Stabilizability and control co-design for discrete-time switched linear systems

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Outline

Collegues

Thanks to my collegues:

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Stabilizability of DT linear switched systems

Discrete-time autonomous switched system

$$x_{k+1} = A_{\sigma(k)} x_k,$$

where $\sigma : \mathbb{N} \to \mathbb{N}_q$ selects the transition matrix $\{A_i\}_{i \in \mathbb{N}_q}$, and can be considered as:

- a perturbation: necessary and sufficient condition for asymptotic stability; existence of a polyhedral Lyapunov function *(Molchanov & Pyatnitskiy, SCL89; Blanchini, AUT95)*,
- or as a control input: sufficient condition for stabilizability, Lyapunov-Metzler inequality *(Geromel & Colaneri, IJC06)*, necessary and sufficient *(Sun & Ge, 2011)*.

Objectives and contributions *(F. & Jungers, IFAC13, AUT13)*:

- provide necessary and sufficient condition for stabilizability,
- set-theory and invariance based results,
- computational espects: algorithmic test,
- nonconvex control Lyapunov functions,
- highlight the duality with the perturbation case,
- characterize the class of stabilizing controls.

Necessary and sufficient condition for stabilizability

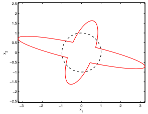
Algorithm 1

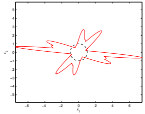
Control λ-contractive C*-set for the switched system.

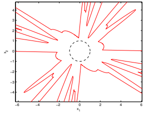
- **Initialization**: given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\begin{aligned} \Omega_{k+1}^i &= A_i^{-1}\Omega_k, \quad \forall i \in \mathbb{N}_q, \\ \Omega_{k+1} &= \bigcup_{i \in \mathbb{N}_q} \Omega_{k+1}^i; \end{aligned}$$

- **Stop** if $\Omega \subseteq \mathrm{int}\left(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j\right)$; denote $\check{N} = k+1$ and $\check{\Omega} = \bigcup_{j \in \mathbb{N}_{\check{N}}} \Omega_j$.







Geometrical interpretation:

- the set Ω_k^i is the set of x that can be stirred in Ω in k steps by a switching sequence beginning with $i \in \mathbb{N}_q$;
- then Ω_k is the set of points that can be driven in Ω in k steps;
- and hence $\check{\Omega}$ the set of those which can reach Ω in $\check{N}$ or less steps, by an adequate switching law.

Necessary and sufficient condition for stabilizability.

Theorem

There exists a control Lyapunov function for the switched system if and only if the Algorithm 1 ends with finite $\check{N}$.

Robustness-control duality

Uncertain linear systems

Robust λ-contractive C-set for an uncertain system.

- **Initialization**: given the C-set $\Gamma \subseteq \mathbb{R}^n$ and $\lambda \in [0,1]$, define $\Gamma_0 = \Gamma$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\begin{aligned} \Gamma^i_{k+1} &= \lambda A_i^{-1} \Gamma_k, \quad \forall i \in \mathbb{N}_q, \\ \Gamma_{k+1} &= \Gamma \cap \bigcap_{i \in \mathbb{N}_q} \Gamma^i_{k+1}; \end{aligned}$$

- **Stop** if $\Gamma_k \subseteq \Gamma_{k+1}$; denote $\hat{N} = k$ and $\hat{\Gamma} = \Gamma_k$.

Theorem (*Blanchini, AUT95*)

There is a Lyapunov function for the parametric uncertain linear system if and only if there exists a polyhedral Lyapunov function for the system.

Then, the family of convex, homogeneous functions induced by a C-set are a class of universal Lyapunov functions for parametric uncertain linear systems.

Switched linear systems

Control λ-contractive C*-set for the switched system.

- **Initialization**: given the C*-set $\Omega \subseteq \mathbb{R}^n$, define $\Omega_0 = \Omega$ and $k = 0$;
- **Iteration** for $k \geq 0$:

$$\begin{aligned} \Omega^i_{k+1} &= A_i^{-1} \Omega_k, \quad \forall i \in \mathbb{N}_q, \\ \Omega_{k+1} &= \bigcup_{i \in \mathbb{N}_q} \Omega^i_{k+1}; \end{aligned}$$

- **Stop** if $\Omega \subseteq \operatorname{int}\Big(\bigcup_{j \in \mathbb{N}_{k+1}} \Omega_j\Big)$; denote $\check{N} = k+1$ and

$$\check{\Omega} = \bigcup_{j \in \mathbb{N}_{\check{N}}} \Omega_j.$$

Theorem (*F. & Jungers, AUT13*)

There exists a control Lyapunov function for the switched linear system if and only if the Algorithm ends with finite $\check{N}$.

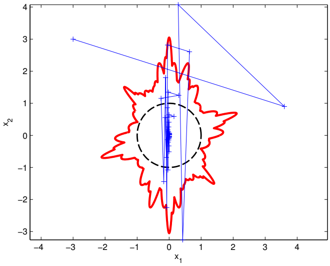
Then, the family of nonconvex, homogeneous functions induced by a C*–set are a class of universal Lyapunov functions for switched systems.

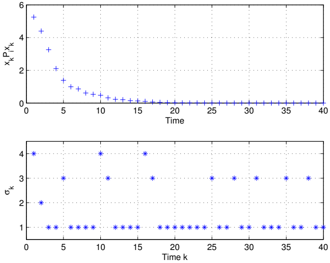
Example 1

System with $q = 4$, $n = 2$ and

$$A_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.8 \end{bmatrix}, \qquad A_2 = 1.1\,R(\tfrac{2\pi}{5})$$
$$A_3 = 1.05\,R(\tfrac{2\pi}{5} - 1), \qquad A_4 = \begin{bmatrix} -1.2 & 0 \\ 1 & 1.3 \end{bmatrix}.$$

The matrices A_i, with $i \in \mathbb{N}_4$, are not Schur. Notice: only one stable eigenvalue!





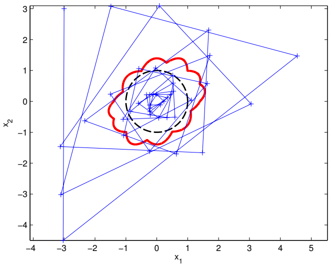
Example 2

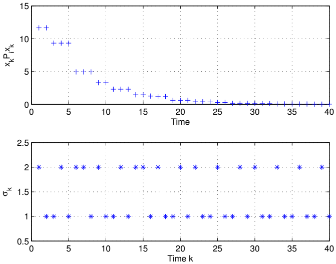
Switched system with

$$A_1 = \begin{bmatrix} 0 & -1.01 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -1.01 \\ 1 & -0.5 \end{bmatrix}.$$

The technique based on Lyapunov-Metzler inequalities *(Geromel & Colaneri, IJC06)* has been numerically checked (gridding) and it results not feasible.

Nevertheless...



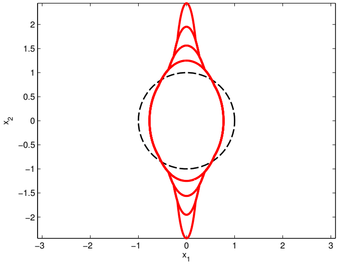


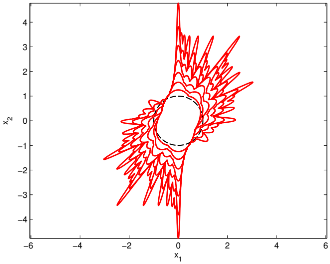
Example 3

Switched system with

$$A_1 = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.8 \end{bmatrix},$$

for $\theta = 0$ (left) and $\theta = \frac{\pi}{5}$ (right).



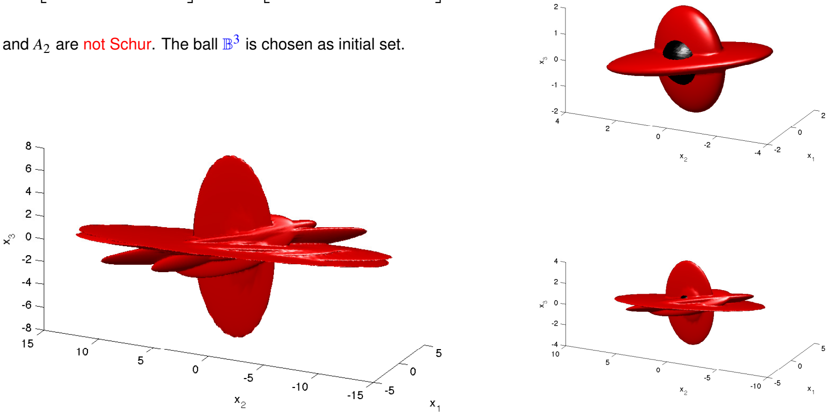


Example 4

Switched system with $q = 2$, $n = 3$ and

$$A_1 = \begin{bmatrix} 1.2 & 0 & 0 \\ -1 & 0.8 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & -0.6 & -2 \\ 0 & 0 & -1.2 \end{bmatrix}.$$

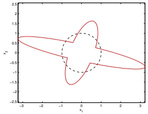
A_1 and A_2 are not Schur. The ball $\mathbb{B}^3$ is chosen as initial set.



Necessary and sufficient condition

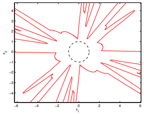
Necessary and sufficient condition for stabilizability: existence of $N \in \mathbb{N}$ such that

$$\Omega \subseteq \text{int}\Big(\bigcup_{i \in \mathscr{I}^{[1:N]}} \Omega_i\Big) \quad \text{with} \quad \begin{array}{l} \mathbb{A}_i = \prod_{j=1}^{k} A_{i_j} = A_{i_k} \cdots A_{i_1}, \\ \Omega_i = \Omega_i(\Omega) = \{x \in \mathbb{R}^n : \ \mathbb{A}_i x \in \Omega\}, \end{array}$$

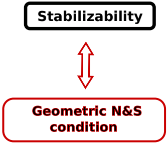


with Ω a given C*-set. It does not depend on the C*-set Ω, then for $\Omega = \mathbb{B}$ we have

$$\mathbb{B} \subseteq \text{int}\Big(\bigcup_{i \in \mathscr{I}^{[1:N]}} \mathbb{B}_i\Big), \quad \text{with} \quad \begin{array}{l} \mathbb{B}_i = \{x \in \mathbb{R}^n : \ x^T \mathbb{A}_i^T \mathbb{A}_i x \leq 1\}, \\ \mathbb{B} = \{x \in \mathbb{R}^n : \ x^T x \leq 1\} \end{array}$$



- stabilizability does not depend on Ω, but N does.
- The set inclusions are the stop conditions to checked numerically at every step;
- i.e. if a C*-set Ω is in the interior of the union of C*-sets $\Rightarrow$ very complex in general;
- but, it provides the exact characterization of the complexity of the problem.



The objectives:

- alternative, computationally suitable, conditions for stabilizability;
- provide geometrical and numerical insights;
- analyze their conservatism by comparison with the necessary and sufficient one.

Lyapunov-Metzler BMI conditions

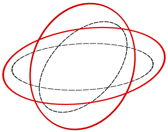
Lyapunov-Metzler condition is sufficient and given by BMI inequalities involving the Metzler matrices.

Theorem (*Geromel & Colaneri, IJC06*)

If there exist $P_i > 0$, with $i \in \mathscr{I}$, and $\pi \in \mathscr{M}_q$ i.e. matrices $\pi \in \mathbb{R}^{q\times q}$ whose elements are nonnegative and $\sum_{j=1}^{q} \pi_{ji} = 1$ for all $i \in \mathbb{N}_q$, such that

$$A_i^T \left(\sum_{j=1}^{q} \pi_{ji} P_j \right) A_i - P_i < 0, \qquad \forall i \in \mathscr{I},$$

holds, then the switched system is stabilizable.



The condition implies that the homogeneous function induced by $\bigcup_{i\in\mathscr{I}} \mathscr{E}(P_i)$ is a control Lyapunov function.

Theorem

If the Lyapunov-Metzler condition holds then the geometric one holds with $N = 1$ and $\Omega = \bigcup_{i\in\mathscr{I}} A_i \mathscr{E}(P_i)$.

Proposition

For $q = 2$, the Lyapunov-Metzler condition is equivalent to $A_1\mathscr{E}(P_1) \cup A_2\mathscr{E}(P_2) \subseteq int(\mathscr{E}(P_1) \cup \mathscr{E}(P_2))$.

Thus, the Lyapunov-Metzler condition for $q = 2$ is equivalent to the existence of a contractive set formed by two ellipsoids, and then to a Lyapunov function given by the pointwise minimum of two quadratic functions.

Generalized Lyapunov-Metzler conditions

First generalization: remove the link between the number of ellipsoids (and matrices P_i) and the system modes.

Proposition

If there exist $M \in \mathbb{N}$ and $P_i > 0$, with $i \in \mathscr{I}^{[1:M]}$, and $\pi \in \mathscr{M}_{\bar{M}}$ such that

$$\mathbb{A}_i^T \left(\sum_{j \in \mathscr{I}^{[1:M]}} \pi_{ji} P_j \right) \mathbb{A}_i - P_i < 0, \qquad \forall i \in \mathscr{I}^{[1:M]},$$

holds, then the switched system is stabilizable.

Meaning: Lyapunov Metzler condition for switched system with one fictitious mode for every matrix $\mathbb{A}_i$ with $i \in \mathscr{I}^{[1:M]}$.

Second generalization: to maintain the sequence length in 1 but increase the number of ellipsoids.

Proposition

If for every $i \in \mathscr{I}$ there exist a set of indices $\mathscr{K}_i = \mathbb{N}_{h_i}$, with $h_i \in \mathbb{N}$; a set of matrices $P_k^{(i)} > 0$, with $k \in \mathscr{K}_i$, and there are $\pi_{m,k}^{(p,i)} \in [0,1]$, satisfying

$$\sum_{p \in \mathscr{I}} \sum_{m \in \mathscr{K}_p} \pi_{m,k}^{(p,i)} = 1,$$

for all $k \in \mathscr{K}_i$, such that

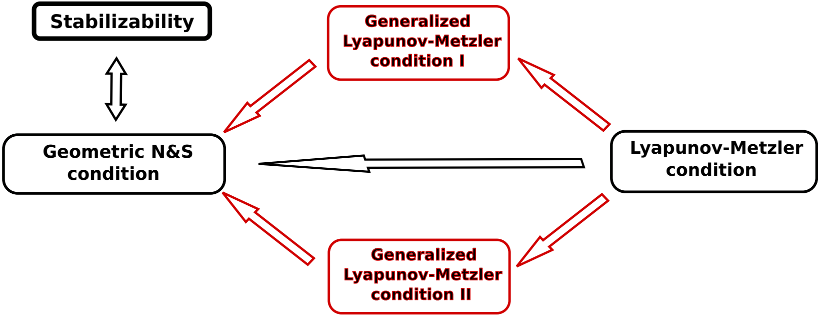
$$A_i^T \left(\sum_{p \in \mathscr{I}} \sum_{m \in \mathscr{K}_p} \pi_{m,k}^{(p,i)} P_m^{(p)} \right) A_i - P_k^{(i)} < 0, \qquad \forall i \in \mathscr{I}, \forall k \in \mathscr{K}_i$$

holds, then the switched system is stabilizable.

Geometrically: there exists a C*-set composed by a finite number of ellipsoids ($P_k^{(i)}$ with $k \in \mathscr{K}_i$) contractive.

Notice: the classical Lyapunov Metzler condition is a particular case.

Stabilizability conditions relations



LMI sufficient condition

Problem with Lyapunov-Metzler: non convex condition.

Our next aim is to formulate an alternative condition that could be checked efficiently, a convex one.

Theorem

The switched system is stabilizable if there exist $N \in \mathbb{N}$ and $\eta \in \mathbb{R}^{\bar{N}}$ such that $\eta \geq 0$, $\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i = 1$ and

$$\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I.$$

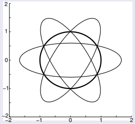
The condition is just sufficient (except for particular cases), is it also necessary? No!

Counterexample

Consider the three modes given by the matrices

$$A_1 = AR(0), \quad A_2 = AR\left(\frac{2\pi}{3}\right), \quad A_3 = AR\left(\frac{-2\pi}{3}\right), \quad \text{with} \quad A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

and $a = 0.6$. The geometric condition holds with $N = 1$.

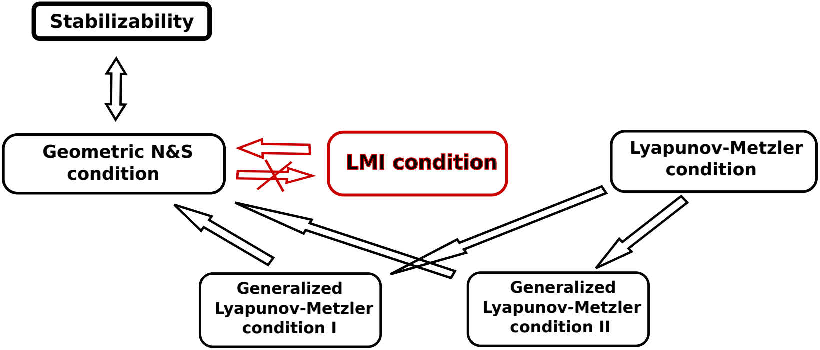


For every N and every $\mathbb{B}_i$ with $i \in \mathscr{I}^{[1:N]}$, the related $\mathbb{A}_i$ is such that $\det(\mathbb{A}_i^T \mathbb{A}_i) = 1$ and $\mathrm{Tr}(\mathbb{A}_i^T \mathbb{A}_i) \geq 2$.

Notice that, for all the matrices $Q > 0$ in $\mathbb{R}^{2\times 2}$ such that $\det(Q) = 1$, then $\mathrm{Tr}(Q) \geq 2$ and $\mathrm{Tr}(Q) = 2$ if and only if $Q = I$.

Thus, for every subset $K \subseteq \mathscr{I}^{[1:N]}$, we have that $\sum_{i \in K} \eta_i \mathbb{A}_i^T \mathbb{A}_i < I$, cannot hold, since either $\mathrm{Tr}(\mathbb{A}_i^T \mathbb{A}_i) > 2$ or $\mathbb{A}_i^T \mathbb{A}_i = I$.

Stabilizability conditions relations



LMI-based control Lyapunov functions

The LMI condition provides control Lyapunov functions and the controller synthesis:

- at $p \in \mathbb{N}$ get $\quad i_p = \arg \min_{i \in \mathscr{I}^{[1:N]}} (x_{k_p}^T \mathbb{A}_i^T \mathbb{A}_i x_{k_p})$.
- the next instant k_{p+1} is given by $\quad k_{p+1} = k_p + l(i_p)$, with $l(i_p)$ length of i_p
- the controller inputs are $\quad \sigma_{k_p+j-1} = i_{p,j}, \quad \forall j \in \{1, \dots, l(i_p)\}$.

Theorem

Assume the LMI condition holds, and the control above is applied. Then, for all $x_0 \in \mathbb{R}^n$, for all $k \in \mathbb{N}$,

$$\|x_k\| \leq \mu^{k/N-1} L^{N-1} \|x_0\| \qquad \text{with } \mu \in [0,1) \text{ such that} \qquad \sum_{i \in \mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i \leq \mu^2 I,$$

where $L \geq \|A_i\|$, for all $i \in \mathscr{I}$. Then, the controlled switched system is globally exponentially stable.

Nevertheless, neither the Euclidean norm of x nor the function $\min_{i \in \mathscr{I}^{[1:N]}} (x^T \mathbb{A}_i^T \mathbb{A}_i x)$ are monotonically decreasing.

On the other hand a positive definite homogeneous non-convex function decreasing at every step can be inferred.

Proposition

Suppose the LMI condition hold. Then there is $\lambda \in [0,1)$ such that defining

$$V(x) = \min_{i \in \mathscr{I}^{[1:N]}} (x^T \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i x) \qquad \text{and} \qquad i^*(x) = \arg \min_{i \in \mathscr{I}^{[1:N]}} (x^T \lambda^{-n_i} \mathbb{A}_i^T \mathbb{A}_i x)$$

where n_i is the length of $i \in \mathscr{I}^{[1:N]}$ and $\sigma(x) = i_1^*(x)$, are such that $V(A_{\sigma(x)} x) \leq \lambda V(x)$ for all $x \in \mathbb{R}^n$.

LMI-condition and periodic stabilizability

A periodic switching law is given by $\sigma(k) = i_{p(k)}$ and

$$p(k) = k - M\lfloor k/M \rfloor + 1,$$

with $M \in \mathbb{N}$ and $i \in \mathscr{I}^M$, which means that the sequence of modes given by i repeats cyclically in time.

The stabilizability through periodic switching law, i.e. periodic stabilizability, is formalized below.

Definition

The switched system is periodic stabilizable if there exist a periodic switching law $\sigma : \mathbb{N} \to \mathbb{N}_q$, $c \geq 0$ and $\lambda \in [0,1)$ such that the system is stabilizable for all $x \in \mathbb{R}^n$.

Notice that for stabilizability the switching function might be state-dependent, hence a state feedback, whereas for having periodic stabilizability the switching law must be independent on the state.

Is there an equivalence relation between periodic stabilizability and the LMI condition? The answer is below.

Theorem

A stabilizing periodic switching law for the switched system exists if and only if the LMI condition holds.

Sketch of the proof

Necessity: stabilizing periodic switching law $\Rightarrow$ implies satisfaction of LMI is direct.

Sufficiency: from LMI condition then

$$\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i \mathbb{A}_i^T \mathbb{A}_i \leq \lambda I \quad \Rightarrow \quad \sum_{i\in\mathscr{I}^{[1:N]}} \sum_{j\in\mathscr{I}^{[1:N]}} \eta_i \eta_j \mathbb{A}_i^T \mathbb{A}_j^T \mathbb{A}_j \mathbb{A}_i \leq \lambda^2 I \quad \Rightarrow \quad \sum_{I\in\mathscr{I}^{[q,Nq]}} \eta_I \mathbb{A}_I^T \mathbb{A}_I \leq \lambda^q I,$$

with $\lambda \in [0,1)$ and for all $q \in \mathbb{N}$, where for every $I = (i_1,\ldots,i_q) \in \mathscr{I}^{[q,Nq]}$ we define

$$\eta_I = \prod_{k\in\mathbb{N}_q} \eta_{i_k}, \qquad \mathbb{A}_I = \prod_{k\in\mathbb{N}_q} \mathbb{A}_{i_k}.$$

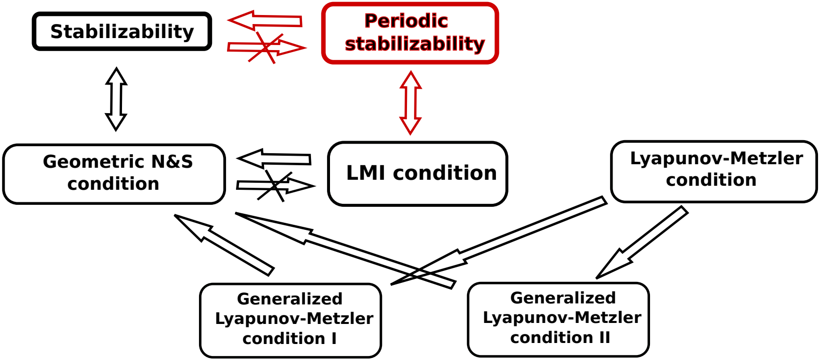
From the linearity of the trace and the fact that $\sum_{I\in\mathscr{I}^{[q,Nq]}} \eta_I = 1$, we have that

$$\min_{I\in\mathscr{I}^{[q,Nq]}} \mathrm{Tr}(\mathbb{A}_I^T \mathbb{A}_I) = \sum_{I\in\mathscr{I}^{[q,Nq]}} \eta_I \min_{I\in\mathscr{I}^{[q,Nq]}} \mathrm{Tr}(\mathbb{A}_I^T \mathbb{A}_I) \leq \sum_{I\in\mathscr{I}^{[q,Nq]}} \eta_I \,\mathrm{Tr}(\mathbb{A}_I^T \mathbb{A}_I) = \mathrm{Tr}\Big(\sum_{I\in\mathscr{I}^{[q,Nq]}} \eta_I \mathbb{A}_I^T \mathbb{A}_I\Big) \leq \lambda^q n,$$

since $\mathrm{Tr}(\mathbb{A}_I^T \mathbb{A}_I) > 0$ for all $I \in \mathscr{I}^{[q,Nq]}$.

Thus, for q big enough, for which $\lambda^q n < 1$, there exists a $I^* \in \mathscr{I}^{[q,Nq]}$ such that $\mathrm{Tr}(\mathbb{A}_{I^*}^T \mathbb{A}_{I^*}) < 1$, which implies that $\mathbb{A}_{I^*}$ is Schur.

Stabilizability conditions relations



Relations between LMI and Lyapunov Metzler conditions

Relation between LMI and the generalized Lyapunov Metzler conditions.

Theorem

Generalized Lyapunov Metzler I $\Leftrightarrow$ LMI condition $\Leftrightarrow$ generalized Lyapunov Metzler II.

Relation between LMI and the Lyapunov Metzler condition.

Lemma

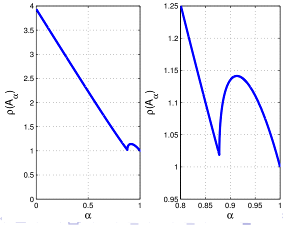
If the Lyapunov Metzler condition holds, then $\exists \alpha \in (0,1)$ such that $A_\alpha = \alpha A_1 + (1-\alpha)\frac{\sqrt{\rho(A_2)^2-1}}{\rho(A_2)} A_2 A_1$ is Schur.

Counterexample

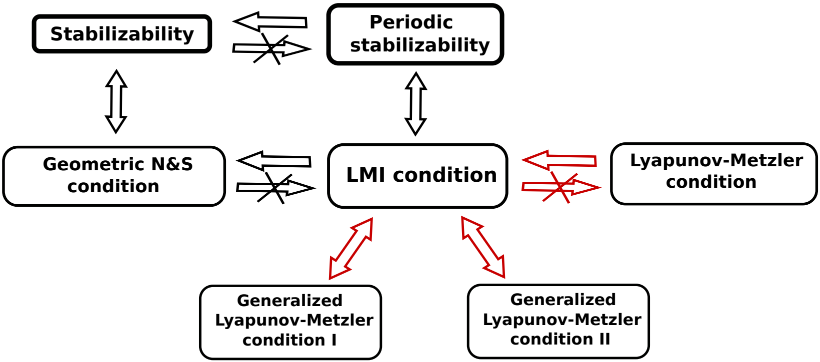
Consider the matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$.

Since $A_1^{11} A_2^2 = \begin{bmatrix} 0 & -32 \\ 1/64 & 0 \end{bmatrix}$ then the LMI condition holds with $N = 13$.

But, from the Lemma, the Lyapunov-Metzler inequalities cannot hold.



Stabilizability conditions relations



Numerical example

Consider the switched system with $q = 2$, $n = 2$, $x_0 = [-3, 3]^T$ and the non-Schur matrices

$$A_1 = 1.01 R\left(\frac{\pi}{5}\right), \qquad A_2 = \begin{bmatrix} -0.6 & -2 \\ 0 & -1.2 \end{bmatrix}.$$

Four different stabilizing switching laws are designed and compared.

Lyapunov-Metzler BMI condition:

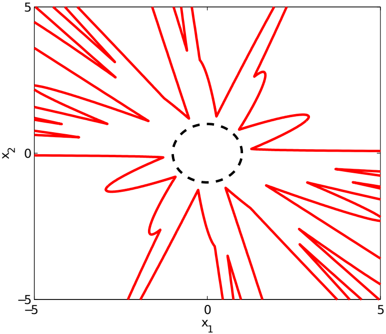
- for systems with $q = 2$ the Lyapunov-Metzler inequalities become two linear matrix inequalities once two parameters, both contained in $[0, 1]$, are fixed.
- the Lyapunov-Metzler inequalities being bilinear, there is no generic numerical method to solve them when a solution exists, except the gridding approach.
- Such LMIs have been checked for this example to be infeasible on a grid of these two parameters, with step of $0.01 \quad \Rightarrow \quad$ the Lyapunov-Metzler inequalities are (probably...) infeasible.
- The computational complexity is unmanageable as q increases.
- Moreover to circumvent the conservatism of the classical Lyapunov-Metzler inequalities, one should increase the problem dimension. Even worse.

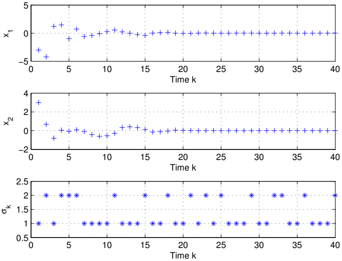
Conclusion: employing the Lyapunov-Metzler inequalities to prove stabilizability might often be computationally intractable, also for systems with few modes.

Numerical example

The geometric condition:

- holds with $N = 5$;
- the homogeneous function induced by the set is a control Lyapunov function and provide a stabilizing switching law.

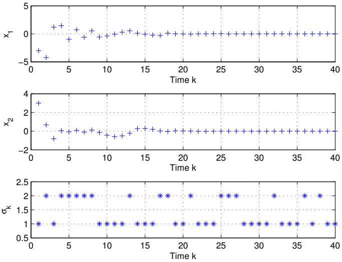


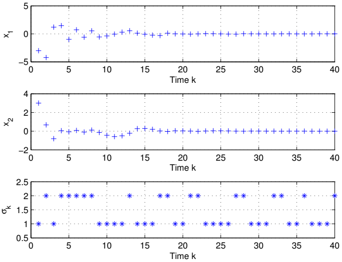


Numerical example

The LMI condition:

- is solved with $N = 7$;
- the results of the time-varying length, i.e. $\{7, 6, 5, 7, 7, \dots\}$ switching control are shown on the left;
- those of the optimization-based control law with $\lambda = 0.9661$, on the right.

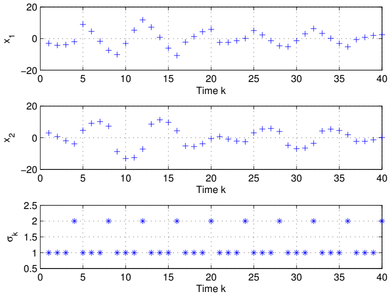




Numerical example

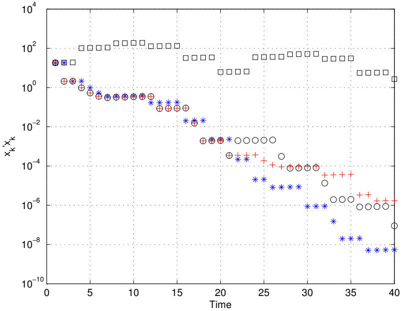
The periodic switching law:

- the shorter sequence of modes, length $M = 4$, which yields a decreasing of the Euclidean norm after its whole application.



Comparison between the different switching laws:

- Euclidean norm as a common measure to compare the convergence performances;
- geometric condition in blue star; LMI lenght-varying law in red cross; LMI and optimization control in black circle and periodic rule in black square.



Control co-design

Controlled discrete-time switched linear system

$$x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k,$$

Objective: design time-varying control policy $\nu : \mathbb{R}^n \times \mathbb{N} \to \mathscr{I} \times \mathbb{R}^{m \times n}$, is such that

$$\nu(x,k) = \big(\sigma(x,k),\ K(x,k)\big) \in \mathscr{I} \times \mathbb{R}^{m \times n}$$

stabilizes the system, with $u_k(x_k) = K(x_k,k)x_k$.

Remark

As proved in *(Zhang et alt, AUT09)*, no loss of generality to consider static control policies of the form

$$\nu(x) = \big(\sigma(x),\ K(x)\big) \in \mathscr{I} \times \mathbb{R}^{m \times n},$$

such that $\nu(ax) = \nu(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, and to piecewise quadratic Lyapunov functions.

Moreover $K(x)$ belongs to a finite set i.e. $K(x) \in \mathscr{K} = \{\kappa_i\}_{i \in \mathbb{N}_M}$, with $M \in \mathbb{N}$.

We are not interested in determining periodic stabilizing switching laws but on computing a state-dependent control policy whenever the system admits a periodic stabilizing switching sequence.

Control co-design

Given an horizon $N \in \mathbb{N}$, a set of $N_{\mathscr{I}} = \sum_{k=1}^{N} q^k$ matrices $\mathbb{F}_\vartheta$, one for every $\vartheta \in \mathscr{I}^{[1:N]}$, can be defined as

$$\mathbb{F}_\vartheta = \prod_{j=1}^{|\vartheta|} F_{\vartheta_j} = F_{\vartheta_J} \dots F_{\vartheta_1} = (A_{\vartheta_{|\vartheta|}} + B_{\vartheta_{|\vartheta|}} K^\vartheta_{|\vartheta|}) \dots (A_{\vartheta_1} + B_{\vartheta_1} K^\vartheta_1).$$

that are parameterized in $\{K^\vartheta_j\}_{j \in \mathbb{N}_{|\vartheta|}}$.

Then, $K(x)$ is a gain among the $\sum_{k=1}^{N} kq^k$ possible, i.e. $K(x) \in \mathscr{K}$ where

$$\mathscr{K} = \{\kappa_i\}_{i \in \mathbb{N}_M} = \{K^\vartheta_j \in \mathbb{R}^{m \times n} : \ \vartheta \in \mathscr{I}^{[1:N]}, \ j \in \mathbb{N}_{|\vartheta|}\}, \qquad \text{with} \quad M = \sum_{k=1}^{N} kq^k.$$

Periodic ϑ-stabilizability

The system is periodic ϑ-stabilizable if there exist: a periodic $\vartheta : \mathbb{N} \to \mathscr{I}$ and a periodic sequence $K^\vartheta : \mathbb{N} \to \mathbb{R}^{m \times n}$, both of cycle length $D \in \mathbb{N}$; $c \geq 0$ and $\lambda \in [0,1)$ such that $\|x^\vartheta_k(x)\| \leq c\lambda^k \|x\|$ holds for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

$$\text{Then:} \qquad \left.\begin{array}{l} \displaystyle\sum_{i \in \mathscr{I}^{[1:N]}} \eta_i = 1, \quad \eta \geq 0 \\ \displaystyle\sum_{j \in \mathscr{I}^{[1:N]}} \eta_j \mathbb{F}_j^T \mathbb{F}_j < I \end{array}\right\} \quad \Leftrightarrow \quad \vartheta\text{-stabilizability} \quad \Rightarrow \quad \text{stabilizability}$$

Problem: the second constraint is not convex in the variables η and $\{K^\vartheta_j\}_{j \in \mathbb{N}_M}$.

Control co-design

Necessary and sufficient LMI condition for the ϑ-stabilizability of switched systems.

Theorem

The controlled switched system is periodically ϑ-stabilizable if and only if there exist $N \in \mathbb{N}$; $\eta \in \mathbb{R}^{N_{\mathscr{I}}}$ such that $\eta > 0$ and $\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i = 1$ hold; and for every $j \in \mathscr{I}^{[1:N]}$ there are:

- $|j|-1$ *nonsingular matrices* $G_{j,k} \in \mathbb{R}^{n\times n}$, *with* $k \in \mathbb{N}_{|j|-1}$;
- $|j|$ *matrices* $Z_{j,k} \in \mathbb{R}^{m\times n}$ *with* $k \in \mathbb{N}_{|j|}$;
- *a symmetric positive definite matrix* $R_j \in \mathbb{R}^{n\times n}$;

such that

$$\begin{bmatrix} \eta_j I & X_{j,|j|} & 0 & \dots & 0 & 0 & 0 \\ X_{j,|j|}^T & Y_{j,|j|-1} & X_{j,|j|-1} & \dots & 0 & 0 & 0 \\ 0 & X_{j,|j|-1}^T & Y_{j,|j|-1} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & Y_{j,2} & X_{j,2} & 0 \\ 0 & 0 & 0 & \dots & X_{j,2}^T & Y_{j,1} & X_{j,1} \\ 0 & 0 & 0 & \dots & 0 & X_{j,1}^T & R_j \end{bmatrix} > 0, \qquad \sum_{j\in\mathscr{I}^{[1:N]}} R_j < I$$

for every $j \in \mathscr{I}^{[1:N]}$ *with*

$$\begin{cases} X_{j,1} = \eta_j A_{j_1} + B_{j_1} Z_{j,1}, & \\ X_{j,k+1} = A_{j_{k+1}} G_{j,k} + B_{j_{k+1}} Z_{j,k+1}, & \forall k \in \mathbb{N}_{|j|-1}, \\ Y_{j,k} = G_{j,k} + G_{j,k}^T, \quad \forall k \in \mathbb{N}_{|j|-1}, & \end{cases}$$

and gains $K_1^j = \eta_j^{-1} Z_{j,1}$, $K_{k+1}^j = Z_{j,k+1} G_{j,k}^{-1}$, *for all* $k \in \mathbb{N}_{|j|-1}$.

Control co-design

Theorem

Suppose there exist $\alpha > 1$ *and* $N \in \mathbb{N}$; $\eta \in \mathbb{R}^{N\mathscr{I}}$ *such that* $\eta > 0$; *matrices* $G_{j,k} \in \mathbb{R}^{n\times n}$ *with* $k \in \mathbb{N}_{|j|-1}$, $Z_{j,k} \in \mathbb{R}^{m\times n}$ *with* $k \in \mathbb{N}_{|j|}$ *and* $R_j \in \mathbb{R}^{n\times n}$ *satisfy the conditions above and*

$$\sum_{i\in\mathscr{I}^{[1:N]}} \eta_i = \alpha.$$

The system is periodically ϑ*-stabilizable and* $\|\mathbb{F}_{\vartheta(x)}x\|_2 < \lambda\|x\|_2$ *holds for all* $x \in \mathbb{R}^n$*, with*

$$\vartheta = \vartheta(x) = \arg\min_{j\in\mathscr{I}^{[1:N]}} (x^T \mathbb{F}_j^T \mathbb{F}_j x),$$

and $\lambda = \alpha^{-1/2}$*. Given* $x(t) = x$*, the stabilizing control policy is defined within an horizon of length* $|\vartheta|$ *as*

$$\nu(x,k) = (\sigma(x,k), K(x,k)) = \left(\vartheta_k, K_k^{\vartheta}\right)$$

to be applied at time $t+k-1$*, for all* $k \in \mathbb{N}_{|\vartheta|}$*.*

The value of α, is related to λ and then could serve for obtaining the fastest decreasing rate, for a given N.

Remark

A nonconvex control Lyapunov function $V(x)$, decreasing at every step, and a state-dependent control policy $\nu(x)$ can be defined from the solution of the LMI problem:

$$V(x) = \min_{j\in\mathscr{Y}_N} \left(x^T \lambda^{-|j|}\, \mathbb{F}_j^T \mathbb{F}_j x\right), \qquad \hat{j}(x) = \arg\min_{j\in\mathscr{Y}_N} \left(x^T \lambda^{-|j|}\, \mathbb{F}_j^T \mathbb{F}_j x\right).$$

where $\mathscr{Y}_N$ is the set of all suffixes of the elements of $\mathscr{I}^{[1:N]}$, and the control policy is $\nu(x) = (\hat{j}_1(x),\ K_1^{\hat{j}(x)})$.

Comparison with switched LQR method

As a first term of comparison, consider *(Zhang et alt, AUT09; Zhang et alt, TAC12)*:

- exponential stabilizability if and only if the infinite-horizon LQR problem leads to a control Lyapunov function;
- the method is based on the iterative application of the Riccati-like mapping

$$\rho_i(P) = Q_i + A_i^T PA_i - A_i^T PB_i(R_i + B_i^T PB_i)^{-1} B_i^T PA_i, \qquad \forall i \in \mathcal{I};$$

- it generates an increasing set of gains and positive definite matrices that eventually provide (an approximation of) the stabilizing LQR control and the related Lyapunov function;
- a stop condition, in form of contraction test, must be checked at every iteration.

Remarks

- constructive necessary and sufficient condition, no conservatism;
- number of matrices generated might grow exponentially, despite the redundancy reduction;
- redundancy test would entail additional computational burden;
- since the general condition could be overly complex, an only sufficient alternative, analogous to our, is employed in the relaxed version of the algorithm (still necessary?);
- not a “real” co-design.

Comparison

- in our approach the feedback gains are design variables;
- they are effectively computed by solving a single LMI problem, co-design;
- still exponential complexity, but on the “shortest” horizon;
- same conservatism as the stop condition only sufficient (periodicity).

Comparison with Lyapunov-Metzler-like conditions

The methods based on Lyapunov-Metzler conditions, *(Geromel et alt, TAC08, Deaecto et alt, AUT11)*:

- based on BMI conditions analogous to autonomous case:

$$\begin{bmatrix} Y & * & * & * & * & * \\ I & X_i & * & * & * & * \\ 0 & 0 & \rho I & * & * & * \\ YA_i + L_iC_i & M_i & YH_i + L_iD_i & Y & * & * \\ A_i & A_iX_i + B_iW_i & H_i & I & J_i + J_i - \sum_{j \in \mathscr{I}} \pi_{j,i} T_{i,j} & * \\ E_i & E_iX_i + F_iW_i & G_i & 0 & 0 & I \end{bmatrix} > 0, \qquad \forall j \in \mathscr{I}$$

Remarks

- fixed geometric complexity, then more conservative;
- nonconvex conditions, computationally demanding;
- "real" co-design.

Example 1

Given $n = 3$ and $q = 2$, define

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & -4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & a \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly the subsystem x_3 is stabilized by $u = Kx_3$ with $|a + K| < 1$.

The subsystem (x_1, x_2) is stabilizable through the LMI condition but no solution to the Lyapunov-Metzler condition.

Example 2 in *(Zhang et alt, AUT09)*

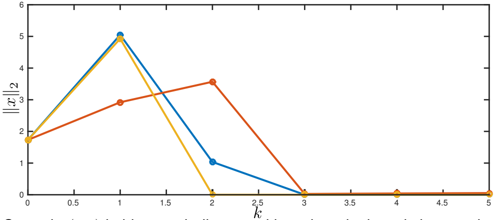
$$A_1 = \begin{bmatrix} 0.5 & -1 & 2 & 3 \\ 0 & -0.5 & 2 & 4 \\ 0 & -1 & 2.5 & 2 \\ 0 & 0 & 0 & 1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & -1 & 2 & 1 \\ 0 & 1.5 & -2 & 0 \\ 0 & 0 & 0.5 & 0 \\ -2 & -1 & 2 & 2.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & -0.5 \\ 0 & 0.5 & 1 & -0.5 \\ 1 & 0 & 0 & 0.5 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.5 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 2 & -2 & 0.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

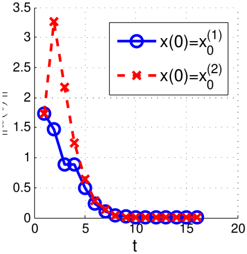
In *(Zhang et alt, AUT09)* stabilizability with horizon 7 and 13 matrices.

In our case, after solving the LMI problem with $N = 3$, we obtain $\alpha = 1149.2$, that implies $\lambda = 0.0295$ (deadbeat?).

The control related to $i = \{3, 4, 4\}$ leads to a Schur matrix whose spectral radius is 0.0069 (deadbeat?)



Control $v(x,k)$ in blue; periodic control in red; and min-switch control in yellow. IC $x(0) = x_0^{(1)}$.



Stabilizability and control co-design for discrete-time switched linear systems

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