Non-smooth and hybrid systems in opinion dynamics

Paolo Frasca

based on joint works with
Francesca Ceragioli (Politecnico Torino),
Sophie Tarbouriech and Luca Zaccarian (LAAS)

Co4 Workshop
LAAS, Toulouse
October 27, 2016

Outline

Basic opinion dynamics

Opinions $x_i(t) \in \mathbb{R}$ for population of individuals $i \in \mathcal{I} = \{1, \dots, N\}$

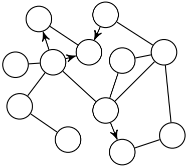
$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(x_j - x_i)$$

Opinions evolve through interactions between agents

- $a_{ij} = 1$ if j influences i; $a_{ij} = 0$ otherwise
- interactions described by the graph with adjacency matrix A

Additional notation:

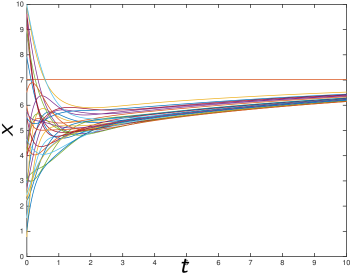
- degree $d_i = \sum_j a_{ij}$
- Laplacian $L = \mathrm{diag}(d) - A$



Opinion dynamics and consensus

If there is one node that can be reached from all other nodes
$\Longrightarrow$ convergence to consensus of opinions

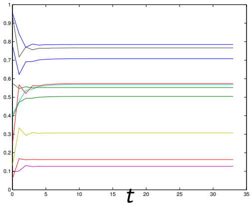
$$x_i(t) \to \alpha \in \mathbb{R} \text{ as } t \to +\infty \text{ for all } i \in \mathcal{I}$$



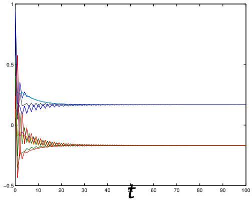
Issue: Societies do not exhibit consensus!

Models for disagreement: some potential causes

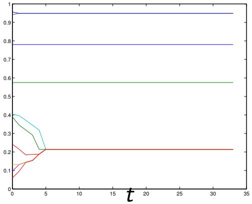
Prejudices



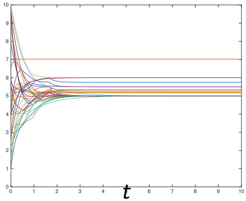
Antagonistic interactions



Bounded confidence



Discretized interactions



In this talk we focus on the last two $\Longrightarrow$ non-smooth systems

Solutions

Let $I \subset \mathbb{R}$ be an interval of the form $(0, T)$.

- A continuously differentiable function $x : I \to \mathbb{R}^N$ is a classical solution if it satisfies $\dot{x} = f(x)$ for all $t \in I$
- An absolutely continuous function $x : I \to \mathbb{R}^N$ is a Carathéodory solution if it satisfies $\dot{x} = f(x)$ for almost all $t \in I$ or, equivalently, if it is a solution of the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\mathrm{d}s$$

- An absolutely continuous function $x : I \to \mathbb{R}^N$ is a Krasowskii solution of $\dot{x} = f(x)$ if, for almost every $t \in I$, it satisfies

$$\dot{x}(t) \in \mathcal{K}f(x(t))$$

where

$$\mathcal{K}f(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}(\{f(y) \ : \ y \text{ such that } \|x - y\| < \delta\})$$

Discrete interactions

Quantized opinions as discrete behaviors

Quantizer $q : \mathbb{R} \to \mathbb{Z}$ defined as $q(s) = \lfloor s + \frac{1}{2} \rfloor$

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \left(q(x_j) - x_i \right) \tag{Q}$$

Motivation: limited verbalization [Urbig'03], discrete actions [Martins'08]

Quantized opinions as discrete behaviors

Quantizer $q : \mathbb{R} \to \mathbb{Z}$ defined as $q(s) = \lfloor s + \frac{1}{2} \rfloor$

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \left(q(x_j) - x_i \right) \tag{Q}$$

Motivation: limited verbalization [Urbig'03], discrete actions [Martins'08]

Comparison with quantized consensus dynamics:

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \left(q(x_j) - q(x_i) \right)$$

[Ceragioli,DePersis&F.'11]
[Wei,Yi,Sandberg&Johansson'16]

$$\dot{x}_i = \sum_{j \in \mathcal{I}} a_{ij} \, q(x_j - x_i)$$

[Dimarogonas&Johansson'10]

these two dynamics approximately converge to consensus

Carathéodory solutions: good and bad news

Solutions to (Q)

From every initial condition there exists a complete Carathéodory solution

Carathéodory solutions: good and bad news

Solutions to (Q)

From every initial condition there exists a complete Carathéodory solution

Pathological attractors

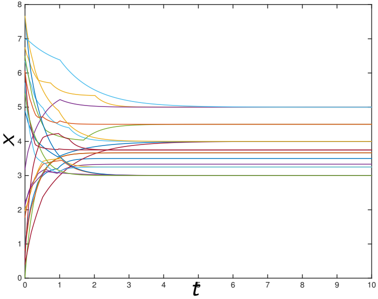
It exists x^* such that $x(t) \to x^*$ but x^* is not equilibrium ?!

Example: $(0, 0.49, 0.51, 1) \to (0, \frac{1}{2}, \frac{1}{2}, 1)$ on path graph

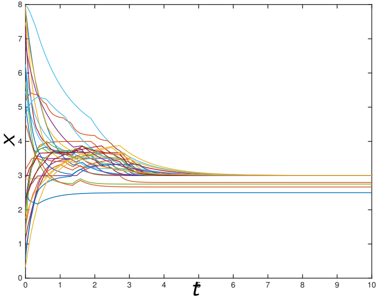
More generally: $(0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \dots, \frac{N-2}{2})$ is attractive non-equilibrium

Examples: far from consensus

Lack of consensus is actually very common:



Geometric random graph



Directed Erdős graph

Krasowskii solutions

Asymptotical distance from consensus

Assume

- $x(t)$ is Krasowskii solution to (Q)
- the graph has symmetric adjacency matrix A
- $M = \left\{ x \in \mathbb{R}^N : \inf_{\alpha \in \mathbb{R}} \|x - \alpha \mathbf{1}\| \leq \frac{||A||}{\lambda_2} \frac{\sqrt{N}}{2} \right\}$
  λ_2 is smallest positive eigenvalue of L

then, $\operatorname{dist}(x(t), M) \to 0$ as $t \to +\infty$

Proof sketch:

- quantization error $x - q(x)$ is bounded
- Lyapunov function $V(x) = \frac{1}{2}||x - x_{\mathrm{ave}}\mathbf{1}||^2$ with $x_{\mathrm{ave}} := \frac{1}{N}\sum_{i=1}^{N} x_i$

Krasowskii solutions

Asymptotical distance from consensus

Assume

- $x(t)$ is Krasowskii solution to (Q)
- the graph has symmetric adjacency matrix A
- $M = \left\{ x \in \mathbb{R}^N : \inf_{\alpha \in \mathbb{R}} \|x - \alpha \mathbf{1}\| \leq \frac{||A||}{\lambda_2} \frac{\sqrt{N}}{2} \right\}$

  λ_2 is smallest positive eigenvalue of L

then, $\operatorname{dist}(x(t), M) \to 0$ as $t \to +\infty$

Proof sketch:

- quantization error $x - q(x)$ is bounded
- Lyapunov function $V(x) = \frac{1}{2}||x - x_{\text{ave}}\mathbf{1}||^2$ with $x_{\text{ave}} := \frac{1}{N}\sum_{i=1}^{N} x_i$

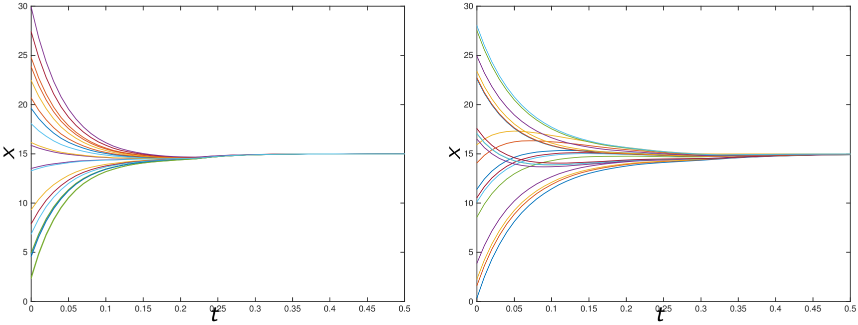
Note: M is tight on path graphs: $\exists x^*$ such that $\frac{1}{\sqrt{N}}\|x^* - x^*_{\text{ave}}\mathbf{1}\| = \Theta(N^2)$

Special graphs

Convergence

Krasowskii solutions to (Q) converge to integer consensus $x^* = k\mathbf{1}$ if

- the graph is complete; or
- the graph is complete bipartite



Conclusions on discrete behaviors

1. This was the simplest possible model...
2. Quantized behaviors can explain disagreement
3. Preferred notion of solutions is Krasowskii (= Filippov in this case)

Open problems:

- Does the dynamics converge?
- Necessary and sufficient conditions for consensus (which topologies?)
- Are there closed solutions?
- Are there non-Caratheodory non-constant solutions with non-negligible basin of attraction?

Bounded confidence

Bounded confidence

Model: Confidence threshold $R > 0$

$$\dot{x}_i = \sum_{j : |x_i - x_j| < R} (x_j - x_i) \tag{BC}$$

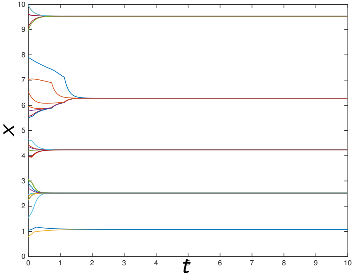
[Hegselmann&Krause'02] [Blondel,Hendricks&Tsitsiklis'10]

Bounded confidence

Model: Confidence threshold $R > 0$

$$\dot{x}_i = \sum_{j:|x_i - x_j| < R} (x_j - x_i) \tag{BC}$$

[Hegselmann&Krause'02] [Blondel,Hendricks&Tsitsiklis'10]



- Discontinuous right-hand side
- Formation of clusters where individuals agree

Existence of solutions

Solutions to (BC)

From almost every initial condition there exists a complete unique Carathéodory solution

From every initial condition there exists a complete Krasowskii solution

Carathéodory solutions $\subsetneq$ Krasowskii solutions

Example: $N = 3,\ R = 1$

$x(0) \in \{x \ : \ |x_1 - x_2| < 1, x_3 - x_2 = 1\}$

$$\dot{x} \in \left\{ \alpha \begin{bmatrix} x_2 - x_1 \\ 1 + x_1 - x_2 \\ -1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 - x_1 \\ x_1 - x_2 \\ 0 \end{bmatrix} \ : \ \alpha \in [0,1] \right\}$$

which can be normal to the discontinuity surface

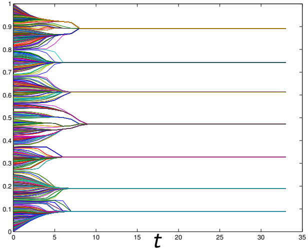
Equilibria and convergence

Convergence of (BC)

1. The set of Krasowskii equilibria is

$$E = \{x \in \mathbb{R}^{\mathcal{I}} \ : \ \text{for every } (i,j) \text{ either } x_i = x_j \text{ or } |x_i - x_j| \geq R\}$$

2. If $x(\cdot)$ is Krasowskii solution, then
   a) $x_{\text{ave}}(t) = x_{\text{ave}}(0)$
   b) $x(t) \to x^* \in E$ as $t \to +\infty$



Equilibria and convergence

Convergence of (BC)

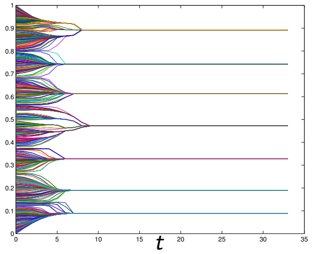
1. The set of Krasowskii equilibria is

$$E = \{x \in \mathbb{R}^{\mathcal{I}} \ : \ \text{for every } (i,j) \text{ either } x_i = x_j \text{ or } |x_i - x_j| \geq R\}$$

2. If $x(\cdot)$ is Krasowskii solution, then
   a) $x_{\mathrm{ave}}(t) = x_{\mathrm{ave}}(0)$
   b) $x(t) \to x^* \in E$ as $t \to +\infty$

Proof sketch:

- Average preservation
- Order preservation
- Contractivity and boundedness
- Lyapunov function $V(x) = \frac{1}{2} x^\top x$
- Invariance Principle [Ceragioli'00]



Properties of the equilibria

Set E is not strongly invariant and is not stable

Example: Take $N = 2$ and $R = 1$ and the solution $x(t) = (\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2t}, \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2t})$

Properties of the equilibria

Set E is not strongly invariant and is not stable

Example: Take $N = 2$ and $R = 1$ and the solution
$x(t) = (\frac{1}{2} + \frac{1}{2}\mathrm{e}^{-2t}, \frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2t})$

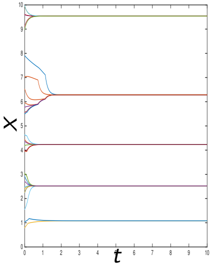
Definition: Equilibrium $x \in E$ is robust if no perturbation consisting in adding one agent causes two of the former clusters to coalesce in the resulting evolution

Let $x \in E$ and consider two clusters in x, denoted by A and B, having values x_A and x_B and cardinalities $n_A \leq n_B$

Robustness (R=1)

For the equilibrium $x \in E$ to be *robust* it is

- *sufficient* that $|x_B - x_A| > 2$ for every A, B
- *necessary* that $|x_B - x_A| > 1 + \frac{n_A}{n_B}$ for every A, B



Bounded confidence: Hybrid systems

Hybrid Laplacian dynamics

Potential edges (i, j) have status of variables $a_{ij} \in \{0, 1\}$: then $x = (y, a)$

$$\begin{cases} \dot{y}_i = \sum_{j \in \mathcal{I} \setminus \{i\}} a_{ij}(y_j - y_i) & \text{for all } i \in \mathcal{I} \\ \dot{a}_{ij} = 0 & \text{for all } (i,j) \in \mathcal{I} \times \mathcal{I} \end{cases} \tag{Flow}$$

$$\begin{cases} y_i^+ = y_i \quad \text{for all } i \in \mathcal{I} \\ a_{hk}^+ = 1 - a_{hk} \\ a_{ij}^+ = a_{ij} \quad \text{for all } (i,j) \neq (h,k) \end{cases} \qquad (y, a) \in D_{hk} \tag{Jump}$$

Jump set: $D = \bigcup_{hk} D_{hk}$

Flow set: $C = \overline{X \setminus D}$

Jump set

Bounded confidence with hysteresis regularization:

$$\begin{aligned} D_{hk}^{\text{on}} &:= \{a_{hk} = 0\} \cap \{(y_h - y_k)^2 \leq R^2 - \varepsilon\} \\ D_{hk}^{\text{off}} &:= \{a_{hk} = 1\} \cap \{(y_h - y_k)^2 \geq R^2 + \varepsilon\} \\ D_{hk} &:= D_{hk}^{\text{off}} \cup D_{hk}^{\text{on}} \end{aligned}$$

where R and ε are positive scalars and ε is (much) smaller that R

Remarks:

- Close approximation of the previous non-smooth model
- Well-posed and chattering-free dynamics

Convergence

Let $\tilde{E} = \{(y,a) \;:\; a_{ij}(y_i - y_j) = 0 \text{ for all } (i,j)\}$

Convergence of hybrid dynamics

If $x(\cdot)$ is hybrid solution then

- $x(t)$ has a finite number of jumps
- $x(t) \to x^* \in \tilde{E}$ as $t \to +\infty$
- $x^* = (y^*, a^*)$ is such that $y_i^* = y_j^*$ if $a_{ij}^* = 1$

Proof sketch:

- Boundedness
- Lyapunov function $V(x) = \frac{1}{2} y^\top y$
- Invariance Principle [Goebel,Sanfelice&Teel'12]

Convergence

Let $\tilde{E} = \{(y,a) \ : \ a_{ij}(y_i - y_j) = 0 \text{ for all } (i,j)\}$

Convergence of hybrid dynamics

If $x(\cdot)$ is hybrid solution then

- $x(t)$ has a finite number of jumps
- $x(t) \to x^* \in \tilde{E}$ as $t \to +\infty$
- $x^* = (y^*, a^*)$ is such that $y_i^* = y_j^*$ if $a_{ij}^* = 1$

Proof sketch:

- Boundedness
- Lyapunov function $V(x) = \frac{1}{2} y^\top y$
- Invariance Principle [Goebel,Sanfelice&Teel'12]

The set $\tilde{E}$ is not invariant and not stable:

take (a,y) such that $a_{ij} = 0$ and $y_i - y_j = R^2 - \varepsilon$

Conclusion

General conclusion

Summary

1. Opinion dynamics naturally lead to discontinuous systems
2. Generalized solutions are an important tool for their analysis
3. The hybrid framework can also be useful
4. Pathologies abound (mainly, convergence without stability)

General conclusion

Summary

1. Opinion dynamics naturally lead to discontinuous systems
2. Generalized solutions are an important tool for their analysis
3. The hybrid framework can also be useful
4. Pathologies abound (mainly, convergence without stability)

Outlook

a. Can we ensure both convergence and stability?
b. Can we control these discontinuous/hybrid models?
c. What is the meaning for social sciences?

References

- F. Ceragioli. *Discontinuous ordinary differential equations and stabilization*. PhD thesis, Università di Firenze, 2000
- R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3):1–33, 2002
- V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis. Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *SIAM Journal on Control and Optimization*, 48(8):5214–5240, 2010
- F. Ceragioli and P. Frasca. Continuous and discontinuous opinion dynamics with bounded confidence. *Nonlinear Analysis: Real World Applications*, 13(3):1239–1251, 2012
- F. Ceragioli and P. Frasca. Continuous-time consensus dynamics with quantized all-to-all communication. In *European Control Conference*, pages 1120–1125, Linz, Austria, July 2015
- F. Ceragioli and P. Frasca. Consensus and disagreement: the role of quantized behaviours in opinion dynamics. June 2016. Submitted. URL: `http://arxiv.org/abs/1607.01482`
- F. Ceragioli, C. De Persis, and P. Frasca. Discontinuities and hysteresis in quantized average consensus. *Automatica*, 47(9):1916–1928, 2011
- P. Frasca, S. Tarbouriech, and L. Zaccarian. A hybrid model of opinion dynamics with limited confidence. In *IFAC Symposium on Nonlinear Control Systems*, Monterey, CA, USA, August 2016
- N. E. Friedkin and E. C. Johnsen. *Social Influence Network Theory: A Sociological Examination of Small Group Dynamics*. Cambridge University Press, 2011
- D. V. Dimarogonas and K. H. Johansson. Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control. *Automatica*, 46(4):695–700, 2010
- R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid dynamical systems: modeling, stability, and robustness*. Princeton University Press, 2012
- A. C. R. Martins. Continuous opinions and discrete actions in opinion dynamics problems. *International Journal of Modern Physics C*, 19(04):617–624, 2008
- D. Urbig. Attitude dynamics with limited verbalisation capabilities. *Journal of Artificial Societies and Social Simulation*, 6(1):1–23, 2003
- J. Wei, X. Yi, H. Sandberg, and K.H. Johansson. Nonlinear consensus protocols with applications to quantized systems. April 2016. URL: `http://arxiv.org/abs/1604.03592`