# Recent developments on the stability of systems with aperiodic sampling


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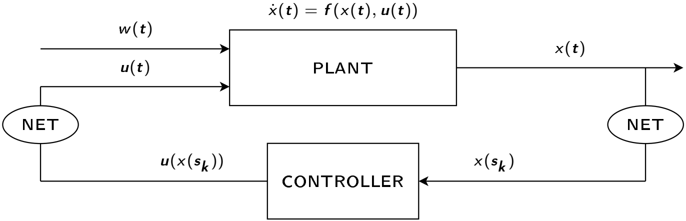
projets ANR ROCC-SYS, LimICoS

# Motivation : Networked Control Systems (NCS)




- Sampling instants $\{s_k\}_{k\in\mathbb{N}}$, $s_{k+1} = s_k + h_k$
- Fluctuations of the transmission (sampling) step

$$h_k = s_{k+1} - s_k \in [\underline{h}, \overline{h}]$$

# Challenges in NCS

$$\left.\begin{array}{l}\textbf{Processor} : \text{limited calculation power}\\ \textbf{Network} : \text{finite bandwidth}\\ \textbf{Sampler} : \text{minimum responding time}\end{array}\right\} \Rightarrow \text{finite number of samples per time unit}$$

How fast SHOULD we sample ? $\leftrightarrow$ How fast CAN we sample ?

# Challenges in NCS

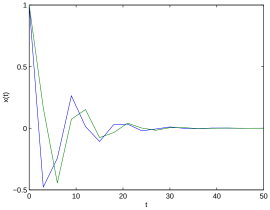
$$\left.\begin{array}{l}\textbf{Sampler clock} : \text{jitter} \\ \textbf{Network} : \text{packet dropouts} \\ \textbf{Scheduling} : \text{interaction between algorithms} \\ \textbf{Real-time computing} : \text{microprocessor latency}\end{array}\right\} \Rightarrow \text{sampling is not necessarily periodic}$$

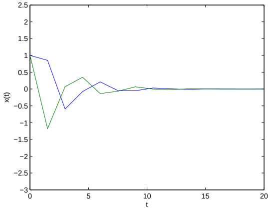
Possible destabilizing effect !

# Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Constant sampling step




$h_k = T_1 = 3s, \forall\; k \in \mathbb{N}$ :
STABLE

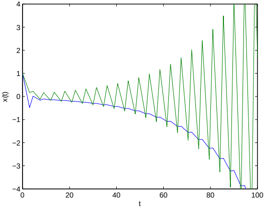
$h_k = T_2 = 1.5s, \forall\; k \in \mathbb{N}$ :
STABLE

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

## Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Periodic sampling sequence




$$\{h_k\}_{k\in\mathbb{N}} = \{3s, 1.5s, 3s, 1.5s, \cdots\}$$

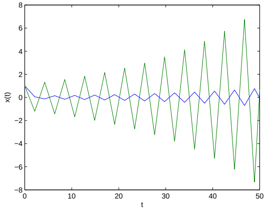
STABLE + STABLE $\Rightarrow$ UNSTABLE !

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

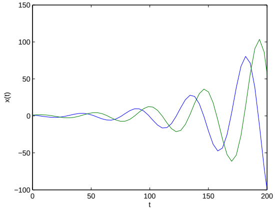
# Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Constant sampling step




$T = T_1 = 2.13s$ : UNSTABLE



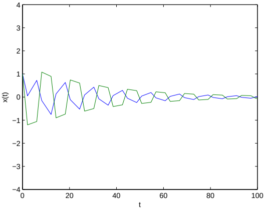

$T = T_2 = 3.95s$ : UNSTABLE

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

# Effects of sampling variation (Zhang, 2001)

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; u(t) = Kx(s_k)$$

Periodic sampling sequence



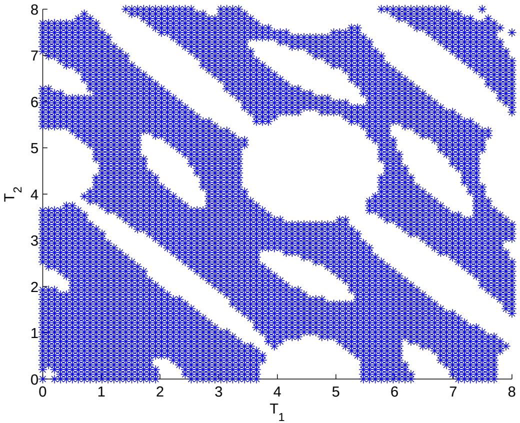

$$T = 3s \to 2.13s \to 3.95s \to 2.13s \to \cdots$$

UNSTABLE + UNSTABLE $\Rightarrow$ STABLE !

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

## Effects of sampling variation (Zhang, 2001)

Stability domain (allowable sampling intervals, in blue) for a periodic sampling sequence $T_1 \to T_2 \to T_1 \to T_2 \to \cdots$



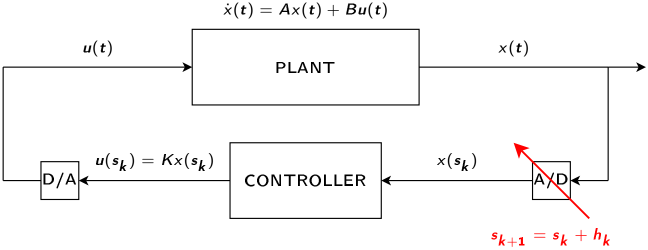
## Question

How to reduce the computational (processor and/or network) load while ensuring the system stability ?

# Research directions

2 main directions :

- Robust stability analysis with respect to time-varying sampling

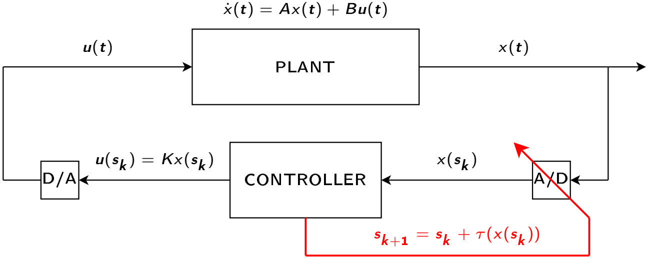



Sampling interval $h_k \in [\underline{h}, \overline{h}]$

# Research directions

2 main directions :

- Control of sampling

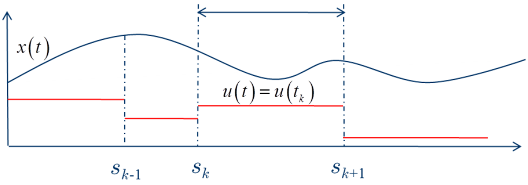



Sampling map $\tau : \mathbb{R}^n \to \mathbb{R}_+$

# Robust stability analysis with respect to time-varying sampling

- Discrete-time and Convex Embedding
- Time-delay approach
- Hybrid (Impulsive) System modelling approach
- I/O approach

# Discrete-time and Convex Embedding




- Basic references
  - Hetel, Daafouz, Iung - IEEE TAC 2006
  - Olaru, Niculescu - IFAC 2008
  - Fujioka - IEEE TAC 2009
  - Cloosterman et. al - TAC 2009
  - Hetel, Kruszewski, Perruquetti, Richard - IEEE TAC 2011
- Continuous-time model

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}), \; h_k = s_{k+1} - s_k \in [\underline{h}, \overline{h}]$$

- Discrete-time model (LPV system)

$$x_{k+1} = \Lambda(h_k)x_k, \quad \Lambda(h) = e^{Ah} + \int_0^h e^{As} ds BK$$

# Discrete-time and Convex Embedding

- Discrete-time model (LPV system)

$$x_{k+1} = \Lambda(h_k)x_k, \quad \Lambda(h) = e^{Ah} + \int_0^h e^{As} ds BK$$

- For quadratic Lyapunov functions

$$V(x) = x^T P x, \;\; P = P^T \succ 0$$

- Stability condition : $V(x_{k+1}) - V(x_k) < 0, \; \forall x \neq 0$

$$x^T \left( \Lambda^T(h) P \Lambda(h) - P \right) x < 0, x \neq 0$$

- Parametric set of Linear Matrix Inequalities (LMIs)

$$\Lambda^T(\tau) P \Lambda(\tau) - P \prec 0, \;\; \tau \in [\underline{h}, \overline{h}]$$

Infinite number of Lyapunov inequalities

## Discrete-time and Convex Embedding

- Parametric set of Linear Matrix Inequalities (LMIs)

$$\Lambda^T(\tau)P\Lambda(\tau) - P \prec 0, \ \tau \in [\underline{h}, \overline{h}]$$

Infinite number of Lyapunov inequalities

- Tractable conditions using a convex embedding :

$$\Lambda(\tau) = e^{A\tau} + \int_0^\tau e^{As} ds BK \in co\{L_1, L_2, \dots, L_N\}, \ \tau \in [\underline{h}, \overline{h}]$$

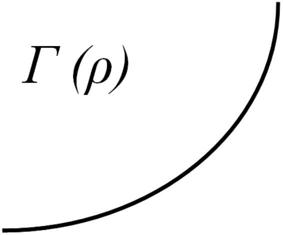
- Finite number of LMI stability conditions for polytopic systems (Daafouz, Bernussou)

$$L_i^T P L_i - P \prec 0, \ i \in \{1, \dots, N\}$$

## Uncertain Exponential Matrix

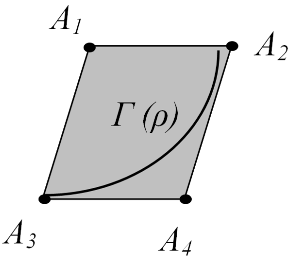
$$\Lambda(\tau) = e^{\tau A} + \int_0^{\tau} e^{sA} ds BK = I + \int_0^{\tau} e^{sA} ds \, (A + BK)$$

$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds, \; \underline{h} < \rho < \overline{h}$$




curve in the space of $\mathbb{R}^{n \times n}$ matrices

# Uncertain Exponential Matrix - Polytopic Embedding




$$\exists \mu_i > 0,\ \forall i = 1, \dots, N,\ \sum_{i=1}^{N} \mu_i = 1$$

$$\Gamma(\rho) = \int_0^{\rho} e^{As} ds = \sum_{i=1}^{N} \mu_i(\rho) A_i$$

**Jordan Forms :**

- (Cloosterman, et. al, TAC 2009),
- (Olaru, Niculescu, IFAC World Congress 2007),

**Cayley-Hamilton :**

- (Gielen, et al. Automatica, 2010)

**Taylor series (+NB approximation) :**

- (Hetel, Daafouz, Iung, TAC 2006)

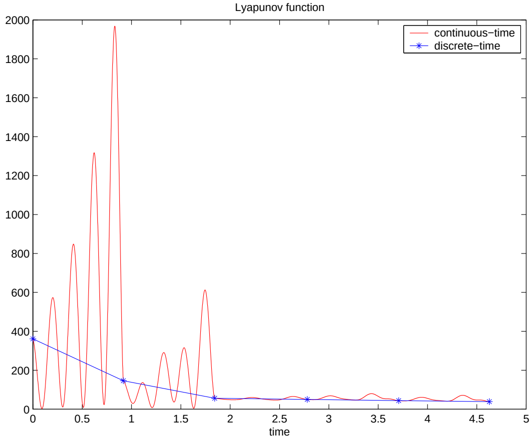
## Remarks

- Quadratic stability = sufficient only stability condition
- Lyap. funct. necessary and sufficient for stability (Molchanov and Pyatnitsky; Hetel et al. TAC 2011) :

$$V(x) = x^T P_{[x]} x, \;\; P_{[x]} = P_{[ax]}, \forall a > 0$$

- Uncertainties in the system matrices $A = A_0 + \Delta A$ ?
- Linear time-varying systems $A = A(t)$ ?

# Discrete-time and Convex Embedding

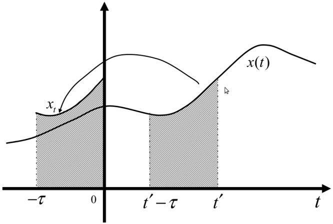



Inter-sampling behaviour ! ! !

## Time-delay approaches

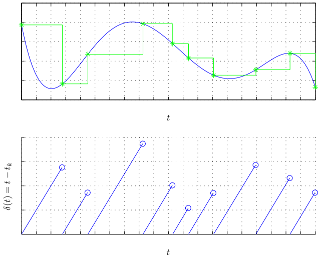
- Basic references
  - Y. Mikheev, V. Sobolev, and E. Fridman. Automation and Remote Control, 1988.
  - A.R. Teel, D. Nesic, and P.V. Kokotovic - IEEE CDC 1998
  - E. Fridman - Automatica, 2010
  - A. Seuret - Automatica, 2012
  - F. Mazenc, M. Malisoff, and T.N. Dinh - Automatica, 2013
  - I. Karafyllis and M. Krstic - IEEE TAC, 2012
- Time-delay system

$$\dot{x} = Ax + A_d x\,(t - \tau)$$




System state : $x_t(\theta) = x(t+\theta),\ \ \theta \in [0, -\tau]$

# Time-delay approaches




- Continuous-time model

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}),\ \ h_k = s_{k+1} - s_k \in [0, \overline{h}]$$

- Time delay system : $x(s_k) = x\left(t - (t - s_k)\right)$ is a past value of $x(t)$

$$\dot{x} = Ax + BKx\left(t - \tau(t)\right)$$

- Sawtooth delay

$$\tau = t - s_k,\ \ \dot{\tau}(t) = 1$$

# Time-delay approaches

- Time delay system

$$\dot{x} = Ax + BKx\left(t - h(t)\right),\ \ h \in [0, \overline{h}]$$

- System state :

$$x_t(\theta) = x(t+\theta),\ \ \theta \in [0, -\overline{h}]$$

- Stability analysis using *Lyapunov-Krasovskii functionals*

$$V(x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{-\underline{h}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta$$

# Time-delay approaches : basic steps

Step 1. *Propose a candidate Lyapunov-Krasovskii functional V*

$$V(x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{-\underline{h}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta$$

Step 2. *Compute the derivative of V.*

$$\frac{\mathrm{d}}{\mathrm{dt}}V(x_t, \dot{x}_t) \;=\; 2\dot{x}^T(t)Px(t) + \bar{h}\dot{x}^T(t)R\dot{x}(t) - \int_{t-\bar{h}}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s$$

Step 3. *Over-approximate the integral terms* (here Jensen Inequality)

$$-\int_{t-\tau}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s \leq -\frac{1}{\tau}\left(x(t) - x(t-\tau)\right)^T R\left(x(t) - x(t-\tau)\right).$$

$$\Downarrow$$

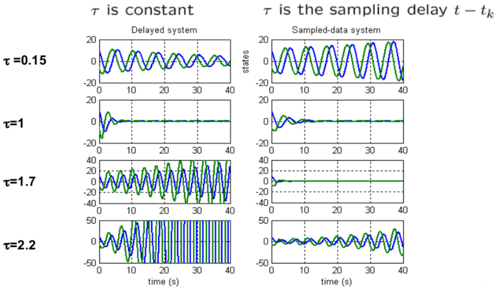
LMI condition

# Remarks

Sampled-data Systems vs. Time-delay systems

Are these two classes of systems equivalent?
Consider the following example

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} x(t_k)$$



## Further improvement

- Take into account the derivative of the delay $\dot{\tau}(t) = 1$
- Other choices of Lyapunov-Krasovskii functionals

$$V(t, x(t), \dot{x}_t) = x^T(t)Px(t) + (h_k - \tau(t)) \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds$$

- Less conservative over-approximations of integral terms

$$\int_{t-\tau}^{t} \dot{x}(s)R\dot{x}(s)\mathrm{d}s$$

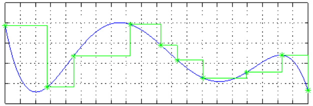
- Can be adapted to deal with uncertain system matrices

# Hybrid system approach

- Basic references
  - G. E. Dullerud and S. Lall, Systems and Control Letters, 1999
  - D. Nesic, A. Teel - IEEE TAC 2004
  - Goebel, Sanfelice, Teel, CSM 2009
  - P. Naghshtabrizi, J.-P. Hespanha, and A.-R. Teel. Systems and Control Letters, 2008
- Impulsive model

$$\begin{cases} \dot{\xi}(t) = \bar{A}\xi(t), & t \neq s_k, \forall k \in \mathbb{N}, , \\ \xi(t) = J\xi(t^-) & t = s_k, \forall k \in \mathbb{N}. \end{cases}$$

# Hybrid system approach




- Sampled-data system

$$\dot{x} = Ax + BKx(s_k)$$

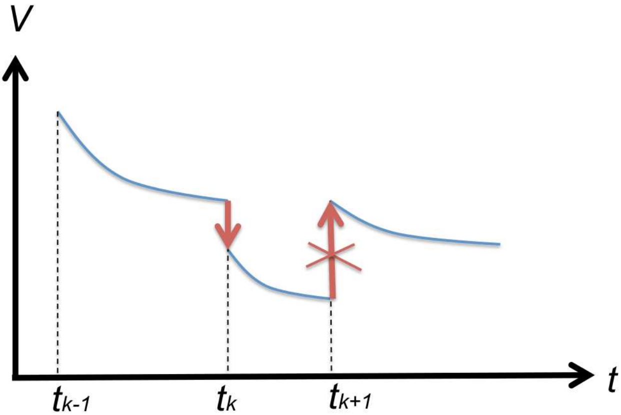
- Augmented state

$$\xi(t) = [x^T(t), z^T(t)]^T, \; z(t) = x(s_k)$$

- Impulsive model

$$\begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BK \\ 0 & 0 \end{bmatrix} \xi(t), & t \neq s_k, \forall k \in \mathbb{N}, , \\ \xi(s_k) = \begin{bmatrix} x(s_k^-) \\ x(s_k^-) \end{bmatrix}, & t = s_k, \forall k \in \mathbb{N}. \end{cases}$$

## Stability of impulsive systems




$$\dot{V}(\xi(t)) < 0, \quad \forall t \neq t_k, \quad \xi \neq 0$$
$$V(\xi(t_k)) \leq V(\xi(t)), \quad t = t_k^-$$

## Hybrid system approach

- Impulsive model

$$\begin{cases} \dot{\xi}(t) = \bar{A}\xi(t), & t \neq s_k, \forall k \in \mathbb{N}, , \\ \xi(t) = J\xi(t^-) & t = s_k, \forall k \in \mathbb{N}. \end{cases}$$

- Stability conditions using time (clock) dependent Lyapunov function with discontinuities at the impulse times

$$V(\tau, \xi) = \xi^T P(\tau)\xi, \;\; P(\tau) = P^T(\tau) \succ 0, \;\; \tau(t) = t - s_k \in [0, \overline{h}]$$

- Stability conditions

$$\dot{V}(\tau, \xi) < 0, \forall t \neq s_k, \xi \neq 0$$

$$V(0, \xi) \leq V(\tau(t), \xi), t = s_k^-$$

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0, \;\; \tau \in [0, \overline{h}], \;\;\; J^T P(0) J - P(\tau) \prec 0, \;\; \tau \in [\underline{h}, \overline{h}].$$

(Sun & Khargonekar ; Toivonen)

## Hybrid system approach

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0,\ \tau \in [0, \overline{h}],\quad J^T P(0) J - P(\tau) \prec 0,\ \tau \in [\underline{h}, \overline{h}].$$

- Examples :
  - polynomial (linear) function

$$P(\tau) = P_1 + (P_2 - P_1)\frac{\tau}{\overline{h}}$$

  - exponential

$$P(\tau) = e^{-\gamma\tau} P_0$$

  - inspired by Lyapunov-Krasovskii functionals

$$P(\tau) = \int_{-\tau}^{0} (\overline{h} + s)\bar{A}^T e^{\bar{A}^T s} R e^{\bar{A}s} \bar{A} ds$$

where

$$\tau(t) = t - s_k \in [0, \overline{h}]$$

## Hybrid system approach

- Parametric LMI conditions

$$\bar{A}^T P(\tau) + P(\tau)\bar{A} + \dot{P}(\tau) \prec 0, \ \tau \in [0, \overline{h}], \quad J^T P(0) J - P(\tau) \prec 0, \ \tau \in [\underline{h}, \overline{h}].$$

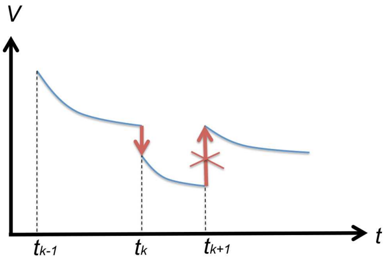
- Example :
  - For linear function

$$P(\tau) = P_1 + (P_2 - P_1)\frac{\tau}{\overline{h}}$$

  - LMI condition

$$\begin{aligned}
&\bar{A}^T P_1 + P_1 \bar{A} + \frac{P_2 - P_1}{\overline{h}} \prec 0, \\
&\bar{A}^T P_2 + P_2 \bar{A} + \frac{P_2 - P_1}{\overline{h}} \prec 0, \\
&J^T P_1 J \prec P_2, \\
&J^T P_1 J \prec P_1 + (P_2 - P_1)\, \underline{h}/\overline{h}.
\end{aligned}$$

# Hybrid system approach : relations with discrete-time approach




(a) $\exists\ V_d(\xi_k) = \xi_k^T L \xi_k$ $(L \succ 0)$ such that $V_d(\xi_{k+1}) < V_d(\xi_k)$

$$\equiv$$

(b) $\exists\ V(\tau,\xi) = \xi^T P(\tau)\xi$ $(P(\tau) = P^T(\tau) \succ 0)$ such that $\dot{V}(\tau,\xi) \leq 0$ and $V(0,\xi) < V(\tau,\xi)$

(Briat, Automatica 2013)

see also the looped functionals in (Seuret, Automatica 2012)
Why do we look for $\tau$ dependent Lyapunov functions ?

# More general hybrid models (Goebel, Sanfelice, Teel)

Sampled-data system

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}),\ \ h_k = s_{k+1} - s_k \in [0, \overline{h}]$$

Hybrid model

$$\left\{ \begin{array}{l} \left. \begin{array}{rcl} \dot{x} & = & Ax + BK\hat{x} \\ \dot{\hat{x}} & = & 0 \\ \dot{\tau} & = & 1 \end{array} \right\} \qquad\qquad \tau \in [0, \overline{h}], \\ \\ \left. \begin{array}{rcl} x^+ & = & x \\ \hat{x}^+ & = & x \\ \tau^+ & = & 0 \end{array} \right\} \qquad\qquad\qquad \tau \in [\underline{h}, \overline{h}]. \end{array} \right.$$

(includes the dynamic of the clock $\tau$)

# More general hybrid models (Goebel, Sanfelice, Teel)

Sampled-data system

$$\dot{x} = Ax + BKx(s_k), \forall t \in [s_k, s_{k+1}),\ h_k = s_{k+1} - s_k \in [0, \overline{h}]$$

Hybrid model with $z = (x^T \quad \hat{x}^T \quad \tau)^T = (\xi^T \quad \tau)^T$

$$F_z(z) = \begin{pmatrix} Ax + BK\hat{x} \\ 0 \\ 1 \end{pmatrix},\ J_z(z) = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}$$

$$C = \left\{ z \in \mathbb{R}^{n_z} : \tau \in [0, \overline{h}] \right\}$$

and

$$D = \left\{ z \in \mathbb{R}^{n_z} : \tau \in [\underline{h}, \overline{h}] \right\}.$$

Stability of the set $\mathcal{A} = \{z^T = (x^T, \hat{x}^T, \tau) \in \mathbb{R}^{n_z} :\ (x, \hat{x}) = (0, 0)\}$ ?

# Necessary and sufficient stability conditions (Cai,Goebel and Teel) :

A set $\mathcal{A}$ is asymptotically stable

$$\begin{aligned} \dot{z} &= F_z(z), \quad z \in C, \\ z^+ &= J_z(z), \quad z \in D, \end{aligned}$$

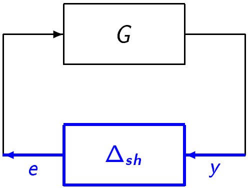
If and only if there exists a $\mathcal{C}^\infty$ function $\tilde{V}(z)$ such that

$$\begin{aligned} &\frac{\partial \tilde{V}}{\partial z} F_z(z) < 0 \text{ for all } z \in C \setminus \mathcal{A}, \\ &\tilde{V}\left(J_z(z)\right) - \tilde{V}(z) < 0 \text{ for all } z \in D \setminus \mathcal{A}. \end{aligned}$$

i.e. a $\mathcal{C}^\infty$ function $V(\tau, \xi)$ for the impulsive model.

# Input/Output stability approaches

- Basic references
  - L. Mirkin - IEEE TAC 2007
  - H. Fujioka - Automatica 2009
  - Y.C. Kao - ACC 2014
  - H. Omran et al. - Automatica 2014
  - H. Omran et al. - ECC 2013



# Input/Output stability approaches

- LTI sampled-data system

$$\dot{x} = Ax + BKx(s_k)$$

- Sampling error : $e(t) = x(s_k) - x(t) = -\int_{s_k}^{t} \dot{x} ds$

$$\dot{x}(t) = [\underbrace{A + BK}_{A_{cl}}]x(t) + \underbrace{BK}_{B_{cl}}\underbrace{(x(s_k) - x(t))}_{e(t)}$$

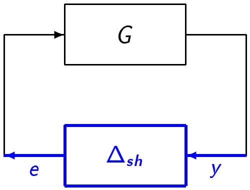
The system can be represented by the interconnection of :

$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \to e$ defined by :

$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t), \quad \forall t \in [s_k, s_{k+1})$$

# Input/Output stability approaches




$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t).$$

- Stability conditions using the properties of $\Delta_{sh}$
- e.g. finite $L_2$ gain (Mirkin, 2007)

$$\|\Delta_{sh}\|_{2,2} = \sup_{y\neq 0} \frac{\|e\|_{L_2}}{\|y\|_{L_2}} \leq \delta_0 := \frac{2}{\pi}\overline{h}$$

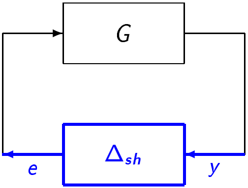
- Small Gain Theorem : the interconnection is $\mathcal{L}_2$ stable if

$$\|\mathbf{G}\|_{2,2}\|\Delta_{sh}\|_{2,2} < 1$$

- Frequency domain condition

$$\|\mathbf{G}\|_{2,2} = \|\mathbf{G}\|_{\infty} := \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left(\hat{\mathbf{G}}(j\omega)\right) < \frac{\pi}{2\overline{h}},$$

# Input/Output stability approaches




$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t).$$

- Scaled Small Gain condition

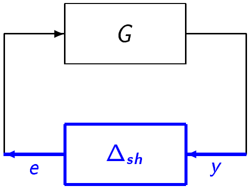
$$\exists\, M \in \mathbb{R}^{n\times n},\ M \succ 0 \quad \text{such that } \|M\hat{\mathbf{G}}(s)M^{-1}\|_\infty < \frac{\pi}{2\bar{h}}.$$

- LMI formulation

$$\begin{bmatrix} XA_{cl} + A_{cl}^T X & \frac{2}{\pi}\bar{h}XBK & A_{cl}^T Y \\ * & -Y & \frac{2}{\pi}\bar{h}K^T B^T Y \\ * & * & -Y \end{bmatrix} \prec 0$$

to be solved for $X, Y \succ 0$ (obtained with $Y = M^2$).

# Input/Output stability approaches




$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

$$e(t) = -\int_{s_k}^{t} y(s)ds := (\Delta_{sh}y)(t).$$

Integral Quadratic Constraints (IQC)

$$\int_0^\infty \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

for all $y \in \mathcal{L}_2^n[0, \infty)$ and $e = \Delta_{sh}y$.

(Megretski & Rantzer, IEEE TAC 1997)

# Input/Output stability approaches

## Theorem (IQC Theorem)

*Suppose that $A_{cl} = A + BK$ is Hutwitz and assume that*

- *there exists a matrix*

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix}$$

  *with $\Pi_{11}, \Pi_{12}, \Pi_{22} \in \mathbb{R}^{n\times n}$, $\Pi_{11} \succeq 0$, $\Pi_{22} \preceq 0$, such that the operator $\Delta_{sh}$ satisfies the IQC defined by $\Pi$;*

- *there exists $\epsilon > 0$ such that*

$$\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \ \forall\ \omega \in \mathbb{R}.$$

*Then the interconnection is $\mathcal{L}_2$ stable.*

# Input/Output stability approaches

$$G := \begin{cases} \dot{x}(t) = & A_{cl}x(t) + B_{cl}e(t) \\ y(t) = & \dot{x}(t) = C_{cl}x + D_{cl}e(t) \end{cases}$$

Equivalent LMI condition (**Kalman-Yakubovich-Popov Lemma**) :

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} < 0$$

to be solved for $P \succ 0$.

# Example of IQCs for $\Delta_{sh}$

- Finite $L_2$ gain (Mirkin, 2007) :

$$\|\Delta_{sh}\|_{2,2} = \sup_{y \neq 0} \frac{\|e\|_{L_2}}{\|y\|_{L_2}} \leq \delta_0 := \frac{2}{\pi}\bar{h}$$

- Time domain formulation :

$$\int_0^{+\infty} \|(\Delta_{sh} y)(\theta)\|^2 d\theta \leq \delta_0^2 \int_0^{+\infty} \|y(\theta)\|^2 d\theta,$$

  for all $y \in \mathcal{L}_2^n[0, \infty)$
- IQC :

$$\int_0^{\infty} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

  for all $y \in \mathcal{L}_2^n[0, \infty)$ and $e = \Delta_{sh} y$ with

$$\Pi = \begin{bmatrix} \delta_0^2 I & 0 \\ 0 & -I \end{bmatrix}.$$

- LMI condition :

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \delta_0^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} < 0$$

## Example of IQCs for $\Delta_{sh}$

- (Anti-)Passivity property (Fujioka, Automatica, 2009)

$$\int_0^{+\infty} y^T(\theta)(\Delta_{sh} y)(\theta) d\theta \leq 0,$$

for all $y \in \mathcal{L}_2^n[0, \infty)$.

- IQC :

$$\int_0^{\infty} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

for all $y \in \mathcal{L}_2^n[0, \infty)$ and $e = \Delta_{sh} y$ with

$$\Pi = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}.$$

- LMI condition :

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} < 0$$

## Example of IQCs for $\Delta_{sh}$

- IQCs can be combined

$$\int_0^{+\infty} y^T(\theta)(\Delta_{sh}y)(\theta)d\theta \leq 0, \quad \int_0^{+\infty} \|(\Delta_{sh}y)(\theta)\|^2 d\theta \leq \delta_0^2 \int_0^{+\infty} \|y(\theta)\|^2 d\theta,$$

- Resulting IQC :

$$\int_0^{\infty} \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix}^T \Pi \begin{bmatrix} y(\theta) \\ e(\theta) \end{bmatrix} d\theta \geq 0$$

for all $y \in \mathcal{L}_2^n[0,\infty)$ and $e = \Delta_{sh}y$ with

$$\Pi = \begin{bmatrix} \delta_0^2 I & -I \\ -I & -I \end{bmatrix}.$$

- LMI condition :

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} & PB_{cl} \\ B_{cl}^T P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \delta_0^2 I & -I \\ -I & -I \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} < 0$$

## Remarks :

- Many other performance specifications and nonlinearities can be described as IQCs (saturations, sector-bounded nonlinearities, etc.)

  (Megretski and Rantzer, IEEE TAC 1997)

- For LTV, polytopic and nonlinear systems : re-interpretation in terms of supply functions

  (Omran et al, Automatica 2014, 2016)

# Dissipativity-based approach

Closed-loop system

$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right) K\left(x(s_k)\right)$$

$$\dot{x}(t) = \underbrace{f\left(x(t)\right) + g\left(x(t)\right) K\left(x(t)\right)}_{f_n(x(t))} + \underbrace{g\left(x(t)\right)}_{g_n(x(t))} \underbrace{\left(K\left(x(t_k)\right) - K\left(x(t)\right)\right)}_{e(t)}$$

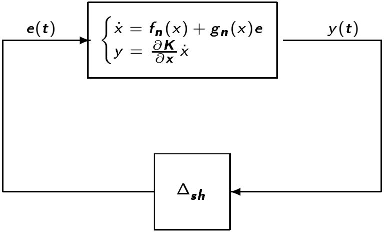
The system can be represented by the interconnection of :

$$\begin{cases} \dot{x}(t) = f_n\left(x(t)\right) + g_n\left(x(t)\right) e(t) \\ y(t) = \frac{\partial K}{\partial x} \dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \rightarrow e$ defined by :

$$e(t) = (\Delta_{sh}\, y)(t) = -\int_{s_k}^{t} y(\tau) d\tau$$

# Dissipativity-based approach




For $(\Delta_{sh}y)(t) := -\int_{s_k}^{t} y(s)ds$, derive "supply functions" $\mathbf{S}(y, \Delta_{sh}y)$ such that

$$\int_{s_k}^{t} \mathbf{S}(y, \Delta_{sh}y)ds \leq 0, \forall t \in [s_k, s_{k+1}).$$

**Exponential stability condition :** $\exists$ "storage function" $V(x)$, $\alpha > 0$ such that

$$\begin{aligned} \dot{V}(x(t)) + \alpha V(x(t)) &\leq \boldsymbol{S}(y(t), e(t)) \\ \dot{V}(x(t)) + \alpha V(x(t)) &\leq \boldsymbol{S}(y(t), e(t)) exp(-\alpha \overline{h}) \end{aligned}$$

# Properties of the operator

**Boundedness property**

For all $y \in L_2[s_k, s_{k+1})$ and $0 < X^* = X \in \mathbb{R}^{n\times n}$ :

$$\int_{s_k}^{t} (\Delta_{sh} y)^* X (\Delta_{sh} y)\, ds - \delta_0^2 \int_{s_k}^{t} y^* X y\, ds \le 0, \quad \forall t \in [s_k, s_{k+1})$$

**(Anti-)Passivity property**

For all $y \in L_2[s_k, s_{k+1})$ and $0 \le Y^* = Y \in \mathbb{R}^{n\times n}$ :

$$\int_{s_k}^{t} (\Delta_{sh} y)^* Y y\, ds + \int_{s_k}^{t} y^* Y (\Delta_{sh} y)\, ds \le 0, \quad \forall t \in [s_k, s_{k+1})$$

$$\Rightarrow \int_{s_k}^{t} \underbrace{\begin{bmatrix} y \\ e \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} y \\ e \end{bmatrix}}_{\text{supply rate } \mathcal{S}(y,e)} ds \le 0, \quad \forall t \in [s_k, s_{k+1})$$

# Robust stability analysis with respect to time-varying sampling

Strong inter-action between approaches

- Time-delay approach
  $\Rightarrow$ less conservative $L_2$ bound for I/O approach
  $\Rightarrow$ continuous-time version of convex embedding approach
- Hybrid (Impulsive) System modelling approach
  $\Rightarrow$ discontinuous LKF
- Discrete-time and Convex Embedding
  $\Rightarrow$ taking into account the sawtooth form of the delay
- I/O approach
  $\Rightarrow$ new LKF based on Wirtingers inequalities

# Control of sampling

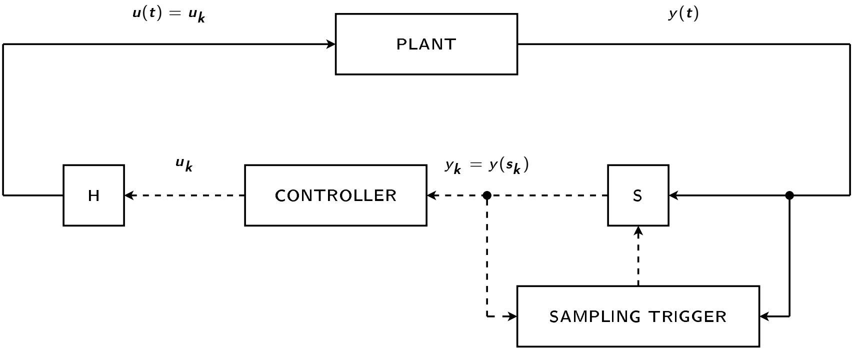



Figure: Generic configuration

- Event-Triggered Control
- Periodic Event-Triggered Control
- Self-Triggered Control

see tutorial paper **(Heemels, Johansson, Tabuada, CDC 2012)**

# Pioneering works

- Best periodic sequence of sampling
  E.I. Jury and F.J. Mullin. IRE TAC, 1959
- Adaptive sampling
  R.C. Dorf, M.C. Farren, and C. Phillips. IRE TAC, 1962
- Lyapunov based self-triggered control
  P. Hsu and S. Sastry. IEEE CDC 1987

# Control of sampling

**Event-Triggered Control**

- E. Hendricks et. al - ACC, 1994
- K. Astrom and B. Bernhardsson - IFAC World Conf., 1999
- K.-E. Arzen - IFAC World Conf., 1999
- Tabuada - IEEE TAC 2007
- Lunze, Lehmann - Automatica 2010

**Self-Triggered Control**

- Wang, Lemmon - IEEE TAC 2010
- Anta, Tabuada - IEEE TAC 2010
- Mazo-Jr, Anta, Tabuada - Automatica 2010
- Fiter et al. - Nolcos 2013

## Basic problems

$$s_{k+1} = s_k + \tau(x_k), \;\; \tau : \mathbb{R}^n \to \mathbb{R}_+$$

- What is the "best" sampling pattern ?

- Which is the "best" trigger function / sampling map $\tau(x)$ ?
- Optimize cost function depending on the sampling sequence $\sigma = \{t_k\}_{k\in\mathbb{N}}$.

$$\mathcal{J}(\sigma) = \liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} (t_{k+1} - t_k).$$

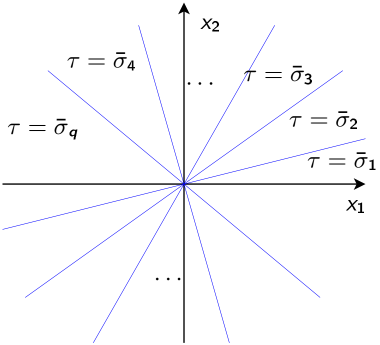
# Goal

Use tools from the *robust control* framework to optimize the design of *sampling maps* !

(Fiter et al. Automatica,2012 ; Automatica 2015)

(R. Postoyan, et al. IEEE TAC 2015)

## Fiter et al. Springer, 2014




- Optimize the lower bound of the sampling map $\tau(x)$
- Intersection of **finite** number of conic regions.

## Example

Batch Reactor system from Mazo et. al ECC 2009

$$\dot{x}(t) = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 5.67 & 0 \\ 1.13 & -3.14 \\ 1.13 & 0 \end{bmatrix} u(t),$$

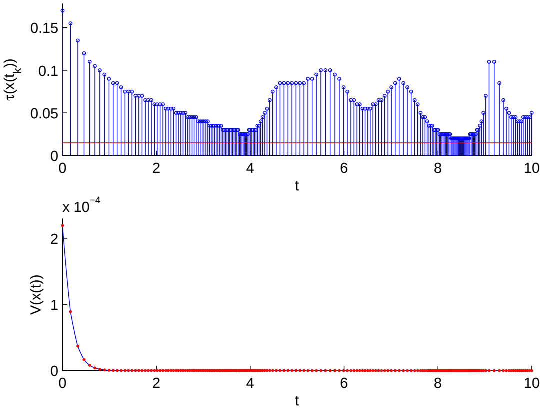
$$u(t) = -\begin{bmatrix} -0.1006 & 0.2469 & 0.0952 & 0.2447 \\ -1.4099 & 0.1966 & -0.0139 & -0.0823 \end{bmatrix} x(t_k).$$

Lower bound of inter-execution time :

- Mazo et. al ECC 2009 : **0.02s**
- Convex embeddings (5th order Taylor approx., 36 subdivisions) : **0.18s**

# Example

Simulations for the Batch Reactor system (with 10% perturbation)




$$\tau^* = 0.015$$

## Conclusion

- Present recent approaches for robust stability analysis of systems with aperiodic sampling
- Indicate the link between the approaches
- Perspectives to event- and self-triggering control

More information : see the tutorial session in the proceedings of ECC 2014 and the survey paper in Automatica 2017.