Event triggered high-gain feedback

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Problem Statement

Consider a physical process described by a dynamical system with x in $\mathbb{R}^n$

$$\dot{x}(t) = f(t, x(t), u_k) \ , \ \ t \in [t_k, t_{k+1}) \ , \ \ f(t, 0, 0) = 0$$

where

- The sequence $(t_k)_{k\in\mathbb{N}}$ is the sampling times
- The control u_k is modified at the time instant t_k
  $\Rightarrow$ The control is constant between two sampling times!

Control Problem:

Based on the model, find the sequence $(t_k, u_k)_{k\in\mathbb{N}}$ such that the origin is globally and exponentially stable.

Problem Statement

The selection of the sequence $(t_k)_{k\in\mathbb{N}}$ follows an event

This is *event-based control*

The event may depend on

- the state at time t, $x(t)$
- an output at time t, $y(t)$
- some new additional variable

Interest of this problematic:

- Computation time on embedded systems
- Actuator limitation
- Network utilization
- Battery power

Event triggered state feedback

Event based control:

- Very active topic of research.
- Main contributor: M. Heemels, R. Tabuada, R. Postoyan, D. Nesic, A. Girard, A. Tanwani, C. Prieur, A. Teel, L. Zaccarian, A. Seuret, N. Marchand...
- Usual techniques are based on hybrid dynamical system theory of Teel and coworkers.
- It is assumed known a continuous time control law. Sampling is made and handled by robustness.

In our work:

- We don't assume that we know a priori a state feedback for the continuous time case.
- The design of the triggered mechanism and the state feedback are made jointly.

Mathematical formulation

Class of nonlinear systems : We consider a dynamical system in a specific form

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & x_2 + f_1(t, x_1) \\ \dot{x}_2 & = & x_3 + f_2(t, x_1, x_2) \\ & \vdots & \\ \dot{x}_n & = & u_k + f_n(t, x_1, \dots, x_n) \end{array}\right.$$

which can be rewritten

$$\dot{x}(t) = Ax(t) + Bu_k + f(t, x(t))$$

The functions f_i may not be well known for the design of the control.

High-gain approach

The model can be rewritten :

$$\underbrace{\dot{x} = A\,x + B\,u_k}_{\text{Chain of integrator part}} \quad + \quad \underbrace{f(t,x)}_{\text{Nonlinearities} \,=\, \text{Disturbances}}$$

HIGH GAIN IDEA: Consider the nonlinear terms as disturbances

Two steps for the design of the sequence $(t_k, u_k)_{k\in\mathbb{N}}$

1. In a first step we synthesize a robust controller for a linear system.
2. Amplify the convergence and robustness to deal with f.

Outline of the presentation

1. The linear case
    - General linear system
    - a chain of integrator
2. The nonlinear case
    - with linearly bounded nonlinearities
    - The most general nonlinear case
3. An example
4. Output feedback design

For the linear parts

Consider a (general) linear system $\dot{x} = Ax + Bu_k$

Step 1: Let K be such that $A + BK$ is hurwitz.

Theorem : A well known robustness result

There exists a strictly positive real number δ^* such that for all δ in $[0; \delta^*)$ the sampled state feedback $(t_k, u_k)_{k \in \mathbb{N}}$ defined as

$$t_0 = 0 \ , \ t_{k+1} = t_k + \delta$$
$$u_k = Kx(t_k)$$

makes the origin a globally and asymptotically stable equilibrium.

For the linear parts

Sketch of the proof : If $A + BK$ is Hurwitz, the origin of the discrete time linear system

$$x_{k+1} = \left[\exp(A\delta) + \int_0^\delta \exp(A(\delta - s))BKds\right] x_k,$$

is asymptotically stable for δ sufficiently small.

Result is well known and based on the robustness of a given controller

For the chain of integrator

Consider the chain of integrator system $\dot{x} = Ax + Bu_k$

Step 1: Let K be such that $A + BK$ is hurwitz.

Theorem :

There exists a positive real number α^* such that for all α in $[0, \alpha^*)$ and for all $\delta > 0$ the sampled state feedback $(t_k, u_k)_{k\in\mathbb{N}}$ defined as

$$t_0 = 0 \ , \ t_{k+1} = t_k + \delta$$

$$u_k = K L^{n+1} \mathcal{L} x(t_k) \ , \ \mathcal{L} = \operatorname{diag}\left(\frac{1}{L}, \ldots, \frac{1}{L^n}\right) \ , \ L = \frac{\alpha}{\delta}$$

makes the origin a globally and asymptotically stable equilibrium.

$\Rightarrow$ For the chain of integrator, we can choose δ as large as we want !

Sketch of the proof : With the change of coordinates:

$$X = \mathcal{L}\, x = \begin{bmatrix} \frac{x_1}{L} & \frac{x_2}{L^2} & \cdots & \frac{x_n}{L^n} \end{bmatrix}'$$

It yields for all t in $[t_k, t_{k+1})$:

$$\dot{X}(t) = L(AX(t) + BKX_k)$$

Which gives

$$\begin{aligned} X_{k+1} &= \left[\exp(AL\delta) + \int_0^\delta \exp(AL(\delta - s))LBKds\right] X_k \\ &= \left[\exp(A\alpha) + \int_0^\alpha \exp(A(\alpha - s))BKds\right] X_k. \end{aligned}$$

$\Rightarrow$ This is the same discrete dynamics then in the robustness result
$\Rightarrow$ With the previous theorem, there exists a positive real number α^* such that $X = 0$ (and thus $x = 0$) is a GAS

Adding the nonlinear disturbances

We consider now the nonlinear system

$$\dot{x} = A x + B u_k \quad + \quad f(t, x)$$

Step 2: Let K and α be given in Step 1 for the linear part part. We consider the controller $(u_k)_{k\in\mathbb{N}}$

$$u_k = K L^{n+1} \mathcal{L} x(t_k) \ , \ \mathcal{L} = \operatorname{diag}\left(\frac{1}{L}, \ldots, \frac{1}{L^n}\right) \ , \ \forall k$$

It remains to select L and the sequences $(t_k)_{k\in\mathbb{N}}$

Nonlinear systems

We consider now the nonlinear system

$$\dot{x} = Ax + Bu_k + f(t, x)$$

We consider two contexts

1. The (well known) linearly bounded context

$$|f_j(t, x(t))| \leqslant c_L \left(|x_1| + \cdots + |x_j|\right)$$

2. The time varying bound context

$$|f_j(t, x(t))| \leqslant c(x_1) \left(|x_1| + |x_2| + \cdots + |x_j|\right)$$

with

$$c(x_1) = c_0 + c_1|x_1|^q$$

The linearly bounded context

The closed loop system is

$$\dot{x} = Ax + BKL^{n+1}\mathcal{L}x(t_k) + f(t,x) \ , \ \ t \in [t_{k-1}, t_k)$$

where $\mathcal{L} = \text{diag}\left(\frac{1}{L}, \ldots, \frac{1}{L^n}\right)$.

Proposition: Continuous discrete state feedback for linearly bounded systems

There exists a positive function $L_{\min}(\cdot)$, such that

- for all $L \geqslant L_{\min}(c_L)$
- with the sequence $(t_k)_{k\in\mathbb{N}}$ defined as

$$t_0 = 0 \ , \ \ t_{k+1} = t_k + \delta \ , \ \ \delta = \frac{\alpha}{L}$$

the origin is a globally and asymptotically stable equilibrium.

Sketch of the proof : With the change of coordinates:

$$X = \mathcal{L}\; x = \begin{bmatrix} \frac{x_1}{L} & \frac{x_2}{L^2} & \cdots & \frac{x_n}{L^n} \end{bmatrix}'$$

It yields for all t in $[t_k, t_{k+1})$:

$$\dot{X}(t) = L\left(AX(t) + BKX_k + \delta(t)\right)\ ,\ \Delta(t) = \frac{\mathcal{L}}{L} f\left(t, \mathcal{L}^{-1}X(t)\right)$$

if $L \geqslant 1$, we have

$$\Delta(t) \leqslant \frac{c_L}{L}\left(|X_1(t)| + |X_2(t)| + \cdots + |X_j(t)|\right)$$

Which gives

$$X_{k+1} = \underbrace{\left[\exp(A\alpha) + \int_0^\alpha \exp(A(\alpha - s))BKds\right] X_k}_{\text{Stabilizing part}} + \underbrace{\int_0^\alpha \exp(A(\alpha - s))\Delta\left(\frac{s}{L}\right) ds}_{\text{get small if } L \text{ is large}}$$

With a Lyapunov analysis we get the result if L is large.

The linearly bounded context

Some remarks on the linearly bounded context

$$|f_j(t, x(t))| \leqslant c_L \left(|x_1| + \cdots + |x_j|\right)$$

1. The parameter L has to be larger then a function of the linear bound c_L
2. If L is large then δ is small $\Rightarrow$ you need to actuate the controller more frequently

How can we do when the linear bound is time varying?

Time varying linear bound

We consider now the case

$$|f_j(t, x(t))| \leqslant c(x_1)\,(|x_1| + |x_2| + \cdots + |x_j|)$$

with

$$c(x_1) = c_0 + c_1|x_1|^q$$

- Now the linear bound is time varying.
- L will have to follow a function of the linear bound.

$\Rightarrow$ We need to adapt the high-gain parameter L

The proposed event triggered control law

The event triggererd state feeddback is

$$u_k = K(L_k)^{n+1}\mathcal{L}_k x(t_k)$$

with $\mathcal{L}(t) = \mathrm{diag}(\frac{1}{L(t)}, \dots, \frac{1}{L(t)^n})$

The sequence $(t_k)_{k\in\mathbb{N}}$ is

$$t_0 = 0\ ,\ \ t_{k+1} = t_k + \delta_k\ ,\ \ \delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k+s)^-) = \alpha\}$$

where $L(t^-) = \lim\limits_{\tau\to t, \tau<t} L(\tau)$

Updated law for the high-gain parameter

$$\left\{\begin{array}{rcl} \dot{L}(t) & = & a_2 L(t) M(t) c(x_1(t)), \quad L(0) \geqslant 1 \\ \dot{M}(t) & = & a_3 M(t) c(x_1(t)) \end{array}\right. , \forall t \in [t_k, t_{k+1}),$$

$$\left\{\begin{array}{rcl} L(t_k) & = & L(t_k^-)(1 - a_1\alpha) + a_1\alpha, \quad a_1\alpha < 1 \\ M(t_k) & = & 1. \end{array}\right.$$

Main Theorem

Theorem (Main theorem):

There exist positive numbers a_1, a_2, a_3, a gain matrix K and α^* such that for all α in $[0, \alpha^*]$, the self-triggered feedback initiated from $L(0) \geqslant 1$ and $M(0) = 1$ renders $x = 0$ a globally and asymptotically stable equilibrium. Moreover there exists a positive real number $\delta_{\min}$ such that $\delta_k > \delta_{\min}$ for all k and so ensures the existence of a minimal inter-execution time.

Some remarks on this algorithm

The model of the high-gain parameter, is composed of two things

1. The continuous-time part

$$\left\{ \begin{array}{rcl} \dot{L}(t) & = & a_2 L(t) M(t) c(x_1(t)), \quad L(0) \geqslant 1 \\ \dot{M}(t) & = & a_3 M(t) c(x_1(t)) \end{array} \right. , \forall t \in [t_k, t_{k+1}),$$

$\Rightarrow$ Learning and upperbounding the linear bound

2. The discrete-time part

$$\left\{ \begin{array}{rcl} L(t_k) & = & L(t_k^-)(1 - a_1\alpha) + a_1\alpha, \quad a_1\alpha < 1 \\ M(t_k) & = & 1. \end{array} \right.$$

$\Rightarrow$ Stabilizing the high-gain parameter

Some remarks on this algorithm

Note that we have for instance,

$$L(\delta_1) = L(\delta_1^-)(1 - a_1\delta_1 L(\delta_1^-)) + a_1\delta_1 L(\delta_1^-)$$

and if δ_1 is small, we have

$$\left\{ \begin{array}{l} \dot{L}(t) = a_2 L(t)M(t)c(x_1(t)) \\ M(0) = 1 \end{array} \right. \quad \Rightarrow L(\delta_1^-) \simeq L(0) + a_2 L(0)c(0)\delta_1$$

Keeping only first order terms in δ_1, we get

$$L(\delta_1) \simeq L(0) + \delta_1 L(0)\Big[a_1(1 - L(0)) + a_2 c(0)\Big]$$

$\Rightarrow$ This is the same structure then the one in Praly 2003 in the continuous time output feedback context:

$$\dot{L} = L\Big[a_1(1 - L) + a_2 c(x_1(t))\Big]$$

$\Rightarrow$ Inspired from a Riccati equation !

Sketch of the proof

To prove the theorem, we have to show:

1. Selection of K
2. The existence of the sequence $(x_k, t_k, L_k, M_k)_{k\in\mathbb{N}}$
3. The decrease to zero of some scaled coordinates
4. Boundedness of the time varying high-gain parameter L
5. Existence of a dwell time and convergence to zero of x
6. Stability analysis.

Sketch of the proof

Step 1 : Selection of K.

Let $D = \mathrm{diag}(b, 1+b, \ldots, n+b-1)$. Let P be a SPD matrix and K a vector such that

$$\begin{aligned} P(A+BK) + (A+BK)'P &\leqslant -I, \\ p_1 I \leqslant P &\leqslant p_2 I, \\ p_3 P \leqslant PD + DP &\leqslant p_4 P, \end{aligned}$$

with $p_1, \ldots, p_4$ positive real numbers.

It can be shown that there always exist a solution

Sketch of the proof

Step 2 : Existence of the sequence $(x_k, t_k, L_k, M_k)_{k\in\mathbb{N}}$

Proposition : Existence of the sequence

Let a_1, a_3 and α be positive, and $a_2 \geqslant \frac{2n}{p_3}$. Then, the sequence $(t_k, x_k, L_k, M_k)_{k\in\mathbb{N}}$ is well defined.

- With a Lyapunov analysis, it can be shown there is no blow up of x before a blow up of L

$$\dot{\overbrace{|x(t)|}} \approx c(x_1(t))\,|x(t)|\ ,\ \ \dot{L}(t) = a_2 L(t) M(t) c(x_1(t))$$

In some sens, L grows at least faster than x

- There exists t_{k+1} since L is strictly increasing and $t_{k+1} = t_k + \delta_k$ and,

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k + s)^-) = \alpha\}$$

It remains to select a_1, a_2, a_3 and α

Sketch of the proof

Step 3 : Decrease of scaled coordinate Let

$$X(t) = \mathcal{S}(t)x(t)$$
$$\mathcal{S}(t) = \mathrm{diag}\left(\frac{1}{L(t)^b}, \cdots, \frac{1}{L(t)^{n+b-1}}\right) = L(t)^{1-b}\mathcal{L}(t)$$

where $b > 0$ is such that $bq < 1$ with q

Decrease of scaled coordinates

Let $V(X) = X^\top PX$. There exist positive real numbers a_1 (sufficiently small), a_2 (sufficiently large), and α^* such that for $a_3 = 2n$ and for all α in $[0, \alpha^*]$ the following property is satisfied:

$$V(X_{k+1}) - V(X_k) \leqslant -\left(\frac{\alpha}{p_2}\right)^2 V(X_k).$$

$\Rightarrow$ implies that $V(X_k)$ goes to zero

Sketch of the proof

This doesn't imply that $V(X(t))$ is decreasing

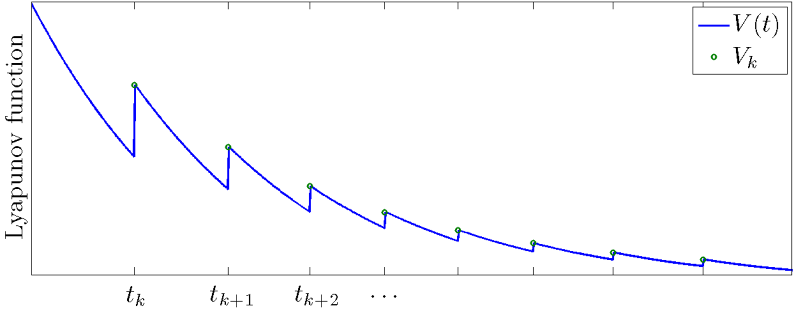


Fig.: Time evolution of Lyapunov function V.

This doesn't imply that $x(t)$ goes to zero since $L(t)$ may go to infinity.

Sketch of the proof

Step 4 : Boundedness of L

Updated law for the high-gain parameter

$$\left\{\begin{array}{rcl} \dot{L}(t) & = & \frac{a_2}{a_3}L(t)\dot{M}(t), \quad L(0) \geqslant 1 \\ \dot{M}(t) & = & a_3 M(t)\left(c_0 + c_1\left|\frac{X_1(t)}{L^b}\right|^q L^{bq}\right) \end{array}\right. ,\forall t \in [t_k, t_{k+1}),$$
$$\left\{\begin{array}{rcl} L(t_k) & = & L(t_k^-)(1 - a_1\alpha) + a_1\alpha, \quad a_1\alpha < 1 \\ M(t_k) & = & 1. \end{array}\right.$$

- It can be shown that for L_k large enough

$$L_{k+1} \leqslant F(L_k)$$

$$F(L_k) = \exp\left(\psi\left(\alpha L_k^{bq-1}\right) - 1\right) L_k(1 - a_1\alpha) + a_1\alpha.$$

  with ψ a continuous function such that $\psi(0) = 1$.
- It can be shown that for L sufficiently large

$$\frac{F(L)}{L} < 1$$

Hence L is bounded.

Sketch of the proof

Step 5 : Conclusion With L bounded

- It implies that $\delta_k = \frac{\alpha}{L_k}$ is lower bounded.

  $\Rightarrow$ There is a dwell time !

- It implies

$$\lim_{k \to +\infty} t_k = +\infty$$

and

$$\lim_{t \to +\infty} |x(t)| = 0$$

Stability requires a bit of work.

An example

Consider the unknown nonlinear systems

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = \theta x_1^2 x_3 + u \end{cases}$$

where θ is a unknown constant parameter with $\theta \leqslant \theta_{\max}$.

An example

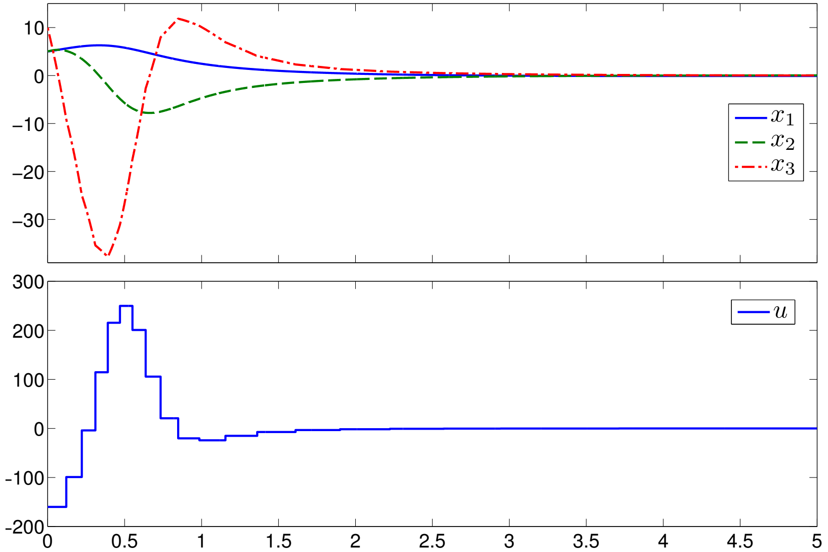
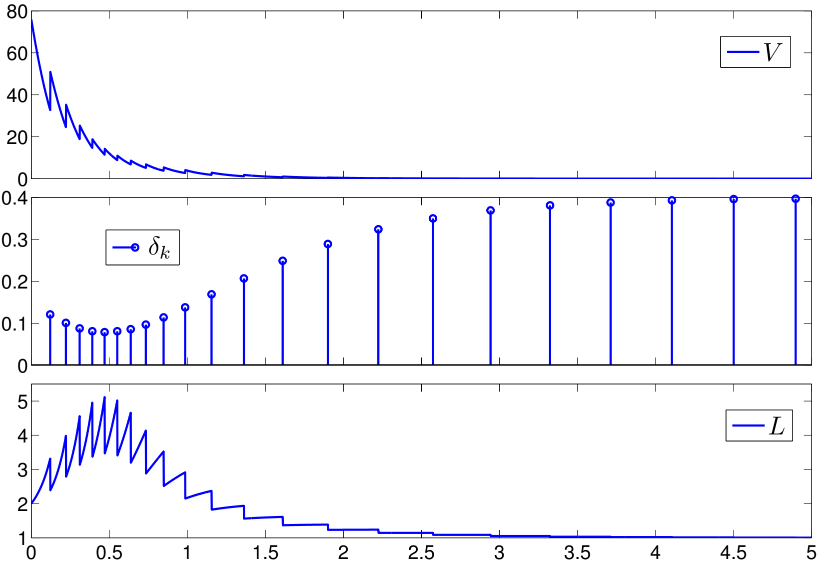


Fig.: Control signal and state trajectories with $(x_1, x_2, x_3) = (5, 5, 10)$ as initial

An example



What about output feedback

What about output feedback ?

1. The control is based on an output measurement.
2. The measurement follows an event

What about output feedback

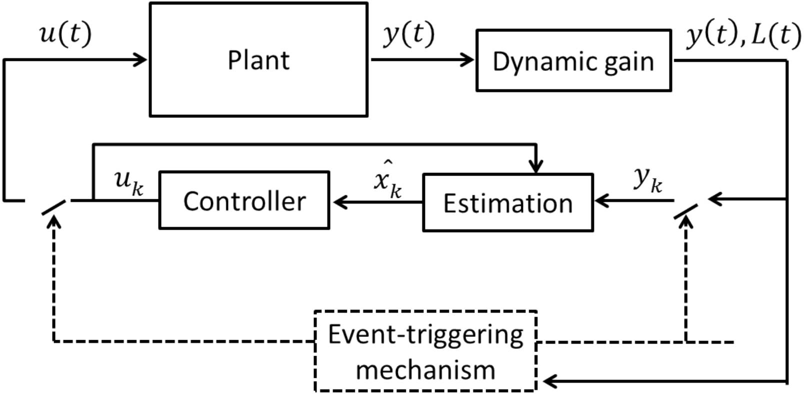


Fig.: Event-triggered output feedback control schematic.

What about output feedback

We consider a dynamical system in a specific form

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 + f_1(t, x_1) \\ \dot{x}_2 & = & x_3 + f_2(t, x_1, x_2) \\ & \vdots & \\ \dot{x}_n & = & u_k + f_n(t, x_1, \dots, x_n) \end{array}\right.$$

with

$$y_k = x_1(t_k)$$

which can be rewritten

$$\dot{x}(t) = Ax(t) + Bu_k + f(t, x(t)) \ , \ y_k = Cx(t_k)$$

The proposed event triggered control law

The event triggered output-feedback $(u_k)_{k\in\mathbb{N}}$ is

$$u_k = K(L_k)^{n+1}\mathcal{L}_k\hat{x}(t_k)$$

where

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu_k, \quad \forall\ t \in [t_k, t_{k+1})\\ \hat{x}(t_k) &= \hat{x}(t_k^-) + \delta_{k-1}\mathcal{L}_k^- K_o(C\hat{x}(t_k^-) - y_k)\end{aligned}$$

The sequence $(t_k)_{k\in\mathbb{N}}$ is

$$t_0 = 0\ ,\ t_{k+1} = t_k + \delta_k\ ,\ \delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k+s)^-) = \alpha\}$$

Updated law for the high-gain parameter

$$\left\{\begin{aligned}\dot{L}(t) &= a_2 L(t)M(t)c(x_1(t)), \quad L(0) \geqslant 1\\ \dot{M}(t) &= a_3 M(t)c(x_1(t))\end{aligned}\right. ,\forall t \in [t_k, t_{k+1}),$$

$$\left\{\begin{aligned}L(t_k) &= L(t_k^-)(1-a_1\alpha) + a_1\alpha, \quad a_1\alpha < 1\\ M(t_k) &= 1.\end{aligned}\right.$$

Depends on $x_1(t)$ and not on y_k

Main Theorem

Theorem (Output feedback theorem):

There exist positive numbers a_1, a_2, a_3, a gain matrix K, K_o and α^* such that for all α in $[0, \alpha^*]$, the self-triggered output feedback initiated from $L(0) \geqslant 1$ and $M(0) = 1$ renders $x = 0$ a globally and asymptotically stable equilibrium. Moreover there exists a positive real number $\delta_{\min}$ such that $\delta_k > \delta_{\min}$ for all k and so ensures the existence of a minimal inter-execution time.

Conclusion

- A new method to design a high-gain event triggered control law.
- Design based on a continuous discrete riccati equations
- It is possible to adress the output feedback case following the same route.