal practical stability]

$\exists \beta \in \mathcal{KL}$ such that $\forall \delta, \Delta > 0$, $\exists \gamma^\star \in (0,1)$ such that $\forall \gamma \in (\gamma^\star, 1)$ and $\forall x \in \{z \in \mathbb{R}^n \, : \, \sigma(z) \leq \Delta\}$, any solution to the system satisfies

$$\sigma(\phi(k,x)) \quad \leq \quad \max\{\beta(\sigma(x),k),\delta\} \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

$\phi(k,x)$: solution at k initialized at x

Link with [Grimm et al., IEEE TAC 2005]

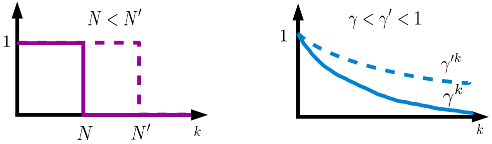
Cost functions of the form

$$J(x, \mathbf{u}) \quad := \quad \sum_{k=0}^{\infty} \xi(k)\ell(x(k), u(k)),$$

- In [Grimm et al., IEEE TAC 2005],

$$\xi(k) \quad = \quad \begin{cases} 1 & \text{when } k \leq N \\ 0 & \text{when } k > N \end{cases}$$

- For us, $\xi(k) = \gamma^k$



Corollaries

Under additional conditions on the comparison functions

- Uniform semiglobal asymptotic stability
- Uniform global exponential stability

Recall:

$$\begin{cases} V_\gamma(x) \leq \overline{\alpha}_V(\sigma(x)) \quad \text{and} \quad W(x) \leq \overline{\alpha}_W(\sigma(x)) \\ W(f(x,u)) - W(x) \leq -\alpha_W(\sigma(x)) + \chi_W(\ell(x,u)) \end{cases}$$

Explicit lower bounds on γ in all cases

Tailored bounds for linear systems with quadratic stage cost

Examples

- Simple example:

$$x(k+1) = 2x(k) + u(k), \quad \ell(x,u) = x^2 + u^2$$

$\rightarrow$ UGES for $\gamma^* = 0.8090$
$\rightarrow$ True value $\gamma^* = \frac{1}{3}$: 142% mismatch

- Linearized inverted pendulum

$$x(k+1) = Ax(k) + Bu(k), \quad \ell(x,u) = x^T Qx + u^T Ru$$

with $Q = C^{\mathrm{T}}C$, $C = [1000\ 0]$ (the pair (A, C) is observable) and $R = 1$
$\rightarrow$ UGES for $\gamma^* = 0.9878$
$\rightarrow$ True estimated value $\gamma^* = 0.9063$: 8% mismatch

Examples

- Simple example:

$$x(k+1) = 2x(k) + u(k), \quad \ell(x,u) = x^2 + u^2$$

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Examples (c'd)

- A nonholonomic integrator[1]

$$\begin{aligned} x_1(k+1) &= x_1(k) + u_1(k) \\ x_2(k+1) &= x_2(k) + u_2(k) \\ x_3(k+1) &= x_3(k) + x_1(k)u_2(k) - x_2(k)u_1(k), \end{aligned}$$

with

$$\ell(x,u) = x_1^2 + x_2^2 + 10|x_3| + |u|^2$$

$\rightarrow$ UGES for $\gamma^* = \frac{22}{25}$

[1] Like in [Grimm et al., IEEE TAC 2005]

Plan

What about robustness?

Arbitrarily small vanishing perturbations might destroy stability

Example in the context of model predictive control in [Grimm et al., Automatica 2004]

Robust stability

[Kellett and Teel, SIAM J. of Contr. and Optim. 2005]

• Continuous Lyapunov function Y_γ
in our case
$$Y_\gamma = \rho_V(V_\gamma) + \rho_W(W),$$
with $\rho_V, \rho_W \in \mathcal{K}_\infty$
$\rightarrow$ continuity of V_γ to be proved

• F_γ^* maps compacts into compacts and is non-empty
(recall $x(k+1) \in f\left(x(k), \mathcal{U}_\gamma^*(x(k))\right) = F_\gamma^*(x(k))$)

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(recall $x(k+1) \in f\big(x(k), \mathcal{U}^*_\gamma(x(k))\big) = F^*_\gamma(x(k))$)

Continuity of V_γ

Theorem

Suppose the following holds.

(a) Previous assumptions are satisfied.

(b) f and ℓ are continuous on $\mathcal{W}$ and $\mathcal{U} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is continuous and locally bounded on $\mathbb{R}^n$ (*recall* $u \in \mathcal{U}(x)$).

(c) For any $M \geq 0$, the set $\{x : \sigma(x) \leq M\}$ is compact.

For any $\Delta > 0$, there exists $\gamma^\star \in (0,1)$ such that for any $\gamma \in (\gamma^\star, 1)$, V_γ is continuous on $\{x \in \mathbb{R}^n : \sigma(x) \leq \Delta\}$. □

Proof based on [Kellett and Teel, SCL 2004]

Remark: $\gamma^\star$ independent of Δ under stronger conditions

Plan

What we mean by near-optimal inputs

[Near-optimality]

- $\exists \widehat{\alpha} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous, positive semi-definite
- $\exists \eta \geq 0$

such that $\forall x \in \mathbb{R}^n$ and $\forall \gamma \in (\overline{\gamma}, 1)$ with $\overline{\gamma} \in [0,1)$, $\exists \widehat{\mathbf{u}}_\gamma(x)$ such that

$$V_\gamma(x) \leq \widehat{V}_\gamma(x) := J_\gamma(x, \widehat{\mathbf{u}}_\gamma(x)) \leq V_\gamma(x) + \widehat{\alpha}(\sigma(x)) + \eta.$$

[Dynamic programming relationship]

$$\widehat{V}_\gamma(x) = \ell(x, \widehat{u}_{\gamma,0}(x)) + \gamma \widehat{V}_\gamma(\widehat{v}) \quad \forall x \in \mathbb{R}^n$$

where $\widehat{u}_{\gamma,0}(x)$ is the first element of $\widehat{\mathbf{u}}_\gamma(x)$ and $\widehat{v} := f(x, \widehat{u}_{\gamma,0}(x))$.

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Stability result

Closed-loop system

$$x(k+1) \quad \in \quad f\left(x(k), \widehat{\mathcal{U}}_\gamma(x(k))\right) =: \widehat{F}_\gamma(x(k)).$$

Theorem [Uniform semiglobal practical stability]

Under the previous assumptions, $\exists \beta \in \mathcal{KL}$ and $\vartheta \in \mathcal{K}_\infty$ such that $\forall \delta, \Delta > 0$, $\exists \gamma^\star \in (\overline{\gamma}, 1)$ such that $\forall \gamma \in (\gamma^\star, 1)$ and $\forall x \in \{z \in \mathbb{R}^n : \sigma(z) \leq \Delta\}$, any solution $\phi(\cdot, x)$ satisfies

$$\sigma(\phi(k,x)) \quad \leq \quad \max\{\beta(\sigma(x), k), \delta, \vartheta(\eta)\} \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Continuity of $\widehat{V}_\gamma$ also ensured

Plan

Reverse-discounted cost

What if $\gamma > 1$?

- Existence of optimal inputs assumed ([Keerthi and Gilbert, IEEE TAC 1985])
- Detectability assumption remains the same (as it is independent of γ)
- Controllability assumption: $V_\gamma(\sigma(x)) \leq \overline{\alpha}_V(\sigma(x))$ for any $\gamma \in [1, \overline{\gamma})$, $\overline{\gamma} \in [0, \infty]$

Reverse-discounted cost (c'd)

Theorem [Uniform global asymptotic stability]

$\forall \check{\gamma} \in (0,1)$, $\exists \beta \in \mathcal{KL}$ such that $\forall \gamma \in (0, \check{\gamma})$, $x \in \mathbb{R}^n$, any solution $\phi(\cdot, x)$ to the system satisfies

$$\sigma(\phi(k,x)) \leq \beta(\sigma(x), k),$$

for any $k \in \mathbb{Z}_{\geq 0}$.

Idea of the proof:

$$\begin{aligned} V_\gamma(x(1)) - V_\gamma(x(0)) &\leq -\alpha_W(\sigma(x(0))) + (1-\gamma)\overline{\alpha}_V(\sigma(x(1))) \\ &\leq -\alpha_W(\sigma(x(0))) \end{aligned}$$

Reverse-discounted cost (c'd)

Similar result for costs

$$J(x, \mathbf{u}) := \sum_{k=0}^{\infty} (1 - \gamma^{k+1}) \ell(x(k), u(k))$$

with $\gamma \geq 1$.

Optimal value functions of the discounted and the undiscounted problems

Assumption on the undiscounted problem

- There exists an optimal sequence of inputs when $\gamma = 1$ for any $x \in \mathbb{R}^n$. The optimal value function is denoted $\overline{V}(x)$.
- $\exists \bar{a}_V > 0$ such that $\forall x \in \mathbb{R}^n$, $\overline{V}(x) \leq \bar{a}_V \sigma(x)$.

Assumption on the discounted problem

- Controllability: $V_\gamma(x) \leq \bar{a}_V \sigma(x)$
- Detectability: $W(x) \leq \bar{a}_W(\sigma(x))$ and $W(f(x,u)) - W(x) \leq -a_W \sigma(x) + \ell(x,u)$.

$\Rightarrow$ UGES

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Optimal value functions of the discounted and the undiscounted problems

Theorem

Let

$$\gamma^\star > \frac{\overline{a}_V}{\overline{a}_V + a_W},$$

then $\forall \gamma \in (\gamma^\star, 1)$ and $\forall x \in \mathbb{R}^n$,

$$V_\gamma(x) \leq \overline{V}(x) \leq V_\gamma(x) + (1-\gamma)\theta(\gamma)(V_\gamma(x) + W(x))$$

where $\theta(\gamma)$ is given.

Tailored result for linear systems with quadratic costs

Plan

Conclusions

- General conditions for the stability of optimal control problems with discounted cost
- Continuity of the value function and robustness
- Results applicable for near-optimal inputs
- Relationship between the optimal value functions $\gamma \in (0,1)$ / $\gamma = 1$
- Reverse-discounted cost
- Results for uniformly bounded stage costs [Postoyan et al., IEEE CDC 2014]

R. Postoyan, L. Busoniu, D. Nešić and J. Daafouz, *Stability analysis of discrete-time infinite-horizon optimal control with discounted cost*, IEEE Transactions on Automatic Control, available on IEEE Xplore

Definitions

Definition [Rockafellar & Wets, 1998]

The mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *locally bounded* when for any $\bar{x} \in \mathbb{R}^n$, for some neighborhood $\mathcal{V}$ of $\bar{x}$, the set $S(\mathcal{V}) \subset \mathbb{R}^m$ is bounded.

Definition [Rockafellar & Wets, 1998]

We denote

$$\limsup_{x \to \bar{x}} S(x) := \left\{ u \;:\; \exists \{x_n\}_{n \in \mathbb{Z}_{\geq 0}} \; \exists \{u_n\}_{n \in \mathbb{Z}_{\geq 0}} \text{ s.t. } x_n \underset{n \to \infty}{\to} \bar{x},\; u_n \underset{n \to \infty}{\to} u \text{ with } u_n \in S(x_n) \right\}$$

$$\liminf_{x \to \bar{x}} S(x) := \left\{ u \;:\; \forall \{x_n\}_{n \in \mathbb{Z}_{\geq 0}} \text{ s.t. } x_n \underset{n \to \infty}{\to} \bar{x},\; \exists \phi \in \mathcal{N} \; \exists \{u_{\phi(n)}\}_{n \in \mathbb{Z}_{\geq 0}} \text{s.t. } u_{\phi(n)} \underset{n \to \infty}{\to} u \text{ with } u_n \in S(x_n) \right\}$$

where $\mathcal{N}$ is the set of strictly increasing functions from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$.

The set-valued mapping S is *continuous* at $\bar{x} \in \mathbb{R}^n$ when

$$\limsup_{x \to \bar{x}} S(x) = \liminf_{x \to \bar{x}} S(x) = S(\bar{x}) \text{ as } x \to \bar{x},$$

and it is continuous on $X \subseteq \mathbb{R}^n$ when it is continuous at any $\bar{x} \in X$.