Robust Hybrid Feedback Control Design for Networked Systems

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Hybrid Systems Laboratory
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Prevalent Network Control Applications

Control of Groups of Neurons [ACC 14, TCNS 16]

Multi-agent Systems with Limited Information [TCNS 16, Automatica 16]

Coordination of Underactuated Vehicles [Automatica 15, TAC 16]

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Key Features:

- Nonlinearities
- Fast time scales / events
- Limited information

Motivation and Approach

Common features in applications:

- Variables changing continuously (e.g., physical quantities) and discretely (e.g., logic variables, resetting timers).
- Abrupt changes in the dynamics (changes in the environment, control decisions, communication events, or failures).

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Driving Question:

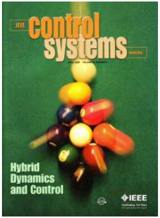
How can we systematically design such systems with provable robustness to uncertainties arising in real-world environments?

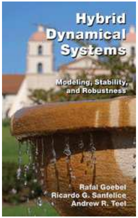
Approach:

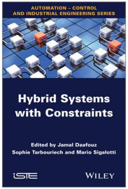
- Capture continuous and discrete behavior using dynamical modeling.
- Analysis of stability and control design using control theoretical tools.
- Numerical (and sometimes experimental) validation.

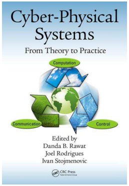
Recent Contributions to Hybrid Systems Theory

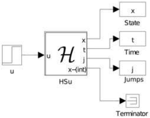
- Autonomous Hybrid Systems
  - Notion of Solution
  - Lyapunov Theory and Invariance
  - Robustness to Small Perturbations
- Nonautonomus Hybrid Systems
  - Notion of Solution
  - Control Lyapunov Functions
  - I/O Stability
  - Interconnections
  - Small gain Theorems
- Hybrid Control Design
  - Minimum-norm Control
  - Passivity-based Control
  - Backstepping
  - Tracking Control
- Numerical Simulation
  - Simulation theory
  - Simulation toolbox











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1. **Introduction**

    - Motivation, Approach, and Recent Contributions
    - Prevalent Network Control Applications

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   - Robustness Tools
   - Applications to Network Estimation and Synchronization

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Related Work

A (rather incomplete) list of related contributions:

- Differential equations with multistable elements

  [Witsenhausen - TAC 66]
- Differential automata [Tavernini - NA 87]
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- Measure-driven differential equations [Moreau 88] [Silva and Vinter - J. Math. Anal. Appl. 96]
- Systems with unilateral constraints [Brogliato 96]
- ⋮

Modeling Trajectories

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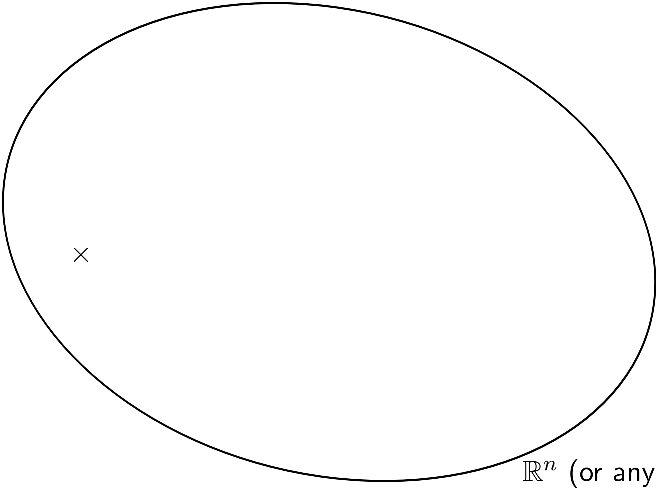

$\mathbb{R}^n$ (or any open set)

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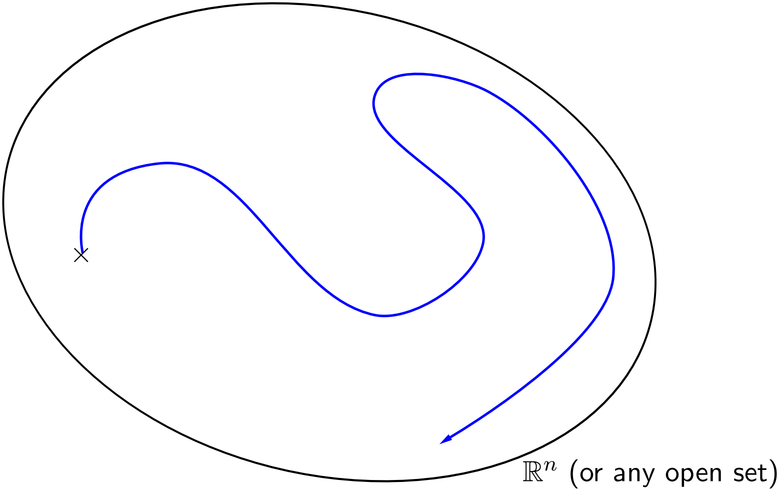

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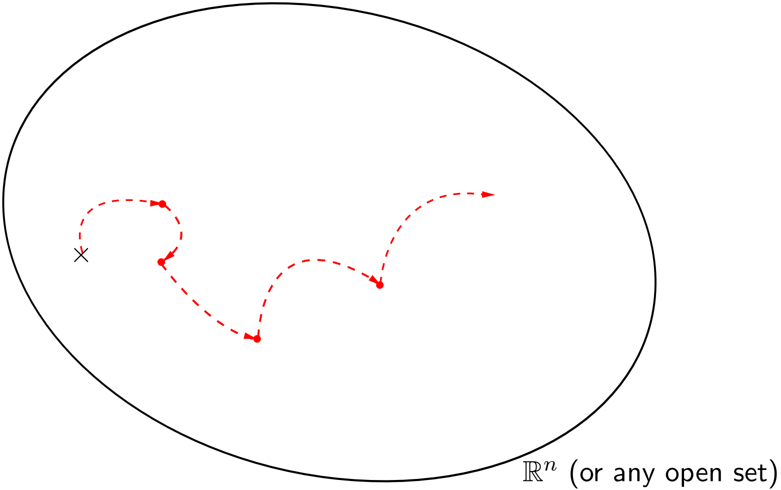

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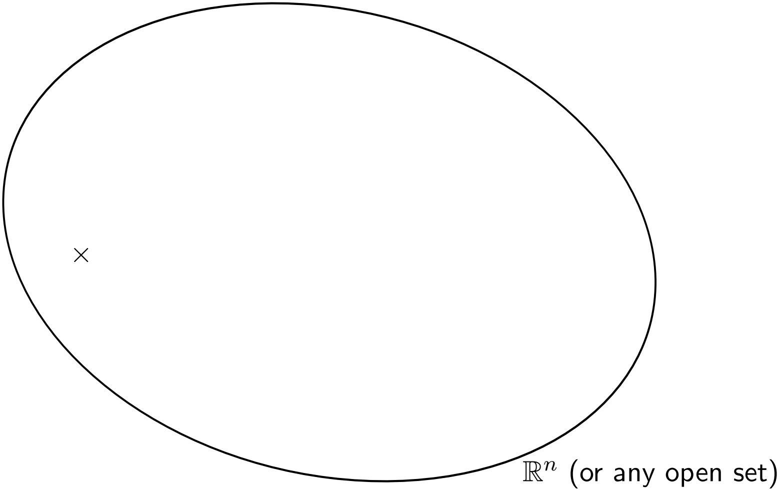

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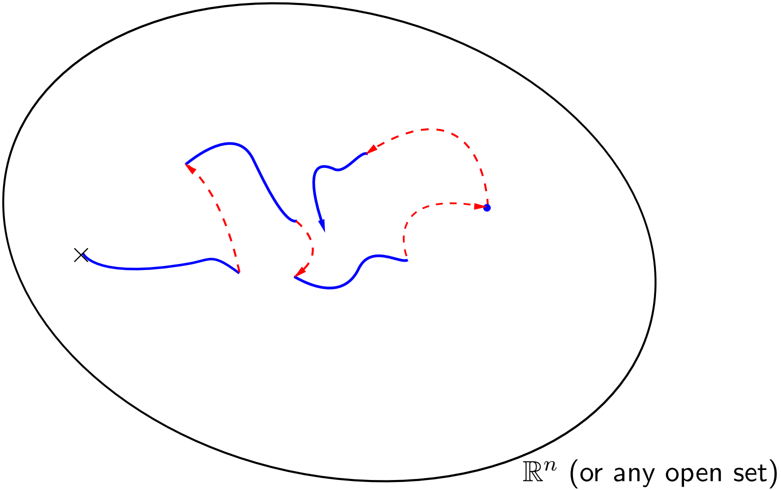

Modeling Hybrid Systems

Hybrid systems are given by *hybrid inclusions*

$$\mathcal{H} \quad \left\{ \begin{array}{ccll} \dot{x} & = & f(x,u) & \qquad (x,u) \in C \\ x^+ & = & g(x,u) & \qquad (x,u) \in D \\ y & = & h(x,u) & \end{array} \right.$$

where x is the *state*, u the *input*, y the *output*

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- f is the *flow map*
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The state x can have logic, memory, and timer components.

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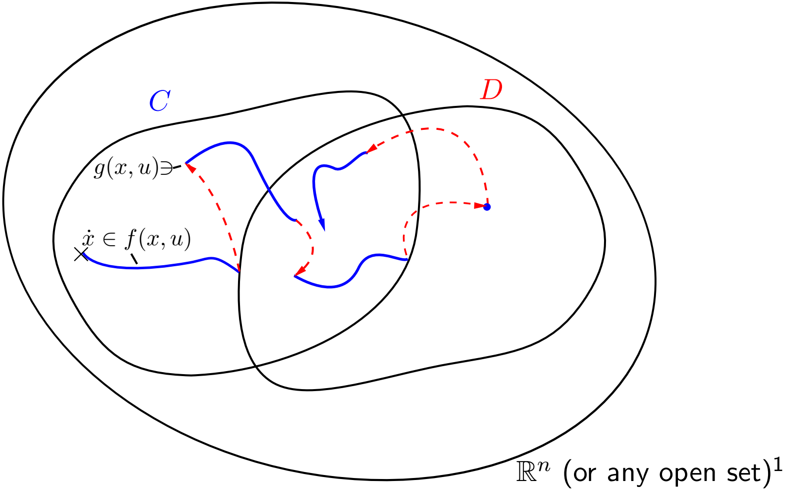

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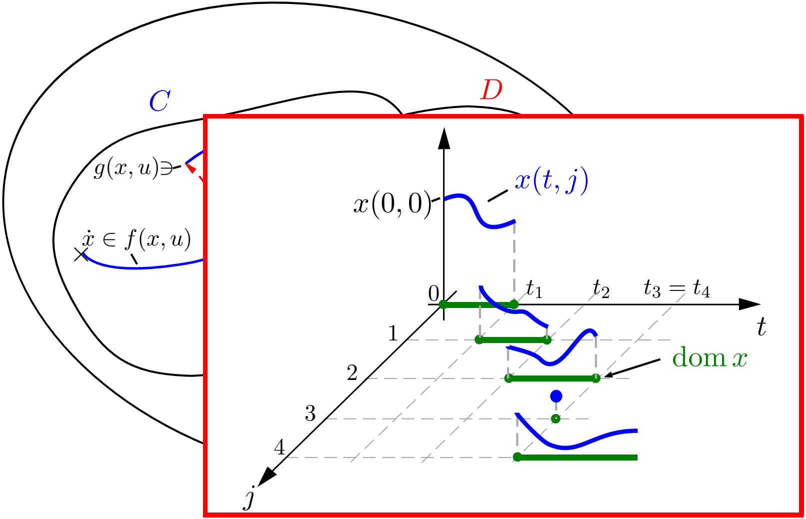
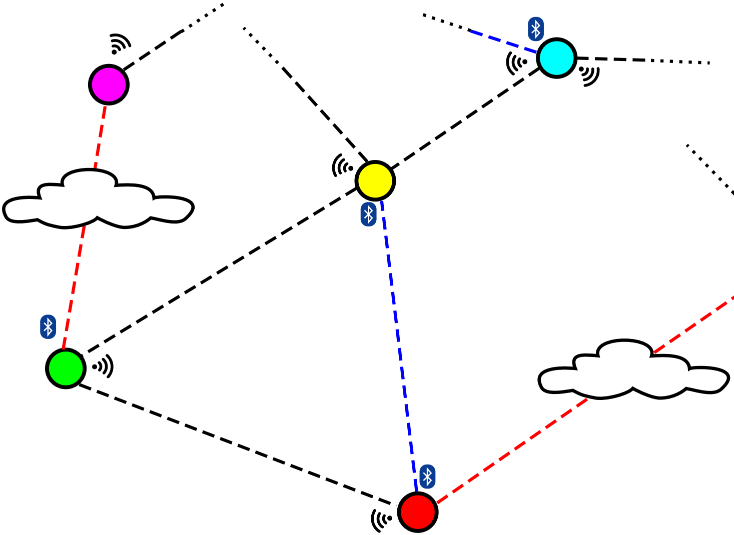


General Network Control Setting



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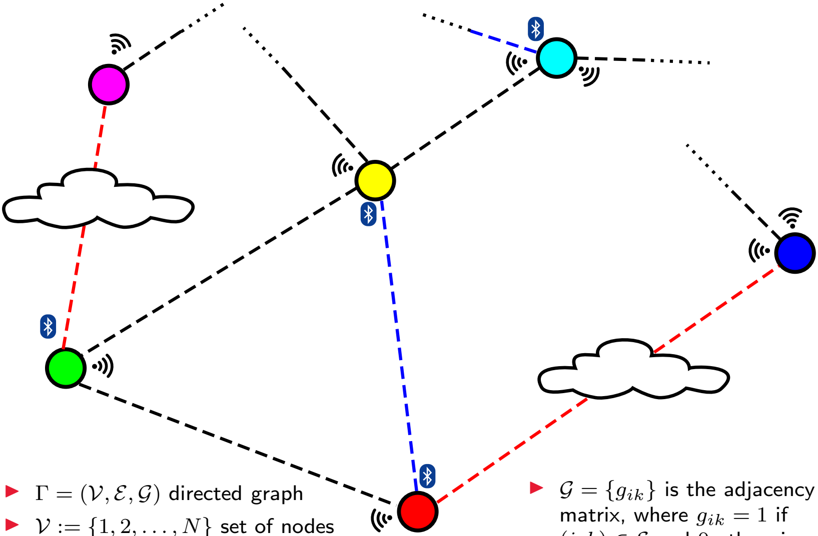

- $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$ directed graph
- $\mathcal{V} := \{1, 2, \ldots, N\}$ set of nodes
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ set of edges
- $\mathcal{J}_i = \{k : (i, k) \in \mathcal{E}\}$ neighbors to agent i
- $\mathcal{G} = \{g_{ik}\}$ is the adjacency matrix, where $g_{ik} = 1$ if $(i, k) \in \mathcal{E}$ and 0 otherwise
- d_i^{in} and d_i^{out} are the indegree and outdegree of agent i

Estimation Over a Network

Let the dynamics of the i-th node of the network be

$$\dot{z}_i = A_i z_i, \qquad y_i = M_i z_i$$

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Goal

Design of an **observer** for z_i that runs at each of the other agents and measures y_i at communication event times $\{t_\ell\}_{\ell=1}^{\infty}$ satisfying

$$T_1 \leq t_{\ell+1} - t_\ell \leq T_2$$

where

- T_1 defines the fastest communication rate
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Proposed Observer: $\hat{z}_k$ stores the estimate of z_i

$$\begin{aligned} \dot{\hat{z}}_k(t) &= A_i \hat{z}_k(t) && \text{when } t \notin \{t_\ell\}_{\ell=1}^{\infty} \\ \hat{z}_k^+ &= \hat{z}_k(t) + L_k(y_i(t) - M_i \hat{z}_k(t)) && \text{when } t \in \{t_\ell\}_{\ell=1}^{\infty} \end{aligned}$$

where L_k is a matrix (gain) to be designed.

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Globally exponentially stabilize the set, denoted $\mathcal{A}_{obs}$, collecting all points such that

$$z_i = \hat{z}_k \qquad \forall k \in \mathcal{V}$$

that is, render the zero estimation error set GES

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Idea: To capture all possible event sequences $\{t_\ell\}_{\ell=1}^{\infty}$ while removing dependency on time and stochastic dynamics, define

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$$\begin{cases} \dot{\tau} & = & -1 & \qquad \tau \in [0, T_2] \\ \tau^+ & \in & [T_1, T_2] & \qquad \tau = 0 \end{cases}$$

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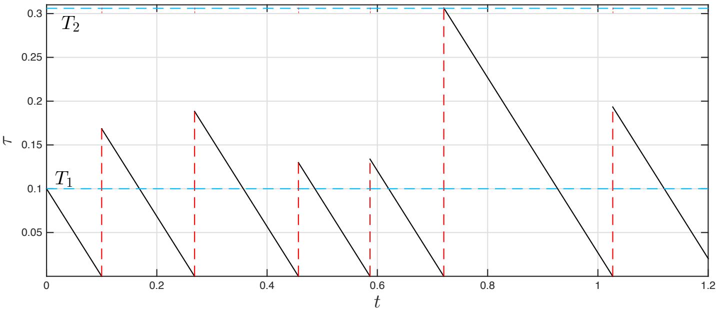

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Then, with $x = (e, \tau)$, we have

$$\mathcal{H}_{obs}^{ik} \begin{cases} \left.\begin{array}{rcl} \dot{e} & = & A_i e \\ \dot{\tau} & = & -1 \end{array}\right\} =: f(x) & (e,\tau) \in C \\ \\ \left.\begin{array}{rcl} e^+ & = & (I - L_k M_i) e \\ \tau^+ & \in & [T_1, T_2] \end{array}\right\} =: g(x) & (e,\tau) \in D \end{cases}$$

with the flow set and the jump set defined as

$$C = \mathbb{R}^n \times [0, T_2], \qquad D = \{(e,\tau) \colon \tau = 0\}$$

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- $\lim_{t \to \infty} |z_i(t) - z_k(t)| = 0$ for each $i, k \in \mathcal{V}$
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Proposed Controller: Controller with state η_i assigns $u_i = \eta_i$

$$\begin{cases} \dot{\eta}_i(t) &= 0 & \text{when } t \in \{t_\ell\}_{\ell=1}^{\infty} \\ \eta_i^+ &= \dfrac{K_i}{d_i^{in}} \displaystyle\sum_{k\in\mathcal{J}_i} (z_i(t) - z_k(t)) & \text{when } t \notin \{t_\ell\}_{\ell=1}^{\infty} \end{cases}$$

where $\mathcal{J}_i = \{k : (i,k) \in \mathcal{E}\}$ collects i-agent neighbors and d_i^{in} is its in-degree.

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General Control Problem

Given a set $\mathcal{A}$ and a hybrid system $\mathcal{H}$ to be controlled

$$\mathcal{H} \quad \left\{ \begin{array}{ccc} \dot{x} & \in & f(x,u) \qquad\qquad (x,u) \in C \\ x^+ & \in & g(x,u) \qquad\qquad (x,u) \in D \\ y & = & h(x,u) \end{array} \right.$$

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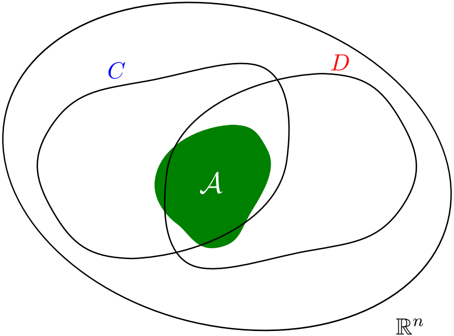

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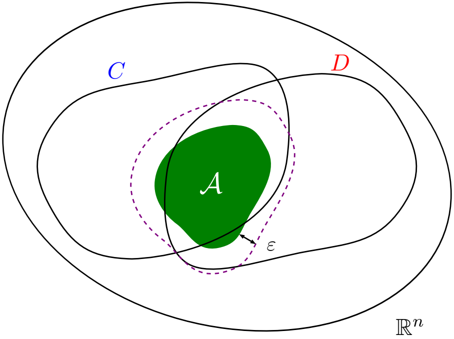

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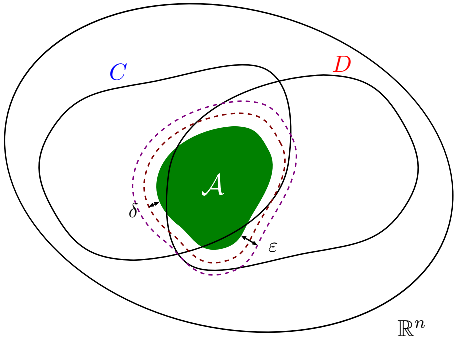

General Control Problem

Given a set $\mathcal{A}$ and a hybrid system $\mathcal{H}$ to be controlled

$$\mathcal{H} \quad \left\{ \begin{array}{ccll} \dot{x} & \in & {\color{blue}f}(x,u) & \qquad (x,u) \in {\color{blue}C} \\ x^+ & \in & {\color{red}g}(x,u) & \qquad (x,u) \in {\color{red}D} \\ y & = & h(x,u) & \end{array} \right.$$

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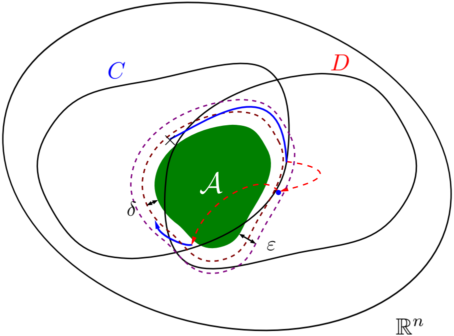

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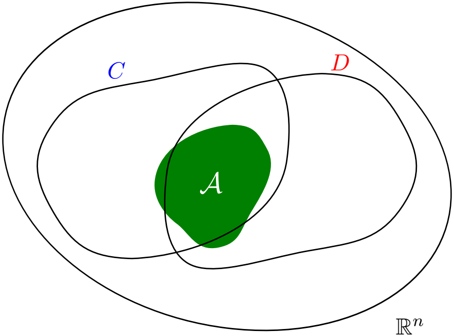

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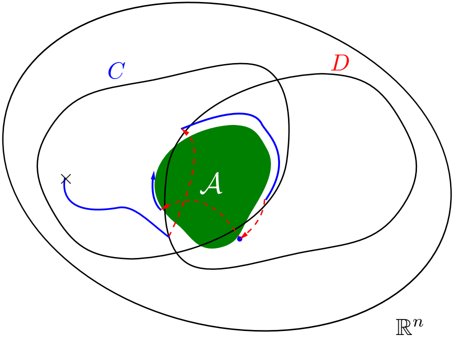

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- solutions starting close to $\mathcal{A}$ stay close (*stability*)
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The **feedback law** could be *output feedback: use y instead of x*

- *static*: $u = \kappa(x)$
- *dynamic*: u is a function of x and other controller states

resulting in a hybrid closed-loop system $\mathcal{H}$ (without inputs)

$$\mathcal{H} \quad \begin{cases} \dot{x} & \in & f(x) & & x \in C \\ x^+ & \in & g(x) & & x \in D \end{cases}$$

Lyapunov Stability Theorem

Theorem (Lyapunov Theorem)

Given a hybrid system $\mathcal{H}$ with state x (and no inputs)

$$\begin{array}{rcl} \dot{x} & \in & f(x) \qquad\qquad x \in C \\ x^+ & \in & g(x) \qquad\qquad x \in D \end{array}$$

a closed set $\mathcal{A} \subset \mathbb{R}^n$ is **asymptotically stable** *if there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that*

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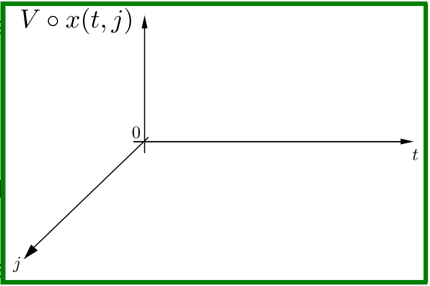

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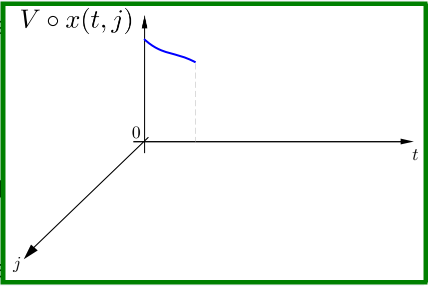

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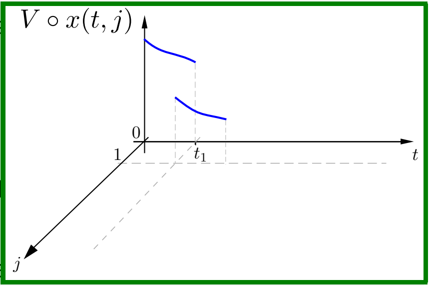

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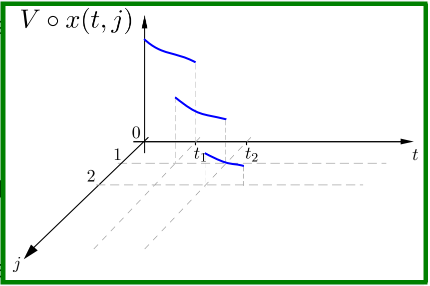

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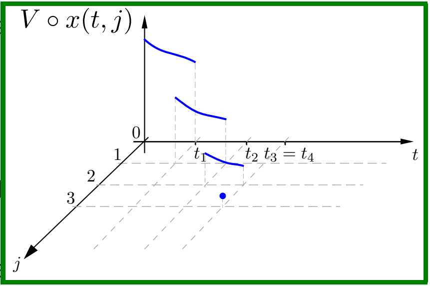

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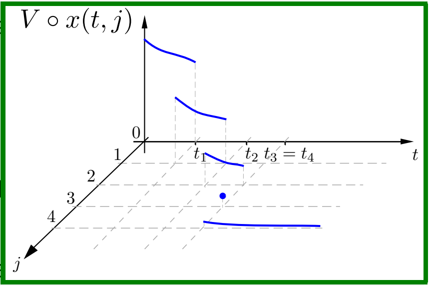

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Lyapunov Theorem Relaxed Flows

Corollary

Given a hybrid system $\mathcal{H}$ with state x (and no inputs)

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- *Every maximal solution has arbitrarily large number of jumps*
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Estimation Over a Network – Modeling

Time-varying and (potentially) stochastic system can be modeled as an autonomous hybrid inclusion

Idea: To capture all possible event sequences $\{t_\ell\}_{\ell=1}^{\infty}$ while removing dependency on time and stochastic dynamics, define

- $\tau \in [0, T_2]$ as a timer that, when expires, generates the communication events from the i-th agent to the j-th agent
- $e = z_i - \hat{z}_k$ as the estimation error used for analysis

Then, with $x = (e, \tau)$, we have

$$\mathcal{H}_{obs}^{ik} \begin{cases} \left.\begin{array}{rcl} \dot{e} & = & A_i e \\ \dot{\tau} & = & -1 \end{array}\right\} =: f(x) & (e,\tau) \in C \\ \\ \left.\begin{array}{rcl} e^+ & = & (I - L_k M_i)e \\ \tau^+ & \in & [T_1, T_2] \end{array}\right\} =: g(x) & (e,\tau) \in D \end{cases}$$

with the flow set and the jump set defined as

$$C = \mathbb{R}^n \times [0, T_2], \qquad D = \{(e,\tau) \colon \tau = 0\}$$

Estimation Over a Network – Stability

Theorem (Global Estimation with Limited Information)

Given two positive scalars T_1 and T_2 such that $T_1 < T_2$, if there exist $P = P^\top > 0$ and a matrix L_k such that

$$(I - L_k M_i)^\top \exp(A_i^\top v) P \exp(A_i v)(I - L_k M_i) - P < 0 \qquad (\star)$$

for all $v \in [T_1, T_2]$, then the set

$$\mathcal{A}_{obs} = \{(z_i, \hat{z}_k, \tau) \ : \ \hat{z}_k = z_i, \tau \in [0, T_2] \ \}$$

is **globally exponentially stable**.

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Construction of V: with $x = (z_i, \hat{z}_k, \tau)$, define

$$V(x) = e^\top \exp(A_i^\top \tau) P \exp(A_i \tau) e$$

F. Ferrante, F. Gouaisbaut, S, and S. Tarbouriech "State Estimation of Linear Systems in the Presence of Sporadic Measurements." To appear in Automatica, 2016.

Estimation Over a Network – Stability

Proposition (LMI version of $(\star)$)

Let T_1 and T_2 be two given positive scalars such that $T_1 < T_2$. If there exist $P = P^\top > 0$, a matrix J, and a matrix F such that for every $v \in [T_1, T_2]$

$$\begin{bmatrix} -(F+F^\top) & F - JM_i & \exp(A_i^\top v)P \\ \star & -P & 0 \\ \star & \star & -P \end{bmatrix} < 0$$

then the matrices P and $L_k = F^{-1}J$ satisfy $(\star)$.

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then the matrices P and $L_k = F^{-1}J$ satisfy $(\star)$.

To reduce the check to finitely many LMIs, we over approximate $\exp(A_i v)$ by

$$\exp(A_i v) \in \mathrm{co}\{F_1, F_2, \ldots, F_\nu\}$$

over $[T_1, T_2]$, and then solve the ν LMIs with appropriate F.

Proof Sketch.

The proof is based on the use of the following Lyapunov function

$$V(x) = e^\top \exp(A_i^\top \tau) P \exp(A_i \tau) e$$

which satisfies

recall that $\mathcal{A}_{obs} = \{(z_i, \hat{z}_k, \tau) \ : \ \hat{z}_k = z_i, \tau \in [0, T_2] \ \}$

$$\alpha_1 |x|^2_{\mathcal{A}_{obs}} \leq V(x) \leq \alpha_2 |x|^2_{\mathcal{A}_{obs}} \qquad \forall x \in C \cup D \cup g(D)$$

The key properties of V are

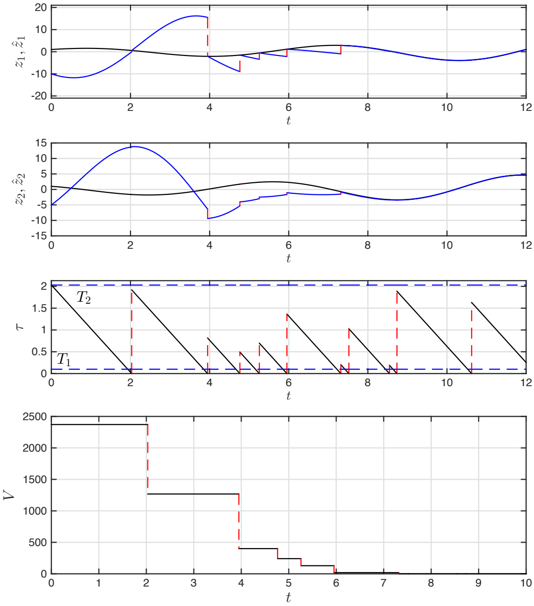
$$\langle \nabla V(x), f(x) \rangle = 0 \qquad \forall x \in C$$

and, by $(\star)$, it follows that for some $\beta > 0$

$$V(g') - V(x) \leq -\beta e^\top e = -\beta |x|^2_{\mathcal{A}_{obs}} \qquad \forall x \in D, g' \in g(x)$$

Then, to apply the Lyapunov Theorem with relaxed flows, we note that, for each solution ϕ, $(t, j) \in \operatorname{dom} \phi$ is such that $t \leq T_2(j+1)$, which is a persistently jumping property.

Estimation with Optimized MATI



i-th Agent dynamics:

$$A_i = \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix}$$
$$M_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

k-th Observer gain:

$$L_k = \begin{bmatrix} 1 \\ 0.173 \end{bmatrix}$$

Communication parameters:

$T_1 = 0.1, \quad T_2 = 2.03$

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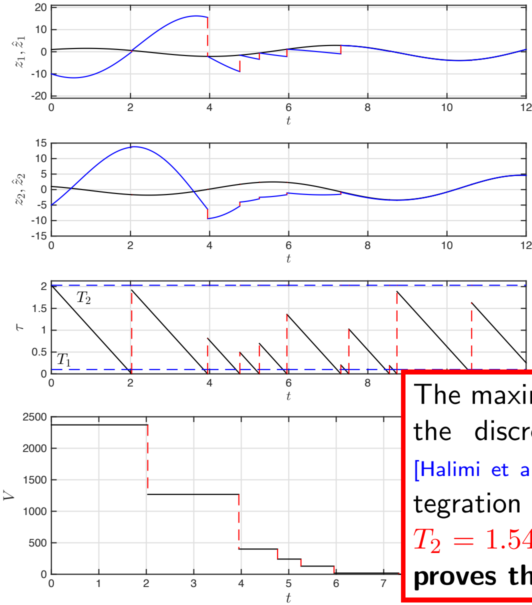

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The maximum MATI (T_2) from the discrete-time observer in [Halimi et al., Springer 13], with integration in between events, is $T_2 = 1.54$. **Our approach improves the MATI by 32%!**

Estimation Over a Network w/Information Fusion

Let the dynamics of the i-th node of the network be

$$\dot{z}_i = A_i z_i, \qquad y_i = M_i z_i$$

Estimation Over a Network w/Information Fusion

Let the dynamics of the i-th node of the network be

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Goal

Design of an **observer** for z_i that runs at each of the other agents and measures y_i at communication event times $\{t_\ell^i\}_{\ell=1}^{\infty}$ satisfying

$$T_1^i \leq t_{\ell+1} - t_\ell \leq T_2^i$$

where

- T_1^i defines the fastest communication rate
- T_2^i represents the Maximum Allowable Transfer Time (MATI)

Related Work

Periodic sampling case

- Observer-protocol pair for LTI networked systems (single agent) [Dacic and Nesic AUT 08]
- Discrete observer for LTI networked systems (common information arrival times) [Park and Martins CDC 12]
- Continuous observer with discrete information (assume local observability) [Dorfler ea. JSTSP 13]
- Continuous-time observer for a class of nonlinear systems (single agent) [Ahmed-Ali ea. SCL 13]

Continuous-discrete observer

- Lipschitz continuous-time systems of small dimensions using reachable sets (periodic) [Farza ea. TAC 14] [Dinh ea. TAC 15]
- Impulsive systems approach (periodic) [Mazenc ea. SIAM 15]
- Hybrid systems approach (single-agent case) [Ferrante ea. AUT 16]

Hybrid Distributed Observer

The i-th agent runs the following (local) observer:

$$\dot{\hat{z}}_i = A\hat{z}_i + \eta_i$$

where η_i is an *information fusion state*, evolves **continuously** and updates **impulsively** at communication events.

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Hybrid Information Fusion

$$\dot{\eta}_i = h_i \eta_i \qquad \tau_i \in [0, T_2^i]$$

$$\eta_i^+ = K_{ii} y_i^e + \sum_{k \in \mathcal{J}_i} K_{ik} y_k^e + \gamma \sum_{k \in \mathcal{J}_i} (\hat{z}_i - \hat{z}_k) \qquad \tau_i = 0$$

$y_i^e = M_i \hat{z}_i - y_i, \quad y_i = M_i z_i + m_i$ measurements w/ noise

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Goal

Guarantee **exponential convergence of $\hat{z}_i - z$ to zero** for each i, robustly to general perturbations and potentially without local detectability (at each node).

Compact Formulation and Error Dynamics

Denote the local estimation error $e_i = \hat{z}_i - z$, $e = (e_1, e_2, \cdots, e_N)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_N)$, it follows that

$$\left.\begin{array}{ll} \dot{e} & = (I_N \otimes A)e + \eta \\ \dot{\eta} & = H(h)\eta \end{array}\right\} \text{ when } \tau_i \in [0, T_2^i] \quad \forall\, i \in \mathcal{V}$$

while when $\tau_i = 0$ for some $i \in \mathcal{V}$,

$$e^+ = e$$

$$\eta_i^+ = K_{ii} y_i^e + \sum_{k \in \mathcal{V}} g_{ik} K_{ik} y_k^e + \gamma \sum_{k \in \mathcal{V}} g_{ik} (\hat{z}_i - \hat{z}_k)$$

- $H(h) = \text{diag}(h_1, h_2, \ldots, h_N)$
- K_{ii}, K_{ik}'s are gains for output errors
- $\gamma \in \mathbb{R}$ is a constant gain for consensus

Compact Formulation and Error Dynamics

Denote the local estimation error $e_i = \hat{z}_i - z$, $e = (e_1, e_2, \cdots, e_N)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_N)$, it follows that

$$\left.\begin{aligned} \dot{e} &= (I_N \otimes A)e + \eta \\ \dot{\eta} &= H(h)\eta \end{aligned}\right\} \text{ when } \tau_i \in [0, T_2^i] \quad \forall\, i \in \mathcal{V}$$

while when $\tau_i = 0$ for some $i \in \mathcal{V}$,

$$e^+ = e$$

$$\eta_i^+ = K_{ii} y_i^e + \sum g_{ik} K_{ik} y_k^e + \gamma \sum g_{ik}(\hat{z}_i - \hat{z}_k)$$

Catch: Complex behavior at communication events

- Which τ_i is zero?
- How many τ_i's are zero?
- What is a Lyapunov function candidate?

Compact Formulation and Error Dynamics

Idea: Define the new coordinates

$$\theta_i = K_{ii} y_i^e + \sum_{k \in \mathcal{J}_i} K_{ik} y_k^e + \gamma \sum_{k \in \mathcal{J}_i} (e_i - e_k) - \eta_i$$

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Denoting $e = (e_1, \cdots, e_N)$, $\theta = (\theta_1, \cdots, \theta_N)$, $\tau = (\tau_1, \cdots, \tau_N)$

- $\theta = ((K_g M_g) * (I_N + \mathcal{G}) + \gamma \mathcal{L} \otimes I_n) e - \eta$
- K_g is a $N \times N$ block matrix with the (i,k)-th entry given by $K_{ik} \in \mathbb{R}^{n \times p}$ for all $i, k \in \mathcal{V}$
- M_g is block diagonal, $M_g = \text{diag}(M_1, M_2, \ldots, M_N)$
- the operation "$*$" denotes the Khatri-Rao product, where the matrix $K_g M_g$ is treated as a $N \times N$ block matrix

Y. Li, S. Phillips, and S, "On Distributed Observers for Linear Time-invariant Systems Under Intermittent Information Constraints", Proceedings of 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS), 2016.

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$$\dot{e} = \underbrace{(I_N \otimes A + \mathcal{K})}_{A_\theta} e - \theta$$

$$\dot{\theta} = \mathcal{K} A_\theta e - \mathcal{K} \theta$$

when $\boxed{\tau \in \mathcal{T}}$, $\mathcal{K} = K_g M_g * (I_N + \mathcal{G}) + \gamma \mathcal{L} \otimes I_n$, and

$$e_i^+ = e_i$$
$$\theta_i^+ = 0$$

when $\boxed{\tau_i = 0}$ for some i, $\mathcal{T} := [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N]$

Compact Formulation and Error Dynamics

Interconnection as a Hybrid System

The interconnection $\mathcal{H}$ has state $\chi = (\sigma, \tau)$, $\sigma = (e, \theta)$, with data

$$f(\chi) := (A_{f\theta}\sigma, -\mathbf{1}_N), \quad A_{f\theta} = \begin{bmatrix} A_\theta & -I_{nN} \\ \mathcal{K}A_\theta & -\mathcal{K} \end{bmatrix}$$

for each $\chi \in C = \mathcal{X} := \mathbb{R}^{2nN} \times \mathcal{T}$,

$$g(\chi) := \{g_i(\chi) : \chi \in D_i, i \in \mathcal{V}\}$$

when $\chi \in D = \bigcup_{i\in\mathcal{V}} D_i$, $D_i = \{\chi \in C : \tau_i = 0\}$,

$$g_i(\chi) = \begin{bmatrix} e \\ (\theta_1, \theta_2, \cdots, \theta_{i-1}, 0, \theta_{i+1}, \cdots, \theta_N) \\ (\tau_1, \tau_2, \cdots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \cdots, \tau_N) \end{bmatrix}$$

Sufficient Conditions for GES

Global Exponential Stability of Zero Estimation Error

Let $0 < T_1^i \leq T_2^i$ be given. Suppose N agents are connected via a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. Moreover, suppose there exist $\gamma \in \mathbb{R}$, $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN \times p}$, $P \in \mathbb{R}^{nN \times nN}$, $Q_i \in \mathbb{R}^{n \times N}$ satisfying $P = P^\top > 0$, $Q_i = Q_i^\top > 0$ for all $i \in \mathcal{V}$, and

$$\mathcal{N} := \begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + \widetilde{A}_\theta^\top \mathcal{K}^\top \widetilde{Q}(\nu) \\ \star & -\delta \widetilde{Q}(\nu) - \mathrm{He}(\widetilde{\mathcal{K}}, \widetilde{Q}(\nu)) \end{bmatrix} < 0$$

$$\forall \nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathcal{T}$$

with

$$\widetilde{Q}(\nu) = \mathrm{diag}\left(\exp(\delta\nu_1)Q_1, \cdots, \exp(\delta\nu_N)Q_N\right).$$

Then, the set $\mathcal{A} = \{0_{nN}\} \times \{0_{nN}\} \times \mathcal{T}$ is GES for the hybrid system $\mathcal{H}$.

Lyapunov-based Analysis

Proof sketch: Consider the Lyapunov function candidate

$$V(x) = e^\top P e + \theta^\top \widetilde{Q}(\tau)\theta \qquad \forall x \in \mathcal{X}$$

with the data of $\mathcal{H}$. It can be shown that

- $\langle \nabla V(x), f(x) \rangle \leq -|\overline{\lambda}(\mathcal{N})||x|^2_{\mathcal{A}}$ for each $x \in C$,
- $V(g') - V(x) \leq 0$ for each $x \in D$ and for each $g' \in g(x)$

which using a Lyapunov Theorem with relaxed jumps leads us to

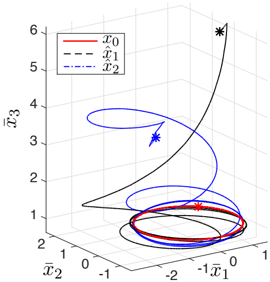
$$|\phi(t,j)|_{\mathcal{A}} \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \exp\left(-\frac{|\overline{\lambda}(\mathcal{N})|}{2\alpha_2} \min\left\{\epsilon, (1-\epsilon)\frac{T_1^{\min}}{2N}\right\}(t+j)\right)|\phi(0,0)|_{\mathcal{A}}$$

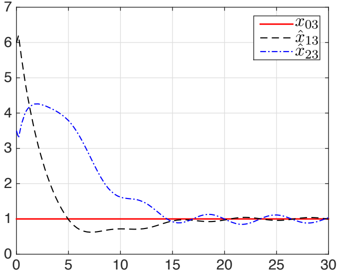
where $\epsilon \in (0,1)$,

$$\alpha_1 = \min\left\{\underline{\lambda}(P), \underline{\lambda}\left(\widetilde{Q}(0)\right)\right\}, \quad \alpha_2 = \max\left\{\overline{\lambda}(P), \overline{\lambda}\left(\widetilde{Q}(\overline{T}_2)\right)\right\},$$

and $\overline{T}_2 = (T_2^1, T_2^2, \ldots, T_2^N)$, $\overline{\lambda}$ gives the maximum eigenvalue, $\underline{\lambda}$ gives the minimum, and $T_1^{\min}$ is the minimum over the T_1^i's

Estimation without Local Detectability





Plant Dynamics:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

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Communication Parameters:

$$T_1^i = 0.2, \quad T_2^i = 0.4$$

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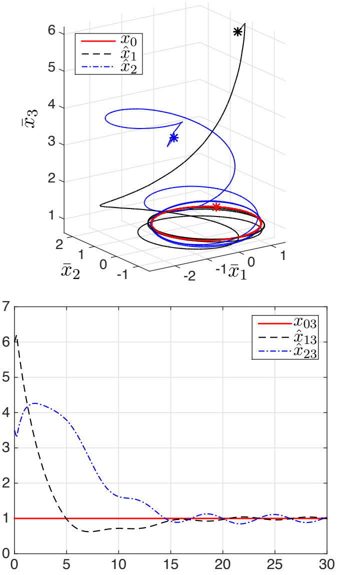

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Without communication,
no agent can estimate x

Synchronization Over a Network – Modeling

As for the estimation problem, we use a timer

- τ as a timer that, when expires, generates the communication events between the agents

Synchronization Over a Network – Modeling

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with dynamics

$$\begin{cases} \dot{\tau} & = & -1 & & \tau \in [0, T_2] \\ \tau^+ & \in & [T_1, T_2] & & \tau = 0 \end{cases}$$

Synchronization Over a Network – Modeling

As for the estimation problem, we use a timer

- τ as a timer that, when expires, generates the communication events between the agents
- and the local average error $e_i = \dfrac{1}{d_i^{in}} \sum_{k \in \mathcal{J}_i} (z_i - z_k)$

Synchronization Over a Network – Modeling

As for the estimation problem, we use a timer

- τ as a timer that, when expires, generates the communication events between the agents

Then, with $x = (\chi, \tau)$, $\chi = (\chi_1, \chi_2, \dots, \chi_N)$, $\chi_i = (e_i, \eta_i)$, we have

$$\mathcal{H}_{sync}: \begin{cases} \dot{x} = \begin{bmatrix} \bar{A}_f \chi \\ -1 \end{bmatrix} =: f(x) & x \in C = \mathbb{R}^{N(n+p)} \times [0, T_2] \\ x^+ \in \begin{bmatrix} \bar{A}_g \chi \\ [T_1, T_2] \end{bmatrix} =: g(x) & x \in D = \mathbb{R}^{N(n+p)} \times \{0\} \end{cases}$$

where

$$\bar{A}_f = I_N \otimes A_f - (\mathcal{G}^\top \mathcal{J}^{-1}) \otimes B_f \qquad \bar{A}_g = \operatorname{diag}(A_{g1}, A_{g2}, \dots, A_{gN})$$

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad B_f = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \qquad A_{gi} = \begin{bmatrix} I & 0 \\ K_i & 0 \end{bmatrix}$$

Synchronization over Networks – Stability

Theorem (Global Synchronization with Intermittent Information)

Given two positive scalars T_1 and T_2 such that $T_1 \leq T_2$ and the digraph Γ, then the set

$$\widetilde{\mathcal{A}}_{sync} = \{(\chi, \tau) \; : \; \chi = 0, \; \tau \in [0, T_2] \; \}$$

is **globally exponentially stable** when

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- the graph Γ is strongly connected and there exist $\sigma > 0$, $P = P^\top > 0$, and, for each $i \in \mathcal{V}$, a matrix K_i such that

$$\exp(\sigma v)\bar{A}_g^\top \exp(\bar{A}_f^\top v) P \exp(\bar{A}_f v)\bar{A}_g - P < 0 \qquad (\star\star)$$

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- the graph Γ is completely connected and there exist $\sigma > 0$, $P = P^\top > 0$, and a matrix K such that

$$\exp(\sigma v) A_g^\top \exp(A_f^\top v) P \exp(A_f v) A_g - P < 0 \qquad (\star\star\star)$$

for each $v \in [T_1, T_2]$.

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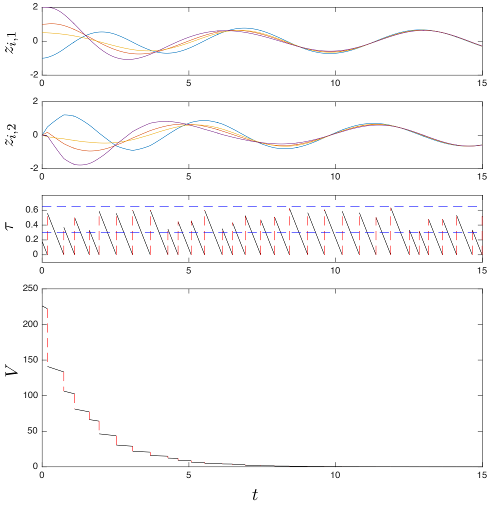
Construction of V (for strongly connected case):
with $x = (\chi, \tau)$ and $\sigma > 0$, define

$$V(x) = \exp(\sigma\tau)\chi^\top \exp(\bar{A}_f^\top \tau) P \exp(\bar{A}_f \tau)\chi$$

S. Phillips, and S. "Robust Synchronization of Interconnected Linear Systems over Intermittent Communication Networks", In Proceedings of the American Control Conference (ACC), 2016.

Synchronization Over a Network - Simulation



i-th **Agent Dynamics:**

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

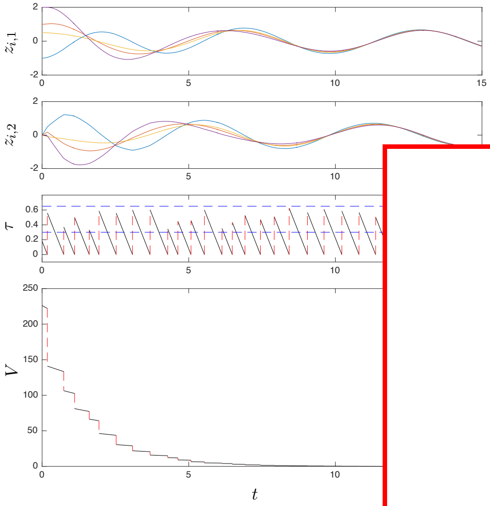
i-th **Controller Gain:**

$$K_i = -\begin{bmatrix} 0.5 & 0.7 \end{bmatrix}$$

Communication Parameters:

$$T_1 = 0.3, \quad T_2 = 0.65$$

Synchronization Over a Network - Simulation



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Local Distributed Hybrid Controller, η_i

The i-th agent assigns its input to the following local controller state:

$$u_i = \eta_i$$

where the dynamics of η_i is given by a zero-order hybrid protocol that updates impulsively at communication events.

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Controller Algorithm

$$\dot{\eta}_i = 0 \qquad\qquad \tau \in [0, T_2]$$

$$\eta_i^+ = -\gamma \sum_{k=1}^{N} g_{ik}(x_i - x_k) \qquad\qquad \tau = 0$$

where g_{ik} are the components of the adjacency matrix.

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Goal

Guarantee **pointwise exponential stability** of the set of points $x_i = x_k$ for each i, k with robustness to communication noise.

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S. Phillips, Y. Li, and S. "On Distributed Intermittent Consensus for First-Order Systems with Robustness." In Proceedings of 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS), 2016.

Partial Pointwise Global Exponential Stability

Definition

Consider a hybrid system $\mathcal{H}$ with state $x = (p, q) \in \mathbb{R}^n$. The closed set $\mathcal{A} \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$ where $r \in \mathbb{N}$ and $0 < r \leq n$ is **partially pointwise globally exponentially stable** with respect to the state component p if

1. $\mathcal{A}$ is exponentially attractive
2. every maximal solution ϕ to $\mathcal{H}$ is complete and has a limit belonging to $\mathcal{A}$
3. for each $p^* \in \mathbb{R}^r$ s.t. there exists $q \in \mathbb{R}^{n-r}$ satisfying $(p^*, q) \in \mathcal{A}$, it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}$ is such that its p component ϕ_p satisfies

$$|\phi_p(0,0) - p^*| \leq \delta \implies |\phi_p(t,j) - p^*| \leq \varepsilon$$

for all $(t, j) \in \operatorname{dom} \phi$.

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Stability Results

Main Results

Let $0 < T_1 \leq T_2$, and Γ be strongly connected and weight balanced. If there exists a positive scalar γ and $P = P^\top > 0$ such that

$$A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < 0$$

$$A_f = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

$$A_g = \begin{bmatrix} I & 0 \\ -\gamma \bar{\mathcal{L}} & 0 \end{bmatrix}$$

for each $\nu \in [T_1, T_2]$ where $\bar{\mathcal{L}} = \text{diag}(\lambda(\mathcal{L}) \setminus \{0\})$, then

- the hybrid system $\mathcal{H}_{sync}$ has the set $\mathcal{A}_{sync}$ GES.

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- the hybrid system $\mathcal{H}_{sync}$ has the set $\mathcal{A}_{sync}$ GES.
- every solution ϕ is complete and

$$\lim_{t+j \to \infty} z_i(t,j) = \frac{1}{N} \sum_{i=1}^{N} z_i(0,0) + \eta_i(0,0)\tau(0,0)$$

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- the set $\mathcal{A}$ is partially pointwise globally exponentially stable with respect to (z, η) for the hybrid system $\mathcal{H}_{sync}$.

Proof Sketch.

Let $x = (\chi, \tau)$, $\chi = (e, \eta)$ and $e_i = z_i - \sum_{k=1}^{N} z_i$. The proof is based off the following Lyapunov function

$$V(x) = \chi^\top \widetilde{T} \exp(A_{f2}^\top \tau) P \exp(A_{f2} \tau) \widetilde{T}^\top \chi$$

where $\widetilde{T} = \text{diag}(T, T)$ and T is an orthonormal diagonalizing matrix for the Laplacian $\mathcal{L}$.

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$$\alpha_1 |x|^2_{\widetilde{\mathcal{A}}_{sync}} \leq V(x) \leq \alpha_2 |x|^2_{\widetilde{\mathcal{A}}_{sync}}$$

Then, the key properties of V are

- $\langle \nabla V(x), f(x) \rangle = 0$ for each $x \in C$,
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Then, the key properties of V are

- $\langle \nabla V(x), f(x) \rangle = 0$ for each $x \in C$,
- $V(g) - V(x) \leq -\beta |x|^2_{\widetilde{\mathcal{A}}_{sync}}$ for each $x \in D$.

which leads us to

$$|\phi(t,j)|_{\widetilde{\mathcal{A}}_{sync}} \leq \exp\left(\frac{R}{2}\right) \sqrt{\frac{\alpha_2}{\alpha_1}} \exp\left(-\frac{\alpha}{2}(t+j)\right) |\phi(0,0)|_{\widetilde{\mathcal{A}}_{sync}}$$

where $\alpha \in \left(0, \frac{|\beta|}{1+T_2}\right)$, $R = \left[\frac{T_2|\beta|}{1+T_2}, \infty\right)$

Asynchronous Update Times

A local timer for each agent triggering communication

$$\begin{aligned} \dot{\tau}_i &= -1 & \qquad \tau_i &\in [0, T_2], \\ \tau_i^+ &\in [T_1, T_2] & \qquad \tau_i &= 0. \end{aligned}$$

when τ_i expires, information is transferred to agent i.

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Consensus Algorithm

$$\begin{aligned} \dot{\eta}_i &= h\eta_i & \qquad\qquad \tau_i &\in [0, T_2] \\ \eta_i^+ &= -\gamma \sum_{k=1}^{N} g_{ik}(z_i - z_k) & \qquad\qquad \tau_i &= 0 \end{aligned}$$

where g_{ik} are the components of the adjacency matrix.

- the gains h and γ are to be determined.

Asynchronous Update Times

A local timer for each agent triggering communication

$$\begin{aligned} \dot{\tau}_i &= -1 & \tau_i &\in [0, T_2], \\ \tau_i^+ &\in [T_1, T_2] & \tau_i &= 0. \end{aligned}$$

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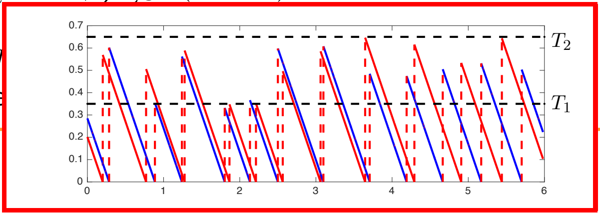
- the



Synchronization Over a Network

Hybrid System, $\mathcal{H}$, with asynchronous update times

$\mathcal{H}_{sync}$ has state $x = (z, \eta, \tau) \in \mathcal{X} := \mathbb{R}^{2N} \times [0, T_2]^N$ with the following dynamics

$$\dot{x} := \begin{bmatrix} \eta \\ h\eta \\ -1_N \end{bmatrix} \qquad\qquad \forall x \in \mathcal{X}$$

jumps are induced when a single timer state reaches 0, i.e. when $x \in \bigcup_{i\in\mathcal{V}} D_i$, $D_i = \{x \in C : \tau_i = 0\}$. Then,

$$x^+ \in \{G_i(x) : x \in D_i, i \in \mathcal{V}\}$$

where

$$G_i(x) = \begin{bmatrix} z \\ (\eta_1, \eta_2, \ldots, \eta_{i-1}, -\gamma \sum_{k=1}^{N} g_{ik}(z_i - z_k), \eta_{i+1}, \ldots, \eta_N) \\ (\tau_1, \tau_2, \ldots, \tau_{i-1}, [T_1, T_2], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix}$$

Asynchronous Update Times - Stability

Proposition

Given $0 < T_1 \leq T_2$ and a strongly connected and weight balanced digraph Γ. If there exist scalars $\gamma, h \in \mathbb{R}$, and $\sigma > 0$, positive definite diagonal matrices P and Q such that

$$\begin{bmatrix} \gamma \mathrm{He}(P, \mathcal{L}) & -P\Pi + \mathcal{K}_1^\top Q E(\tau) \\ \star & -\sigma Q E(\tau) - \mathrm{He}(QE(\tau), \mathcal{K}_2) \end{bmatrix} \leq 0 \qquad (3)$$

for each $\tau \in [0, T_2]^N$, where $\Pi = I - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^\top$, $\mathcal{K}_1 = \gamma \mathcal{K}_2 \mathcal{L}$, $\mathcal{K}_2 = \gamma \mathcal{L} - hI$, and $E(\tau) = \mathrm{diag}(e^{\sigma\tau_1}, e^{\sigma\tau_2}, \ldots, e^{\sigma\tau_N})$, then the set $\mathcal{A}$ is globally asymptotically stable for the hybrid system $\mathcal{H}_{sync}$.

Proof Sketch.

Let $x = (e, \theta, \tau)$, where $e_i = z_i - \sum_{k=1}^{N} z_k$ and $\theta_i = \gamma \sum_{k \in \mathcal{J}_i} (x_i - x_k) - \eta$. Consider the Lyapunov function candidate

$$V(x) = e^\top P e + \theta^\top Q E(\tau) \theta \qquad \forall \chi \in \mathcal{X}$$

which satisfies

$$\alpha_1 |x|^2_{\mathcal{A}_{sync}} \leq V(x) \leq \alpha_2 |x|^2_{\mathcal{A}_{sync}}$$

with the data of $\mathcal{H}$. As a consequence of (4), it can be shown that

- $\langle \nabla V(x), f(x) \rangle \leq 0$ for each $x \in C$
- $V(g') - V(x) \leq 0$ for each $x \in D$ and $g' \in g(x)$

Then, we apply the Invariance Principle involving a nonincreasing function, which results in global asymptotic stability of $\widetilde{\mathcal{A}}_{sync}$.

Outline

1. **Introduction**
    - Motivation, Approach, and Recent Contributions
    - Prevalent Network Control Applications
2. **Hybrid Systems Tools for Control of Networks**
    - Hybrid Inclusion Models
    - Lyapunov Stability Tools
    - Robustness Tools
    - Applications to Network Estimation and Synchronization
3. **Conclusion**

Nominal Robustness For Networked Systems

Further results on robustness to $\rho > 0$

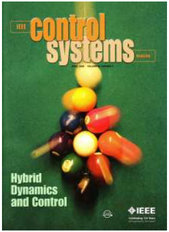
- Parameter uncertainty
  - Skewed clocks
  $$\dot{\tau}_i = -1 \qquad \to \qquad \dot{\tau}_i \in -1 + \rho\mathbb{B}$$
  - Rate uncertainty
  $$[T_1^i, T_2^i] \qquad \to \qquad [T_1^i - \rho, T_2^i + \rho]$$
- Unmodeled dynamics
  - Additive dynamics
  $$\dot{z}_i = f_i(z_i, u_i) \qquad \to \qquad \dot{z}_i \in f_i(z_i, u_i) + \rho\mathbb{B}$$
  - Event conditions
  $$\tau_i = 0 \qquad \to \qquad \tau_i \in \rho\mathbb{B}$$
- Disturbances
  - Actuator noise (ISS)
  - Measurement noise (ISS)

Conclusion

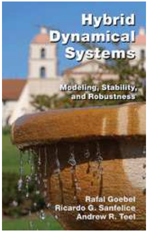
- Introduction to Hybrid Systems and Modeling for Control of Networks
- Control Design via Lyapunov and its Relaxations
- Nominal Robustness via Checkable Properties
- Applications and Current Projects

Future Directions:

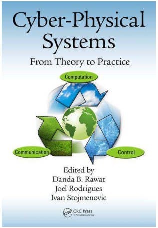
- Temporal logic
- Optimality conditions
- Hybrid games
- Computation-based learning



IEEE 2009



Princeton U. Press 2012



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Conclusion

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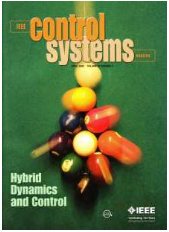

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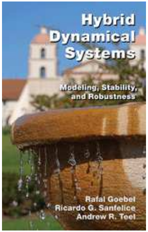

Princeton U. Press 2012

References at hybrid.soe.ucsc.edu

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One-day Workshop @ 2016 CDC in Las Vegas
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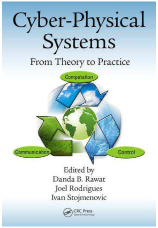

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