Linearization techniques for MINLP

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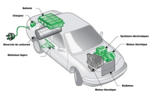
Séminaire LIPN-AOC - 11/02/2021

Outlook

1. Illustrative cases studies and key issues
2. From iterative to a non-iterative methods
3. nnc PWL approximation/bounding and relative tolerance
4. Corridors fitting and generalization to classes of errors
5. Conclusion

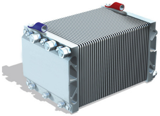
Energy in hybrid electric vehicles

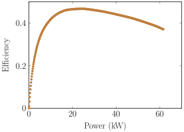


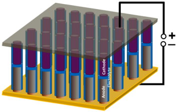
Electric propulsion motor powered by:

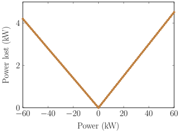
- onboard generator: e.g. hydrogen fuel cell (FC)
- reversible source: e.g. supercapacitator (SE)

Energy sources characteristics: power limits(kW), efficiency(%), capacity(kWs) ...

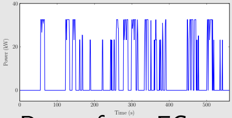






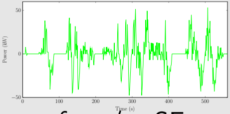


Find at each instant the optimal power split between the energy sources to minimize the total fuel consumption.



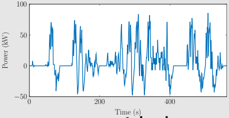
Power from FC

+



Power from/to SE

=



Total power provided

Mathematical model

$$\min \sum_{i=1}^{n} f^{\mathrm{FC}}(x_i) \quad (1)$$

s.t. Power demand satisfaction

$$x_i + y_i - z_i \geq P_i, \qquad \forall i \in \mathbb{I} \quad (2)$$

Final SE energy level higher or equal to initial energy level

$$\sum_{i=1}^{n} f^{\mathrm{SE}^+}(y_i) - f^{\mathrm{SE}^-}(z_i) \leq 0 \quad (3)$$

SE energy level within bounds

$$E_0^{\mathrm{SE}} - E_{\max}^{\mathrm{SE}} \leq \sum_{k=1}^{i} f^{\mathrm{SE}^+}(y_k) - f^{\mathrm{SE}^-}(z_k) \leq E_0^{\mathrm{SE}} - E_{\min}^{\mathrm{SE}}, \qquad \forall i \in \mathbb{I} \quad (4)$$

Variables domains

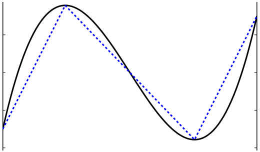
$$x_i \in \{0\} \cup [P_{\min}^{\mathrm{FC}}, P_{\max}^{\mathrm{FC}}], y_i \in [0, P_{\max}^{\mathrm{SE}}], z_i \in [0, P_{\min}^{\mathrm{SE}}], \qquad \forall i \in \mathbb{I} \quad (5)$$

Classical MINLP solution methods

MILP-based solution methods on similar problems

Camponogara *et al.* 2011; Borghetti *et al.*, 2008

- approximate with piecewise linear functions



+ (more) tractable problems

- try and error approach: No guarantees on the solution quality or iterative process with an undefined number of iterations

- global optimality cannot be guaranteed

Generic MINLP solution methods / Hybrid algorithms and frameworks

Grossmann 2002 / Bonami *et al.*, 2008

+ global optimality guaranteed if carried out to completion

- only for small/medium instances

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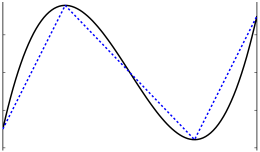

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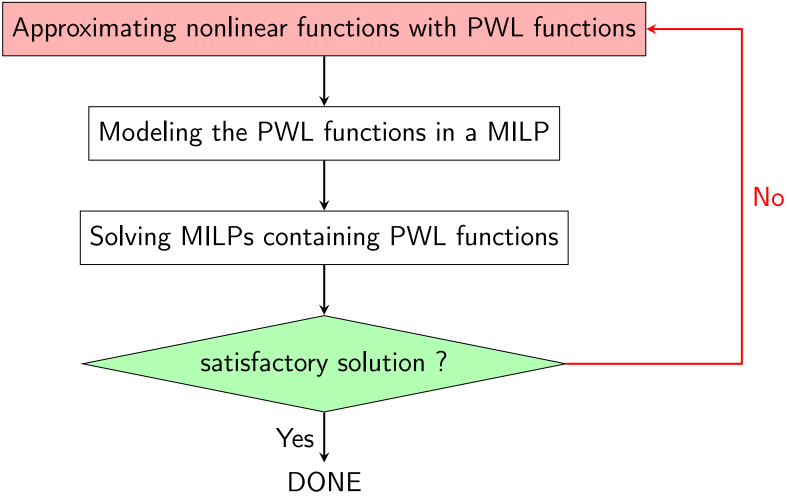
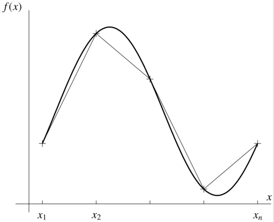
MILP-based methods for MINLPs



PWL approximation

Sampling



Given a number of breakpoints on the curve, find the PWL functions which minimizes an error metric.

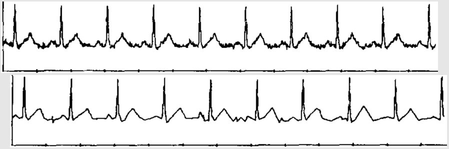
D'Ambrosio et al 2010, 2015

Fitting a discrete data set

Applications : ECG, pattern recognition, data reduction, ...

in

out



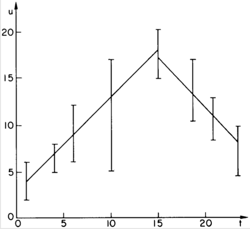
Bellman and Roth 1969 ... Tomek 1974 ...

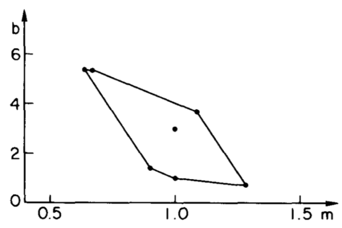
O'Rourke 1981 ... Toriello and Vielma 2012

Rebennack and Krasko 2019

- input = triplets $(t_k, u_k^{\min}, u_k^{\max})$
- output = coefficients m, b of the fitting line $\mathcal{L}$

$\mathcal{L}$ fits the k^{th} data set iff: $u_k^{\min} \leq mt + b \leq u_k^{\max}$



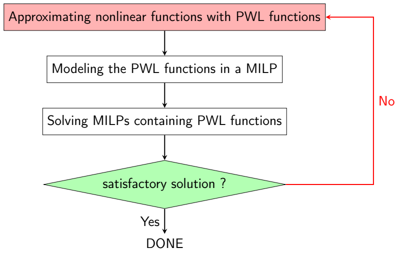


Idea: switch to the $m - b$ parameter space

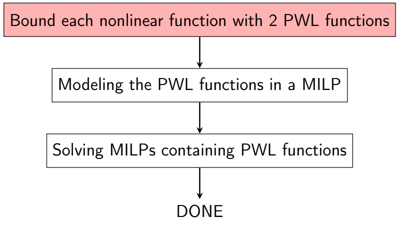
- $b \geq (-t_k)m + u_k^{\min}$ and $b \leq (-t_k)m + u_k^{\max}$
- Half planes intersection

Not sufficient for continuous functions

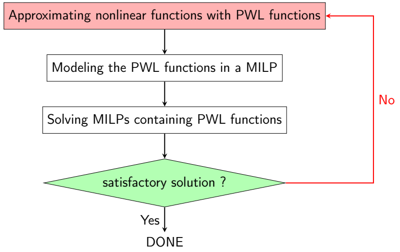
Iterative ⇒ non-iterative method ?



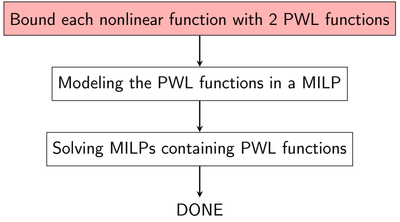
⇒

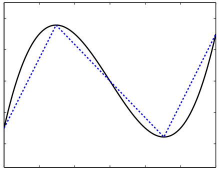


Iterative ⇒ non-iterative method ?

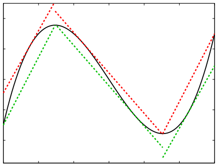


⇒





(a) PWL approximation



(b) PWL bounding

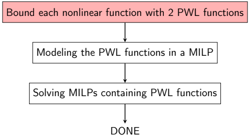
Resulting PWL approx/bounding Problem

Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a tolerance bound, find a PWL function that:

- Respects the imposed bounded error
- Minimizes the number of pieces of the PWL function

Benefits

- predetermined error criteria $\rightarrow$ non-iterative approach

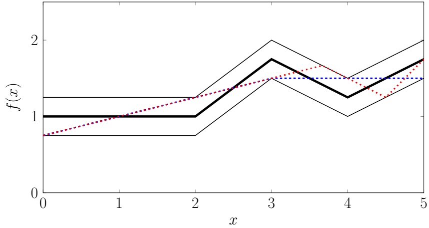


- Smallest MILP possible!

Why is the Problem Hard?

Bounded tolerance constraints → semi-infinite programming (SIP)

Maximizing the length of the projection on x-axis of each segment is not optimal



Approx functions with PWL

Minimize number of linear pieces for a bounded error or distance metric

Rosen and Pardalos, 1986. Global minimization of large-scale constrained concave quadratic problems by separable programming. **Math. Prog**

Frenzen, Sasao and Butlerc, 2010. On the number of segments needed in a piecewise linear approximation. **J. of Comp. and Applied Math.**

Geibler, Martin, Morsi, Schewe, 2012. Using piecewise linear functions for solving MINLPs. **The IMA Volumes in Mathematics and its applications**

Rebennack and Kallrath, 2015. Continuous piecewise linear δ-approximations for univariate functions: computing minimal breakpoint system. **J. Optim. Theory Appl**

Ngueveu, 2019. Piecewise linear bounding of univariate nonlinear functions and resulting mixed integer linear programming-based solution methods **EJOR**

Rebennack and Krasko, 2019. Piecewise linear function fitting via mixed-integer linear programming. **INFORMS Journal on Computing**

L. Kong and C.T. Maravelias, 2020. On the derivation of continuous piecewise linear approximating functions. **INFORMS Journal on Computing**

Continuous PWL approx of univariate functions

Optimal Breakpoint System using a Continuum approach for x [RK2015]

Decision variables

- $x_b \in [X_-, X_+]$: breakpoint value
- $s_b \in [-\delta, +\delta]$: deviation on bpt b
- $\chi_b \in [0, 1] = 1$ iff bpt b is used
- $y_b \geq \frac{1}{M}$: $= x_b - x_{b-1}$ if $x_b - x_{b-1} > 0$ and $= |X_-, X_+|$ otherwise
- $\xi^x_{bx} \in [0, 1]$: =1 iff $x \in [x_{b-1}, x_b]$
- $l_b(x) \in \mathbb{R}$: $= l(x)$ if $x \in [x_{b-1}, x_b]$

Semi-infinite programming model

Obj = minimize number of active breakpoints

s.t. (c1) Order active breakpoints

(c2) Link x_b and χ_b

(c3) Compute y_b and $l_b(x)$

(c4) Compute ξ^x_{bx}

(c5) Compute $l(x)$

(c6) Ensure the δ-approximation: $|l(x) - f(x)| \leq \delta, \forall x \in D$

Continuous PWL approx of univariate functions

[Rebennack and Kallrath 2015], [Rebennack and Krasko 2019]

Solution 1: Semi-infinite programming models (large and difficult)

Iterative solution method

- enforce the δ gap constraint on discrete points $|\mathbb{I}|$
  - RK2015 $\Rightarrow$ solve a MINLP
  - RK2019 $\Rightarrow$ solve a MILP (extension of Toriello&Vielma 2012)
- compute the real maximum error $\Rightarrow$ solve an NLP

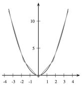
Solution 2: Greedy heuristics computing x_b given x_{b-1},s_{b-1} and δ

- Maximize $x_b - x_{b-1}$(projection on x-axis): not optimal

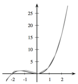
Numerical comparisons

RK2015 vs RK2019 vs KM2020

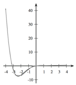
Some results from the literature



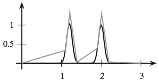
x^2



$e^{-x}\sin(x)$



$e^{-100(x-2)^2}$



$1.03e^{-100(x-1.2)^2} + e^{-100(x-2)^2}$

			[RK2015]				[RK2019]			[KM2020]		
	p	δ	n_*	n_-	n_+	cpu	n_*	n_-	cpu	n_*	n_+	cpu
I	1	0.05	8				-	-	-	8		350 s
		0.01	12				-	-	-		15	t.o.
		0.005	25				-	-	-			
VI	2	0.05	15			few days	15		107.7 s			
		0.01		15	34			31	6819.1 s			
		0.005		15	47			40	35787.4 s			
VII	3	0.05		4	19		19		14514.6 s			
		0.01		4	43			34	35411.0 s			
		0.005		4	61			35	70313.1 s			
IX	5	0.05		7	11		9		12 s			
		0.01		7	21		21		13873.4 s			
		0.005		7	28		27		42068.4 s			

RK finding: non equidistant breakpoints $\Rightarrow$ cardinality reduction 1.3 to 14.3

What about multivariate functions ?

- separable or non-separable ?

What about multivariate functions ?

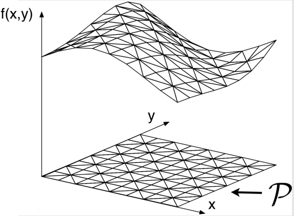
- separable or non-separable ?

Bivariate/Multivariate functions

Bivariate/Multivariate input function

$f(x,y) = x^2 + y^2$, $f(x,y) = xe^{(-x^2-y^2)}$, $f(x) = x\sin(x)\sin(y)$, ...

Multivariate PWL function



$$f(x,y) = \begin{cases} 0.48x + 0.03y + 6 & ,(x,y) \in P_1 \\ \vdots & \\ -0.4x - 0.04y + 8.45 & ,(x,y) \in P_{128} \end{cases}$$

$P_1 = \{(x,y) \in \mathbb{R} : y \geq 0, x \leq 1, y - x \leq 0\}$

$\vdots$

$P_{128} = \{(x,y) \in \mathbb{R} : y \geq 0, x \geq 7, x + y \leq 8\}$

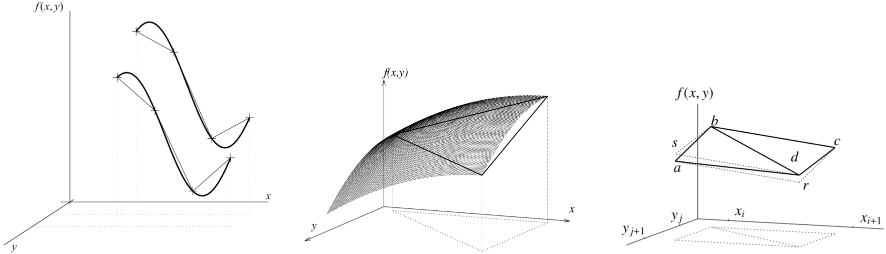
Definition: Piecewise linear $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$:

$f(x) := \{a_P x + b_P, x \in P, \forall P \in \mathcal{P}$.

for finite family of polytopes $\mathcal{P}$ such that $D = \bigcup_{P \in \mathcal{P}} P$

What about multivariate functions ?

- One dimentional method (discretization of one variable)
- Triangle method
- Rectangle method with a corrective term



D'ambrosio, Lodi and Martello (2010). Burlacu, Geibler and Schewe (2019). Keller and Karl (2019).

What about multivariate functions ?

Rebennack 2016

- Dimension-reduction techniques

Function	Transformation	Condition
$\sum_{i=1}^{n} \pm f_i(x_i)$	Treat each term $f_i(x_i)$ individually	
$\prod_{i=1}^{n} f_i(x_i)$	$ln(f(x)) = \sum_{i=1}^{n} \ln(f_i(x_i))$	$f_i(x_i) > 0, \forall i$
$f_1(x)^{f_2(x)}$	$ln(ln(f(x))) = ln(f_1(x)) + ln(ln(f_2(x)))$	$f_1(w), f_2(x) > 1$
$f_1(f_2(x))$	$f_1(u)$ and $f_2(x)$	

- allow shift from the fonction at extremities of the triangles

What about multivariate functions ?

An open question

- Is it worthwhile to exploit special properties, e.g., separability, of the functions to reduce the dimensionality, or is it more efficient to approximate the function directly in its dimensionality ?

Non necessarily continuous (nnc) PWLA

[Ngueveu, 2019] Getting rid of the continuity

Theorem

$\forall$ continuous function $f : \mathbb{D} = [X_{-}, X_{+}] \to \mathbb{R}$ and any scalar $\delta \in \mathbb{R}^{+}$, there exists an optimal nnc δ-PWLA g defined by $G = \bigcup_{i=1}^{n_g}([a_i, b_i], [x_i^{\min}, x_i^{\max}])$ such that each line-segment i has a maximal length projection on the interval $[x_i^{\min}, X_{+}]$.

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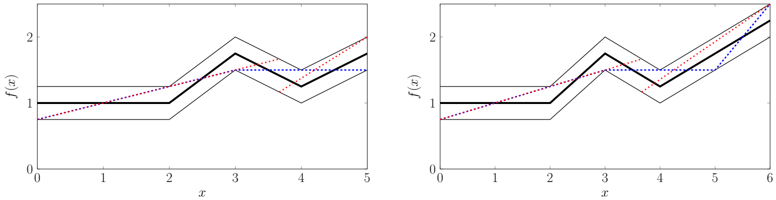
The greedy algorithm becomes optimal



nnc vs continuous approximation

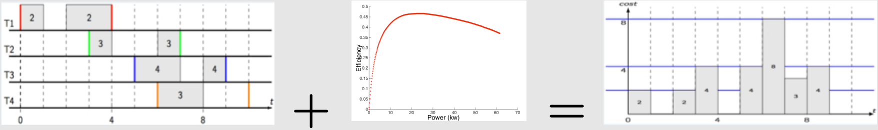
$f(x) = e^{-x}\sin(x)$: interval $= [-4; +4]$: 3 convex/concave sub-intervals

f		δ	continuous approximation [RK2019]			nnc approximation [Ng2019] Exact		Heuristic		
	p		n_*	n_-	cpu	n_*	cpu	n_-	n_+	cpu
f	3	0.05	19		14514.6 s	19	287 s	19	21	28 s
		0.01		34	35411.0 s	43	268 s	43	45	52 s
		0.005		35	70313.1 s	61	869 s	61	63	72 s

What about the limitations of such approximation/bounding ?

- dependent on the instance: e.g. dependent on time horizon
- dependent on the solution: e.g. solution cost
- difficult to choose a relevant value of δ for a new instance/problem

Scheduling problem with a non-linear cost function



$$(\mathrm{CF})\ \min \quad F = \sum_{t \in T} f\Big(\sum_{i \in \mathcal{A}} b_i x_{it}\Big) \tag{6}$$

$$s.t. \sum_{t \in T} a_{it} x_{it} \geq p_i, \qquad \forall i \in \mathcal{A} \tag{7}$$

$$x_{it} \in \{0, 1\}, \qquad \forall i \in \mathcal{A}, t \in T \tag{8}$$

Use relative ϵ-tolerance

Principle

For a function $f : \mathbb{D} \rightarrow \mathbb{R}^*$ and a tolerance value $\epsilon \in [0, 1]$, identify two piecewise linear functions $(\overline{f}^{\epsilon}, \underline{f}^{\epsilon})$ that verify:

$$\underline{f}^{\epsilon}(x) \leq f(x) \leq \overline{f}^{\epsilon}(x), \quad \forall x \in \mathbb{D} \tag{9}$$

$$|f(x) - \underline{f}^{\epsilon}(x)| \leq \epsilon |f(x)|, \quad \forall x \in \mathbb{D} \tag{10}$$

$$|\overline{f}^{\epsilon}(x) - f(x)| \leq \epsilon |f(x)|, \quad \forall x \in \mathbb{D} \tag{11}$$

Consequences

- Guarantees on the quality of the resulting lower and upper bounds
- Upper and lower bounding PWL functions $\overline{f}^{\epsilon}(x)$ and $\underline{f}^{\epsilon}(x)$ may not share the same breakpoints and number of pieces (no shift !)
- if f is convex, the further the tangent point, the larger the error: no longer true !

$$\Downarrow$$

- ~~need to solve NLPs during the dichotomic search~~: Codsi, Gendron and Ngueveu 2020

Relative vs absolute tolerance

From absolute to relative tolerance

$\Rightarrow$ 37% to 78% less linear pieces ! (energy optimization in HEV)

Class (R)	using relative tol			using absolute tol		Gap
instance	ϵ	n^ϵ	UB	δ	n^δ	$\frac{n^\delta - n^\epsilon}{n^\epsilon}$
S_40	1 %	6	454.3	0.1136	14	133.33 %
	0.01 %	56	453.4	0.0011	133	137.50 %
I_561	1 %	6	8756.6	0.1561	12	100.00 %
	0.01 %	56	8741.0	0.0016	110	96.43 %
H_734	1 %	6	18626.0	0.2538	10	66.67 %
	0.01 %	56	18569.0	0.0025	89	58.93 %
U_811	1 %	6	2613.5	0.0322	25	316.67 %
	0.01 %	56	2607.7	0.0003	255	355.36 %
N_1200	1 %	6	23137.7	0.1928	11	83.33 %
	0.01 %	56	23114.9	0.0019	102	82.14 %
E_1400	1 %	6	27088.9	0.1935	11	83.33 %
	0.01 %	56	27065.9	0.0019	102	82.14 %

Comparison of lower bounds

Instances		old lower bound [NCMG2019]		ϵ-PWLB+MILP [Ng2019]		
set	#	ratio	cpu	ϵ	ratio	cpu
(R)	6	97.81 %	2 s	1 %	99.18 %	20 s
				0.01 %	**99.99** %	58 s
(A1)	6	97.36 %	2 s	1 %	99.25 %	29 s
				0.01 %	**99.99** %	216 s
(A2)	6	78.50 %	2 s	1 %	99.50 %	26 s
				0.01 %	**99.99** %	405 s

Instances			antigone	baron	couenne	lindoglobal
class	#	optCR	ratio	ratio	ratio	ratio
(R)	6	1 %	24.66 %	- % (6)	- % (6)	- % (6)
		0.01 %	21.32 %	- % (6)	- % (6)	- % (6)
(A1)	6	1 %	49.41 %	37.89 % (0)	33.03 % (3)	35.99 % (5)
		0.01 %	49.60 %	37.72 % (0)	33.07 % (3)	35.80 % (5)
(A2)	6	1 %	53.21 %	- % (6)	- % (6)	- % (6)
		0.01 %	53.21 %	- % (6)	- % (6)	- % (6)

PWL bounding with relative ϵ-tolerance

Strength

- efficiency for problems with a nonlinear function per data set
- various fields and domains of application

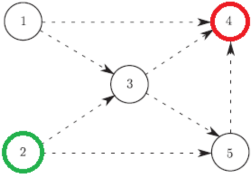
To be improved

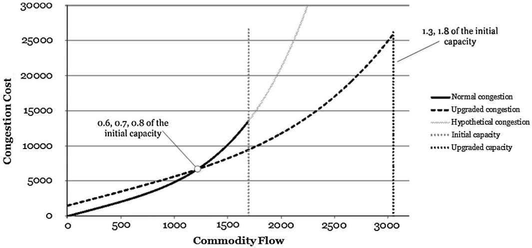
- access/ease-of-use for non-technical users
- speed improvement to tackle problems with multiple nonlinear functions

Minimize Total cost = design cost + routing cost + capacity augmentation cost + congestion cost

s.t.

- Flow conservation
- Maximum capacity (with/without upgrade)
- all commodities get to destination



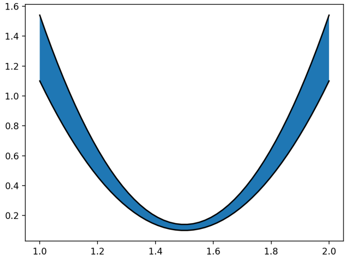


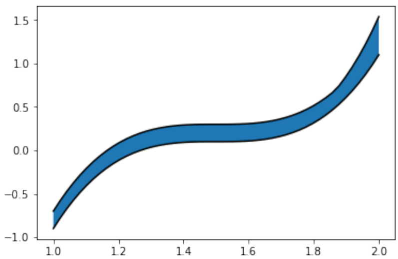
Paraskevopoulos, Sinan Gürel and Bektas, 2016

New perspective

Definition (Corridor)

Let $h, l : [a, b] \to \mathbb{R}$ C^1 $h(x) > l(x), \forall x \in [a, b]$. We call $\mathcal{C} = \{(x, y) | x \in [a, b], l(x) \leq y \leq h(x)\}$ a corridor between h and l .





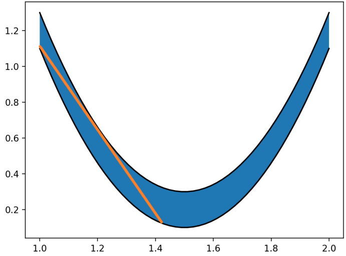
Codsi, Gendron, Ngueveu (2019-2020)

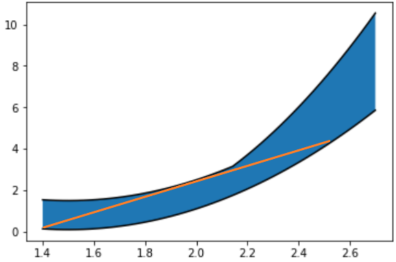
Convex corridor

Theorem (Convex corridor segment characterization)

On convex corridor $\mathcal{C}$ there exists an optimal linear segment such that

- *Both ends lie on the lower curve*
- *it is tangent to the upper curve*



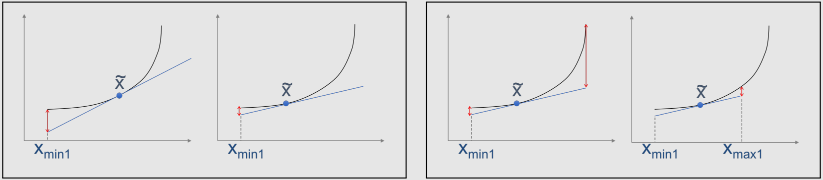


Convex corridor algorithm

generalisation of the algorithm of Ngueveu 2019

- Use dichotomy to find the tangent point
- find the resulting segment
- repeat 1 and 2 for the next segment

optimal 2-steps dichotomic search : No NLP or MINLP solved !



step a: max $\tilde{x}$ step b: max x_{max_i}

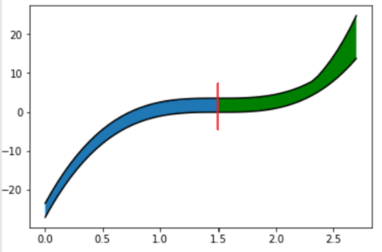
Strength of the algorithm

- logarithmic convergence (for each segment)
- Works on concave corridors too
- not limited to absolute error
- can be used to approximate, underestimate and overestimate functions

Corridors Without Constant Convexity

Splitting the corridor

+ parallelizable
+ Efficient
- Heuristic Not necessarily optimal but the error is tightly bounded



$$n^* \leq n \leq n^* + \#\text{Sub-corridors}$$

O'Rourke adaptation

based on function sampling and constraints on the line coefficient space

+ Exact
- Not as efficient

function	absolute error	Continuous		ncc			
		time					
		exact		exact		Heuristic	
		[RK15]	[RK19]	[Ngu19]	New	[Ngu19]	New
VI	0.1	Min	24.4 s	115 s	2.9 s	11 s	0.8 s
	0.05	Few days	107.7 s	88 s	3.0 s	17 s	0.8 s
	0.01	*	*	164 s	3.5 s	36 s	0.8 s
	0.005	*	*	195 s	2.8 s	59 s	0.8 s
VII	0.1	*	311.1 s	226 s	4.0 s	17 s	0.7 s
	0.05	*	14 514.6 s	287 s	3.9 s	28 s	0.8 s
	0.01	*	*	268 s	4.9 s	52 s	0.9 s
	0.005	*	*	869 s	5.0 s	72 s	0.9 s
VIII	0.1	sec	1.7 s	74 s	4.6 s	4 s	0.4 s
	0.05	*	5.3 s	83 s	5.4 s	6 s	0.4 s
	0.01	*	59.6 s	138 s	4.7 s	11 s	0.4 s
	0.005	*	247.2 s	1466 s	4.5 s	16 s	0.4 s
IX	0.1	Few days	1.9 s	77 s	8.3 s	8 s	0.8 s
	0.05	*	12 s	64 s	8.1 s	12 s	0.8 s
	0.01	*	13 873.4 s	114 s	8.1 s	23 s	0.8 s

Results

Network design with congestion

- from a multivariate objective-function ...
  min $\sum_{(i,j)\in A} O_{ij} y_{ij} + \sum_{(i,j)\in A} \sum_{p\in P} D_{ij}^p x_{ij}^p + \sum_{i\in N} E_i z_i + \sum_{i\in N} g_i(\sum_{j\in N} \sum_{p\in P} x_{ij}^p, z_i)$
- ... to a univariate objective-function
  min $\sum_{(i,j)\in A} O_{ij} y_{ij} + \sum_{(i,j)\in A} \sum_{p\in P} D_{ij}^p x_{ij}^p + \sum_{i\in N} f_i(v_i)$

instance	litterature	new
c35_0.3_0.6	6,615 s	6,92 s
c36_0.8_0.8	21,528 s	14,59 s
c49_0.8_0.6	172,255 s	118,17 s
c50_0.8_0.6	2609,568 s	2575,08 s

- as good as advanced state of the art solution methods
- no consideration on the problem structure
- easy to implement

Implementation: LinA package

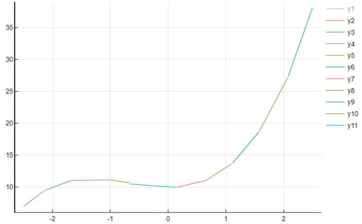
LinA Package

Implementing both the exact and heuristic methods and include many classical error metrics!

- Julia based

$2x^2 + x^3 + 10$

```
LinA.Linearize(:(2x^2+x^3+10), -2.5, 2.5, Relative(1.0); bounding = Under())
```



With an error guarantee!

Conclusion

Done

- PWLA+MILP efficient for solving certain classes of MINLPs
  - Non-necessarily continuous piecewise linear functions
  - Relative $\epsilon-$ tolerance
  - Bounding instead of approximation
- 2 similar MILPs to solve
- Various applications

What next ?

- ~~Coming soon~~ (available now): opensource toolbox LinA
- Extension to non-separable functions
- Other classes of problems ? (stochastic programming ?)

Useful Tools / Julia Packages

LinA: Computing a PWL approximation, over-/under-estimators with minimum # linear segments

- link: `http://homepages.laas.fr/sungueve/LinA.html`
- input : a univariate continuous nonlinear function
- output : a nnc PWL function with minimum number of pieces
- related reference: Codsi, Gendreau, Ngueveu (2019-HAL)

PiecewiseLinearOpt: Modeling efficiently a given continuous PWL function in MILP

- `https://github.com/joehuchette/PiecewiseLinearOpt.jl`
- input : a continuous PWL function (or sampled nonlinear fct)
- output : variables and constraints to insert in a MILP
- related reference: Huchette and Vielma (2018-arXiv)

Bibliography I

Bellman, R. and Roth, R. (1969)
Curve fitting by segmented straight liness.
Journal of the American Statistical Association, 64:1079 - 1084.

Bonami, P., Biegler, L.T., Conn, A.R., Cornuéjols, G., Grossmann, I.E., Laird, C.D., Lee, J., Lodi, L, Margot, F., Sawaya, N. and Wächter, A., (2008)
An algorithmic framework for convex mixed integer nonlinear programs.
Discrete Optimization, 5(2): 186 - 204. In Memory of George B. Dantzig.

Borghetti, A. C. D'Ambrosio, A. Lodi, and S. Martello (2008).
An milp approach for short-term hydro scheduling and unit commitment with head-dependent reservoir.
Power Systems, IEEE Transactions on, 23(3):1115 - 1124.

Burlacu, R., Geibler, B., and Schewe, L. (2019)
Solving mixed-integer nonlinear programmes using adaptively refined mixed-integer linear programmes.
Optimization Methods and Software, DOI: 10.1080/10556788.2018.1556661.

Camponogara, E., M.P. de Castro, and A. Plucenio (2007).
Compressor scheduling in oil fields: A piecewise-linear formulation.
In Automation Science and Engineering. CASE 2007. IEEE International Conference on, p. 436 - 441.

Bibliography II

Codsi, J., Gendron, B. and Ngueveu S.U. (2019-2020).
Fitting a piecewise linear function within a corridor: application to mixed integer optimization.
working paper.

D'Ambrosio, C., Lodi, A. and Martello, S. (2010).
Piecewise linear approximation of functions of two variables in MILP models.
Operations Research Letters, 38:39 - 46.

D'Ambrosio, C., Lodi, A. Wiese, S. and Bragalli, C. (2015).
Mathematical programming techniques in water network optimization.
European Journal of Operational Research, 243(3):774 - 788.

Floudas C.A. and C.E. Gounaris (2009).
A review of recent advances in global optimization.
Journal of Global Optimization, 45: 3 - 389.

Frenzen, C.L., Sasao, T. and Butler J.T. (2010).
On the number of segments needed in a piecewise linear approximation.
J. of Comp. and Applied Math., 45: 3 - 389.

Bibliography III

Gaoua, Y., Caux, S. and Lopez P. (2013).
Energy management for an electric vehicle based on combinatorial modeling.
In 2013 IEEE Int. Conf. on Ind. Eng. and Systems Management (IESM), p. 1 - 6.

Geißler, B., Martin, A., Morsi, A. and Schewe, L. (2012).
Using Piecewise Linear Functions for Solving MINLPs, pages 287–314.
Springer New York, New York, NY.

Grossmann I.E. (2002).
Review of nonlinear mixed-integer and disjunctive programming techniques.
Optimization and Engineering, 3:227 - 252.

Huchette, J. and Vielma, J.P. (2018).
Nonconvex piecewise linear functions: Advanced formulations and simple modeling tools
arXiv:1708.00050v2.

Kong, L. and Maravelias, C.T. (2020).
On the derivation of continuous piecewise linear approximating functions.
INFORMS Journal On Computing, 32(3):1–16, 2020.

Bibliography IV

Ngueveu, S. U. (2019).
Piecewise linear bounding of non linear univariate functions and mixed integer linear programming-based solution methods.
European Journal of Operational Research, 1058-1071.

Ngueveu, S. U., Artigues, C. and Lopez, P. (2016)
Scheduling under a non-reversible energy source: An application of piecewise linear bounding of non-linear demand/cost functions.
Discrete Applied Mathematics, 208:98–113.

O'Rourke, J. (1981)
An on-line algorithm for fitting straight lines between data ranges
in Communications of the ACM, 24(9):574 - 578.

Dimitris C. Paraskevopoulos and Sinan Gürel and Tolga Bektaş, (2016)
The congested multicommodity network design problem
Transportation Research Part E: Logistics and Transportation Review, 85:166 - 187.

Polisetty P.K. and E.P. Gatzke (2006).
A decomposition-based MINLP solution method using piecewise linear relaxations.
Technical report, University of South Carolina.

Bibliography V

Rebennack, S. and Kallrath, J. (2014).
Continuous Piecewise Linear Delta-Approximations for Bivariate and Multivariate Functions.
J. Optim. Theory Appl., 167(1):102 - 117.

Rebennack, S. and Kallrath, J. (2015).
Continuous piecewise linear delta-approximations for univariate functions: Computing minimal breakpoint systems.
J. Optim. Theory Appl., 167(2):617 - 643.

Rebennack, S. and Krasko, V. (2019).
Piecewise linear function fitting via mixed-integer linear programming.
INFORMS Journal on Computing, to appear.

Rosen, J. B. and Pardalos, P. M. (1986).
Global minimization of large-scale constrained concave quadratic problems by separable programming.
Mathematical Programming, 34(2):163–174.

Tong, B., Zhai, Q. and Guan, X. (2013).
An MILP based formulation for short-term hydro generation scheduling with analysis of the linearization effects on solution feasibility.
IEEE Trans. Power Syst., 28(4), 3588– 3599 ()

Bibliography VI

Tomek, I. (1974).
Piecewise-linear approximation with a bound on absolute error.
Computers and Biomedical Research, 7(1):64 - 70.

Toriello, A. and Vielma J.P. (2012).
Fitting piecewise linear continuous functions.
European Journal of Operational Research, 219:86 - 95.