Template complex zonotopes for stability and invariant verification

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Workshop Co4: Control subject to Computational and Communication Constraints
October 2016, LAAS, Toulouse

Overview

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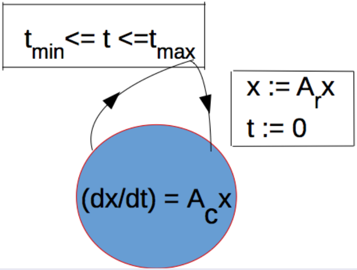
- **Approach:** Introduce set representations called "*complex (template) zonotopes*" to represent *contractive sets* to establish exponential stability of QPLIS.
- **Advantage:** Complex zonotopes can utilize *eigenstructure* to represent contractive sets.

Quasi periodic linear impulsive systems (QPLIS)



$\mathbf{x}(0) = \mathbf{x}_0$ Initial State

$\dot{\mathbf{x}}(t) = A_c\mathbf{x}(t) \quad t \in [t_k, t_{k+1})$ Continuous linear evolution

$\mathbf{x}(t_k^+) = A_r\mathbf{x}(t_k^-)$ Linear Impulse

$t_{k+1} - t_k \in [t_{min}, t_{max}]$ Quasi-periodicity

Global exponential stability

- **Global exponential stability (GES)**: Every state reaches equilibrium at exponential rate.
- Mathematically, GES means there exists $\lambda \in [0, 1)$ and $c > 0$ such that for all $\mathbf{x}_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$

$$\|\mathbf{x}(t_k)\| \leq c\lambda^k \|\mathbf{x}_0\|.$$

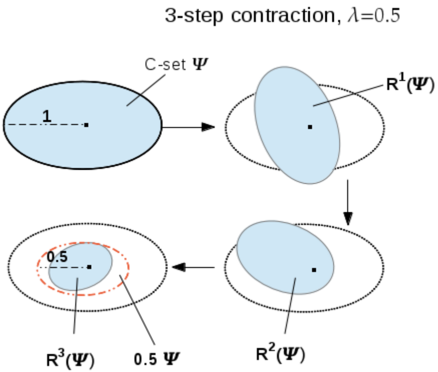
Stability verification problem

Given A_c, A_r and sampling interval $\Delta = [t_{min}, t_{max}]$, verify that the quasi-periodic linear impulsive systems is exponentially stable.

Contractive set

k-step contractive: Contracts after k-impulses if

$$R^k(\Psi) = \{x(t_k^+) : x_0 \in \Psi\} \subseteq \lambda(\Psi) : \lambda \in [0, 1)$$



Reachability operators

$\Delta = [t_{min}, t_{max}]$

- One-step reachability operators

$$O_1(\Delta) = \{H_t = e^{A_c t} A_r : t \in \Delta\}.$$

- Multiple-step reachability operators:

$$O_k(\Delta) = \{H_{t_1} ... H_{t_k} : t_i \in \Delta \ \forall i \in \{1, ..., k\}\}$$

Set theoretic condition for exponential stability

- Define *C-set*: Convex and compact set containing the origin in its interior.
- QPLIS is GES if and only if there is a contractive C-set [Fiacchini2014,Athanasopoulos2014,AlKhatib2015].

Finding a contractive C-set

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- **Earlier approach [Fiacchini2014]**: Starting with an initial polytope, do iterative intersection of inverse images of large number of reachability operators.
  Drawback: Complexity of set representation grows with the number of iterations.
- **Our solution**: Find contractive *complex zonotopes* based on eigenvectors of reachability operators.
  Motivations
  1. Eigenstructure of reachability operators is closely related to stability.
  2. Size of set representation remains constant.
  3. Contraction due to any reachability operator can be checked efficiently using convex optimization (second order conic programming).

Review: Real (usual) Zonotopes

Minkowski sum of line segments (generators)

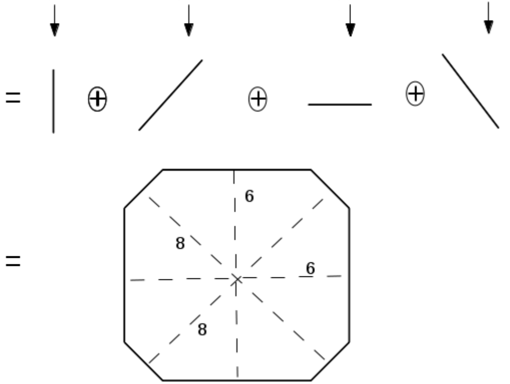
Definition

Let $V_{n\times m}$ be real valued matrix of m generators which are elements in $\mathbb{R}^n$, $b \in \mathbb{R}^n$ (center).
Zonotope is a set: $\mathcal{Z} = \{V\epsilon + b : \epsilon \in \mathbb{R}^m, ||\epsilon||_\infty \leq 1\}$

Example of real zonotope

$$Z = \left\langle \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \end{matrix}, \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle$$

$$\left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle \oplus \left\langle \begin{matrix} 1 \\ 1 \end{matrix} \right\rangle \oplus \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle \oplus \left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle$$

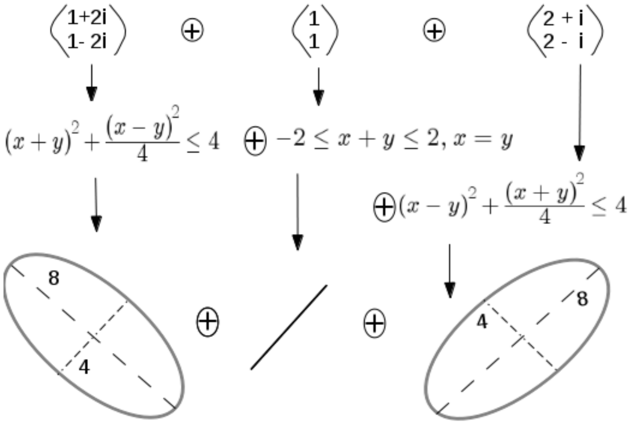


Complex zonotopes: A new class of sets

Definition

Let $V_{n\times m}$ be a matrix of m complex valued generators which are elements of $\mathbb{C}^n$, $c \in \mathbb{C}^n$ (center).
Complex zonotope: $\mathcal{Z} = \{V\epsilon + c : \epsilon \in \mathbb{C}^m, ||\epsilon||_\infty \leq 1\}$

Example of complex zonotope



Reason for complex zonotopes: Utilizing eigenstructure

- **Fixed sampling period**: If sampling period τ is fixed, i.e., has single reachability operator H_τ:
  Exponential stability $\iff$ *generators chosen as eigenvectors of* H_τ *give contractive complex zonotope.*

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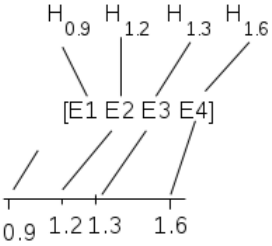
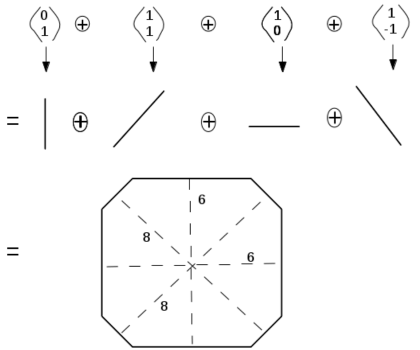
  Exponential stability $\Longleftrightarrow$ *generators chosen as eigenvectors of* H_τ *give contractive complex zonotope.*
- **Quasi-periodic case**: Generators can be choosen from eigenvectors of at a number of $H_t : t \in [\tau_{min}, \tau_{max}]$ and expect that set contracts.

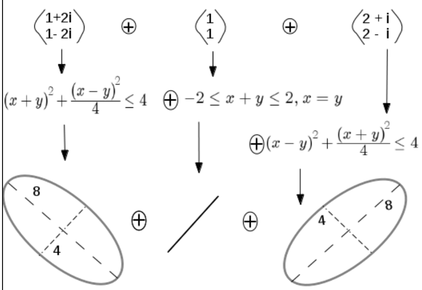


Complex zonotopes instead vs real zonotopes

Real zonotope	Complex zonotope
Only real eigenvectors (not complex) as generators	Can have complex and real eigenvectors as generators

$$Z = \left\langle \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 \end{matrix}, \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle$$





Related Work

- **Classical abstract domains**: **intervals** [Cousot Cousot 1976] and **convex polyhedra** [Cousot Halbwachs 1978], and their variants, such as **zones** [Mine 2001], **octagons** [Mine 2006], **linear templates** [Sankaranarayanan et.al 2005], **zonotopes** [Girard et.al 2009], and **tropical polyhedra** [Allamigeon et.al 2008] (to achieve a good comprise between computational speed and precision)
- **Non-polyhedral set representations**: **ellipsoids** [Kurzhanski Varaiya 2000], **polynomial invariants** [Tiwari 2002] **polynomial inequalities** for invariant computation via reduction to linear inequalities) [Bagnara et al. 2005] and polynomial equalities via Gröner basis method [Rodriguez-Carbonell et al. 2007], **quadratic templates** [Feron et al. 2010, Adje et al. 2010]
- Other extensions of zonotopes: **quadratic** [Adje et. al 2015] and more generally **polynomial zonotopes** [Althoff 2011].
- A polynomial zonotope is a set-valued polynomial function of **intervals**, whereas a template complex zonotope is a set-valued function of **circles** in the complex plane.

Contributions

- Introduce **complex zonotopes** and **template complex zonotopes** for representing sets.

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Plan

1. Stability of Quasi-Periodic Linear Impulsive Systems, as motivation

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2. Complex Zonotopes and Template Complex Zonotopes
3. Stability Verification Algorithm for Quasi-Periodic Linear Impulsive Systems
4. Invariance Verification Algorithm for Switched Systems with Additive Disturbance

Contraction w.r.t. Reachability Operators

- Let J be a k-step reachability matrix, *i.e.*, $J \in O_k(\Delta)$. We define **contraction** $\chi(\Psi, J) = \min\{\lambda \in \mathbb{R}^+ : J\Psi \leq \lambda\Psi\}$.

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- Verify k-step contraction means verify that contraction due to all $J \in O_k(\Delta)$ is less than one, i.e., $\forall J \in O_k(\Delta)$, verify $\chi(\Psi, J) < 1$.

Choice of initial C-set

- **Efficient verification**: find a C-set that contracts in a small number of steps
- In this regard, we introduce **complex zonotopes** whose real projection represents C-set.

Complex zonotopes representing C-sets

- **Complex zonotope**

Definition

Let $m \in \mathbb{Z}_{\geq 0}$ (number of generators), let $V_{n\times m} \in \mathbb{M}_{n\times m}(\mathbb{C})$ (generators are column vectors) be a complex-valued matrix, $c \in \mathbb{C}^n$ (center). Then

$$\mathcal{Z} = \{V\epsilon + c : y \in \mathbb{C}^m, ||\epsilon||_\infty \leq 1\}$$

is a complex zonotope. We denote $\mathcal{Z} = \langle V, c\rangle$.

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- **Sufficient condition for C-sets**: Real projection $Re(\mathcal{Z})$ is a C-set if $c = 0$, and $rank(V) = n$.

Template Complex Zonotopes

Drawback of complex zonotopes: adding generators may make the set unnecessarily large in some directions.
Template Complex Zonotope: allowing to "scale" the set along the directions of the generators $\Rightarrow$ dynamical addition of generators without compromising the precision

Definition

- $V \in \mathbb{M}_{n\times m}(\mathbb{C})$: *template*
- $c \in \mathbb{C}^n$: *center*
- $s \in \mathbb{R}^m_{\geq 0}$: *scaling factors*
- *Template complex zonotope* $\mathcal{Z} = \langle V, c, s\rangle$ *is the set*

$$\{V\zeta + c : \zeta \in \mathbb{C}^m \ \wedge \ \forall i \in \{1, ..., m\}, |\zeta_i| \leq s_i\}.$$

Note: Complex zonotopes are thus template complex zonotopes with (fixed) unit scaling factors.

Template Complex Zonotopes: Linear Transformation and Minkowski Sum

- **Linear transformation** of a template complex zonotope is a template complex zonotope
  $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ is a complex square matrix. Then,
  $$A\langle V, c, s\rangle = \langle AV, Ac, s\rangle \,. \tag{1}$$
- **Minkowski sum** of two template complex zonotopes is another template complex zonotope.
  Let $V \in \mathbb{M}_{n\times m}(\mathbb{C})$, $G \in \mathbb{M}_{n\times r}(\mathbb{C})$ (generators), $c, d \in \mathbb{C}^n$ (centers), $s \in \mathbb{R}^m_{\geq 0}$, $h \in \mathbb{R}^r_{\geq 0}$ (scaling factors). Then
  $$\langle V, c, s\rangle \oplus \langle G, d, h\rangle = \left\langle [V \quad G], (c+d), \begin{pmatrix} s \\ h \end{pmatrix} \right\rangle \tag{2}$$

Template Complex Zonotopes: Inclusion Checking

Problem: checking the inclusion between two template complex zonotopes centered at the origin.

This problem is non-convex in general $\Rightarrow$ we derive an easily verifiable sufficient condition (a set of second order conic constraints)

$\mathcal{Z} = \langle V, 0, s \rangle$ and $\mathcal{Z}' = \langle G, 0, h \rangle$: two template complex zonotopes. Then $\mathcal{Z}' \subseteq \mathcal{Z}$ if

$$\begin{aligned}
&\exists X \in \mathbb{M}_{m \times r}(\mathbb{C}) \text{ such that} \\
&VX = G\mathcal{D}(h) \\
&\left(\sum_{j=1}^{r} |X_{ij}| \right) \leq s_i \quad \forall i \in \{1, \ldots, m\}
\end{aligned} \tag{3}$$

$\mathcal{D}(h)$ is the diagonal matrix with h along the diagonal.

Template Complex zonotopes: Contraction Bound by Linear Transformation

Definition (Contraction)

For a template complex zonotope $\mathcal{Z} = \langle V, 0, s \rangle$, the amount of contraction by a square matrix $J \in \mathbb{M}_{n\times n}(\mathbb{R})$ is

$$\chi(\mathcal{Z}, J) = \min \left\{ \lambda \in \mathbb{R}_{\geq 0} : J \langle V, 0, s \rangle \subseteq \lambda \langle V, 0, s \rangle \right\}.$$

A contraction bound can be derived:

$$\chi(\mathcal{Z}, J) \leq \beta(\mathcal{Z}, J) = \min\{\|X\|_\infty : X \in \mathbb{M}_{m\times m}(\mathbb{C}) \text{ and } V\mathcal{D}(s)X = JV\mathcal{D}(s)\}$$

Stability Condition in terms of Template Complex Zonotopes

Consider again the Quasi-Periodic LIS system

$$\begin{aligned} &\dot{\mathbf{x}}(t) = A_c\mathbf{x}(t) \;\; \forall t \in \mathbb{R}_{\geq 0} : \;\; t \neq t_i \; \forall i \in \mathbb{Z}_{>0} \\ &\mathbf{x}(t_i^+) = A_r\mathbf{x}(t_i^-) \;\; \forall i \in \mathbb{Z}_{>0} \\ &(t_{i+1} - t_i) \in \Delta \;\; \forall i \in \mathbb{Z}_{>0} \end{aligned} \tag{4}$$

Sufficient Condition for Globally Exponentially Stability(GES): if there exists a scalar $\lambda \in [0, 1)$ and template complex zonotope $\langle V, 0, s\rangle$ (centered at the origin) of rank n, such that for all $t \in \Delta$, we have

$$H_t\mathcal{Z} \subseteq \lambda\mathcal{Z}$$

where $H_\tau = e^{A_c\tau}A_r$ is the *one-step reachability operator*.

Stability verification using Template Complex Zonotopes

- **Fixed sampling period** τ: single reachability operator H_τ:
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- **Quasi-periodic case**: Generators can be chosen from the eigenvectors of $H_t : t \in [\tau_{min}, \tau_{max}]$ at a number of time points t, and expect that set contracts.

Bounding contraction for small time uncertainty

- Let $\rho \in [0, \epsilon]$, period uncertainty
- One-step reachability operator $H_{t+\rho}$ can be written using Taylor expansion as $H_{t+\rho} = e^{(A_c \rho)} H_t = P_r^t(\rho) + E_r^t(\delta) : \; 0 \leq \delta \leq \rho$

$$P_r^t(\rho) = \sum_{i=0}^{r} \frac{A_c^i \rho^i}{i!} H_t, \qquad E_r^t(\delta) = \frac{A_c^{r+1} \delta^{r+1}}{(r+1)!} H_t$$

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Contraction in the vicinity of a reachability operator

If $0 \leq \rho \leq \epsilon$, then the contraction $\chi(\mathcal{Z}, H_{t+\rho})$, due to $H_{t+\rho}$, is bounded by

$$\chi(\mathcal{Z}, H_{t+\rho}) \leq \eta_r(\mathcal{Z}, t, \epsilon) = \max_{j \in \{0, \ldots, r\}} \beta\left(\mathcal{Z}, P_j^t(\epsilon)\right) + \frac{\epsilon^{r+1}}{(r+1)!} \beta(\mathcal{Z}, A_c^{r+1})$$

Contraction bounds (due to linear transformation) $\beta(\mathcal{Z}, J)$ can be efficiently computed

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Similar bound obtained for multiple-step reachability operators.

Contraction Verification

Initialize $t = t_{min}$ and $h = tol$.
Initialize r as order of Taylor expansion (typically ≤ 2).
while $h \geq tol$ and $t < t_{max}$ **do**
    if $\eta_r(\mathcal{Z}, t, h) < 1$ **then**
        $t \leftarrow t + h$
        $h \leftarrow h + tol$
    else
        $h \leftarrow h - tol$
    end if
end while
if $t \geq t_{max}$ **then**
    $\mathcal{Z}$ *is contractive w.r.t. all reachability operators*
else
    Inconclusive
end if

Synthesizing the Templates

Define k sampled time points $\Lambda_k = \left\{ t_{min} + i\frac{(t_{max} - t_{min})}{k} : i \in \{1, ..., k\} \right\}$ and set of k sampled reachability operators as $\Gamma_k = \{H_t : t \in \Lambda_k\}$

Find $\mathcal{Z}$ that contracts w.r.t all operators in Γ_k:

- Fix the template E_k (by all eigenvectors of the operators in Γ_k)
- **Synthesize scaling factors** s such that $\mathcal{Z}$ is contractive w.r.t. Γ_k (combining contraction bound and inclusion checking)

$$
\begin{aligned}
&\text{Find } s \in \mathbb{R}^n_{\geq 1} \ \text{ for which} \\
&\exists X \in \mathbb{M}_{m \times m}(\mathbb{C}) \ \text{ such that} \\
&E_k X = H_t E_k \mathcal{D}(s) \ \forall t \in \Lambda_k \ \text{(a finite set of time points)} \\
&\sum_{j=1}^{r} |X_{ij}| \leq \lambda s_i \ \ \forall i \in \{1, ..., m\}
\end{aligned} \tag{5}
$$

Example: Networked control systems modeled as LIS

Networked control system (NCS): Plant interacts with controller.

$$
\begin{aligned}
\dot{x}_p(t) &= A_o x_p(t) + B_o u(t) \\
u(t) &= u(t_k) \ \forall t \in (t_k, t_{k+1}] \\
y(t) &= C_p x_p(t) \\
x_o(t_{k+1}) &= A_o x_o(t_k) + B_o y_p(t_k) \\
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\end{aligned} \tag{6}$$

Has LIS representation where state is $z = \begin{pmatrix} x_p \\ x_o \\ u \end{pmatrix}$,

$$A_c = \begin{pmatrix} A_p & 0 & B_p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A_r = \begin{pmatrix} I & 0 & 0 \\ B_o C_p & A_o & 0 \\ D_o C_p & C_o & 0 \end{pmatrix}$$

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Parameters: $A_p = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$, $B_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C_p = \begin{pmatrix} 0 & 1 \end{pmatrix}$,
$A_o = 0.4286$, $B_o = -0.8163$, $C_o = -1$ and $D_o = -3.4286$.

Implementation

Software: For convex optimization, we use CVX (disciplined convex optimization software) on MATLAB.

Computation platform: For convex optimization, we used CVX version 2.1 with Matlab R2015a on 64 bit GNOME (3.4.2) with Intel Core i5-3470 CPU, 3.20GHz processor, having 3.8 GiB memory and disk space 242.1 GB.

Experiments: Example 1

Software: CVX (disciplined convex optimization software) on MATLAB. We chose the template from 3 sampled reachability operators, and could verify contraction in a sampling interval $[0.08, 0.5]$, with first order Taylor expansion ($r = 1$). Computation time is 2.69s.

The NCS tool [Hetel2013] could only verify until 0.4.

Reference	τ_{min}	τ_{max}
Digital Control Book [Wittenmark02]	0.08	0.22
NCS tool [Hetel2013]	0.08	0.4
Complex zonotope	0.08	0.5
Template complex zonotope	0.08	0.5

Experiments: Example 2

[Hetel2013] $A_c = \begin{pmatrix} 0 & -3 & 1 \\ 1.4 & -2.6 & 0.6 \\ 8.4 & -18.6 & 4.6 \end{pmatrix}$ and $A_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Using $k = 3$ sampled reachability operators, we could verify contraction in a sampling interval $[0.1, 0.495]$. Computation time is 6.9076 seconds.

Reference	τ_{min}	τ_{max}
Lyapunov, parametric LMI [Hetel2013]	0.1	0.3
Polytopic set contractiveness [Fiacchini2014]	0.1	0.475
Khatib et al. [AlKhatib2015]	0.1	0.514
Complex zonotope	0.1	0.49
Template complex zonotope	0.1	0.495

Template Complex Zonotopes vs Complex Zonotopes

Using complex zonotopes, contraction had to be verified after two impulses.
Using the (synthesized) template complex zonotopes, contraction is verified only after one impulse.

		TCZ	CZ
Nb. of impulses		1 (both examples)	2 (both examples)
Comp. time	Ex. 1	2.69s	27.4052s
	Ex. 2	6.9076s	74.0311s

Switched Systems with Additive Disturbance

$\mathcal{S} = \langle L, \mathbb{A}, \Omega, \mathcal{I} \rangle$

- L is a finite set of **modes**
- $\mathbb{A} : L \to \mathbb{M}_{n \times n}(\mathbb{R})$ specifying for each mode a set of **switching matrices**
- Ω is a bounded set called **additive disturbance set**
- $\mathcal{I}$ is the **initial set**

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbb{A}(\sigma(t))\mathbf{x}(t) + w(t) \\ \mathbf{x}(0) &\in \mathcal{I}. \end{aligned} \tag{7}$$

Switching signal $\sigma : \mathbb{Z}_{\geq 0} \to L$, **disturbance signal** $w : \mathbb{Z}_{\geq 0} \to \Omega$

Starting from a set Ψ, reachable set at the next time instant.

$$R(\Psi) = \bigcup_{l \in L} \left(A^l \Psi \oplus \Omega \right) \tag{8}$$

Positive Invariance

Definition (Positive invariant)

A set $\Psi \subseteq \mathbb{C}^n$ is called a positive invariant of the switched system if

$$R(\Psi) \subseteq \Psi \quad \textit{and} \quad \mathcal{I} \subseteq \Psi$$

or equivalently, for all locations $l \in L$,

$$\begin{aligned} &\left(A^l \Psi \oplus \Omega\right) \subseteq \Psi \quad \textit{and} \\ &\mathcal{I} \subseteq \Psi \end{aligned} \tag{9}$$

Sufficient Condition for Positive Invariance

$V \in \mathbb{M}_{n\times m}(\mathbb{C})$, $G \in \mathbb{M}_{n\times r}(\mathbb{C})$, $W \in \mathbb{M}_{n\times p}(\mathbb{C})$, $s \in \mathbb{R}^m_{\geq 0}$ and $h \in \mathbb{R}^r_{\geq 0}$ and $z \in \mathbb{R}^p_{\geq 0}$. Let $\Psi = \langle V, 0, s\rangle$, $\Omega = \langle G, 0, h\rangle$ and $\mathcal{I} = \langle W, 0, z\rangle$ be template complex zonotopes. Then $\langle V, 0, s\rangle$ is a positive invariant if

$$\exists X^l \in \mathbb{M}_{m\times(m+r)}(\mathbb{C}) \text{ for each } l \in L \text{ and } \exists Y \in \mathbb{M}_{m\times p}(\mathbb{C}) \text{ such that}$$

$$VX^l = \begin{bmatrix} A^l V & G \end{bmatrix} \mathcal{D}\begin{pmatrix} s \\ h \end{pmatrix} \quad \forall l \in L$$

$$VY = W\mathcal{D}(z)$$

$$\sum_{j=1}^{m+r} |X^l_{ij}| \leq s_i \quad \forall i \in \{1, ..., m\} \quad \forall l \in L \quad \text{(inclusion after switching)}$$

$$\sum_{j=1}^{p} |Y_{ij}| \leq s_i \quad \forall i \in \{1, ..., m\} \quad \text{(containment of the initial set)}$$

Sufficient Condition for Linear Positive Invariance

A linear invariant is a collection of linear constraints s.t. every trajectory starting in the initial set always satisfies the linear constraints.

Let $k \in \mathbb{Z}_{>0}$, $T \in \mathbb{M}_{k\times n}(\mathbb{R})$ and $b \in \mathbb{R}_{\geq 0}$. A linear invariant property is specified a tuple (T, b) such that $\forall t \in \mathbb{Z}_{>0}$ and $\forall x \in R^t(\Psi)$, we have $\|Tx\|_\infty \leq b$.

Condition for linear invariance

The set of constraints (T, b) is a linear invariant iff there exists a positive invariant Ψ such that $\forall x \in real(\Psi)$, we have $\|Tx\|_\infty \leq b$.

Verifying Linear Invariance

Problem: We want to find a positive invariant as a template complex zonotope satisfying a given set of linear constraints.

Approach: fix the template and derive sufficient conditions on the scaling factors, by combining (1) the (above) condition for positive invariance and (2) the (following) condition for inclusion in a set defined by linear constraints.

Inclusion of a zonotope in set defined by linear constraints

$\forall x \in \langle V, 0, s \rangle$, the inequality $\| Tx \|_\infty \le b$ holds if and only if $\| TV\mathcal{D}(s) \|_\infty \le b$.

Sufficient Condition for Linear Invariance

Disturbance set $\Omega = \langle G, 0, h\rangle$ and initial set $\mathcal{I} = \langle W, 0, z\rangle$ (with scaling factors $h \in \mathbb{R}^r_{\geq 0}$ and $z \in \mathbb{R}^p_{\geq 0}$). A tuple (T, b) (where $T \in \mathbb{M}_{k\times n}(\mathbb{R})$ and $b \in \mathbb{R}_{\geq 0}$) is linear invariant if

$$\exists s \in \mathbb{R}^m_{\geq 0} \text{ (scaling factors)}, \ \exists X^l \in \mathbb{M}_{n\times(m+r)}(\mathbb{C}) \ \exists Y \in \mathbb{M}_{n\times p}(\mathbb{C})$$

$$VX^l = \left[A^l V \quad G\right] \mathcal{D}\begin{pmatrix} s \\ h \end{pmatrix} \quad \forall l \in L$$

$$VY = W\mathcal{D}(z)$$

$$\|TV\mathcal{D}(s)\|_\infty \leq b \quad \text{(satisfaction of linear constraints)}$$

$$\sum_{j=1}^{m+r} |X_{ij}| \leq s_i \quad \forall i \in \{1, \ldots, m\} \wedge \forall l \in L \quad \text{(inclusion after switching)}$$

$$\sum_{k=1}^{p} |Y_{ik}| \leq s_i \quad \forall i \in \{1, \ldots, m\} \quad \text{(containment of initial set)}$$

Choosing the template

For **linear systems without switching**, a positive invariant exists if and only if a template complex zonotope having eigenvectors as the template is a positive invariant.

For a **switched system**, we can choose the template as the collection of eigenvectors of the switching matrices.

We can **add new direction vectors** to the template (to increase the chances of successful verification)

Experiments: Example 1

Example

Switching matrices [Allamigeon et. al. 2015].

$$A^f = \begin{pmatrix} -0.06515 & -0.4744 & 0.3041 \\ -0.4744 & 0.4872 & 0.3732 \\ 0.3041 & 0.3732 & -0.1271 \end{pmatrix} \quad A^g = \begin{pmatrix} 0.04419 & 0.3155 & -0.04247 \\ 0.1451 & -0.04931 & -0.2805 \\ 0.2833 & -0.01418 & 0.1554 \end{pmatrix},$$

We additionally considered an input set and a disturbance set.
Zonotopic disturbance set *is box* $\Omega = \langle \mathbb{I}_{3\times 3}, 0, [0.1\ 0.1\ 0.1]^T \rangle$
Initial set *as box* $\mathcal{I} = \langle \mathbb{I}_{3\times 3}, 0, [0.5\ 0.5\ 0.5]^T \rangle$.

We chose a template of eigenvectors A^f and A^g. For rectangular constraints $(\mathbb{I}_{3\times 3}, b)$ to be a linear invariant, we found $b = 0.9569$. Computation time is 0.6006s.

Note: To handle additive disturbance, it is not easy to extend the ellipsoidal method [Allamigeon *et. al.* 2015] (Minkowski sums need to be abstracted as ellipsoids), and the LMI approach (biconvex optimization required)

Experiments: Example 2

[Kouramas et. al. 2005]
Here we additionally considered an initial zonotopic set around the origin.

Switching matrices $A^f = F_1 + GK$ and $A^g = F_2 + GK$ where
$F_1 = \begin{pmatrix} 1.2 & 1 \\ 0 & 1 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0.8 & 1 \\ -0 & 1 \end{pmatrix}$, $G = [1\ 1]^T$ and $K = [-1.2\ \ -1]$.
Disturbance set is box $\Omega = \left\langle 0.1 \times \mathbb{I}_{3\times 3}, 0, [0.1\ 0.1\ 0.1]^T \right\rangle$. **Initial set** is box $\mathcal{I} = \left\langle \mathbb{I}_{3\times 3}, 0, [1\ 1\ 1]^T \right\rangle$ and rectangular constraints of the form $\|x\|_\infty \leq b$.

We chose a template of eigenvectors A^f and A^g. For rectangular constraints $(\mathbb{I}_{2\times 2}, b)$ is a linear invariant, we found b to be 1.10.
Computation time is 0.3920s.

Random examples

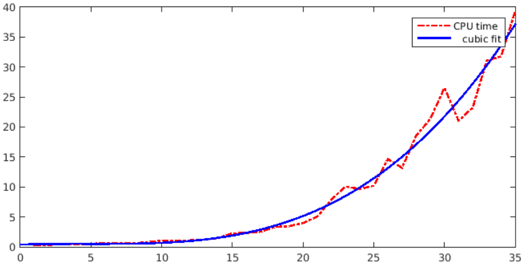

Figure: Computation time (in seconds) for scaling factors vs dimension

Disturbance set (in all the runs) is the unit box centered at the origin.
Initial set is the origin.

Conclusion

Template complex zonotopes are appropriate for verification of stability and invariance properties, since they can capture eigenstructures of linear dynamics $\Rightarrow$ good convergence of fixed point computations

Ongoing Work

Extending the framework to invariant computation of hybrid systems

Template Complex Zonotopes: Inclusion Checking

Problem: checking the inclusion between two template complex zonotopes centered at the origin.
Although this problem is non-convex in general, we derive an easily verifiable sufficient condition, as a set of second order conic constraints.
Let $V \in \mathbb{M}_{n\times m}(\mathbb{C})$, $G \in \mathbb{M}_{n\times r}(\mathbb{C})$, $s \in \mathbb{R}^m_{\geq 0}$ and $h \in \mathbb{C}^r_{\geq 0}$ for some $n, m, r \in \mathbb{Z}_{>0}$. Let $\mathcal{Z} = \langle V, 0, s \rangle$, $\mathcal{Z}' = \langle \bar{G}, 0, g \rangle$ be two template complex zonotopes. Recall that $\mathcal{D}(h)$ is the diagonal matrix with h along the diagonal. Then $\mathcal{Z}' \subseteq \mathcal{Z}$ if all the following constraints are collectively satisfied.

$$\begin{aligned}
&\exists X \in \mathbb{M}_{m\times r}(\mathbb{C}) \text{ such that}\\
&VX = G\mathcal{D}(h)\\
&\left(\sum_{j=1}^{r} |X_{ij}|\right) \leq s_i \quad \forall i \in \{1, \dots, m\}
\end{aligned} \tag{10}$$

Template Complex Zonotopes: Inclusion Checking

Proof.

Any point p inside $\mathcal{Z}'$ can be represented as $p = G\zeta$ for some $\zeta \in \mathbb{C}^r$ such that for all $j \in \{1, \ldots, r\}$, $|\zeta_j| \leq h_j$. Equivalently, p can be written as $G\mathcal{D}(h)\epsilon$ such that $\forall j \in \{1, \ldots, r\}, \epsilon_j = \zeta_j / h_j$. Using the equations in (10), p can be rewritten as $VX\epsilon$ where $\forall j \in \{1, \ldots, r\} : |\epsilon_j| = |\zeta_j| / h_j \leq 1$. So, for p to be included in $\mathcal{Z}$ as well, it is sufficient if the absolute values of components of the vector of coefficients $X\epsilon$ to be less than the corresponding components of s, i.e. $\forall i \in \{1, \ldots, m\} : |\sum_{j=1}^{r} X_{ij}\epsilon_i| \leq s_i$. Since $\forall j \in \{1, \ldots, r\} : |\epsilon_j| \leq 1$, using this bound gives the last inequality in (10). $\square$