Output-reference tracking problem for discrete-time systems with input saturations

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Abstract: Nonlinear state feedback controllers are exhibited for locally stabilising linear discrete-time systems with both saturating actuators and additive disturbances when the output must track a certain time-varying reference level. The objective is to bring the steady-state error due to disturbances to zero, by using a saturated controller and a dead-zone function. The objective is two-fold: to determine both a stabilising controller and a region of the state space over which the stability of the resulting closed-loop system is ensured, when the controls are allowed to saturate.

1 Introduction

With the growing complexity of control specifications, linear models tend to describe more precisely the practical systems from which they are built. Indeed, for technological, economical or safety reasons, the energy delivered by the actuators to the system is bounded. This can be described by a bound on the control input amplitude. If this limitation is omitted in the control design, unexpected phenomena can affect the closed-loop system: a degradation in performance, the occurrence of parasitic equilibrium points or limit cycles, and in the worst case the closed-loop system can become unstable. A major improvement in linear theory thus consists of taking into account actuators' limitations.

There has been renewed interest in the problem of local [1], global [2] or semi-global [3] stabilization of systems subject to input saturations. The problem is to synthesize simultaneously a saturating controller and a domain of the state space where the system can be initialised with guaranteed stability [4, 5].

In addition to the problem of stability induced by actuators' saturations, the control must meet certain performance requirements, including reference tracking in the presence of disturbances. It is well known that, for linear systems, the disturbances can be eliminated by using an integral-error feedback, but when the control saturates, a wind-up phenomenon can occur.

Several approaches, based on the addition of a dead-zone nonlinearity, have been proposed in the continuous-time framework to overcome the wind-up problem [6–9]. Unfortunately, they do not provide a systematic and single-step methodology to compute both a stabilising and saturating controller and a related domain of stability for the closed-loop system. Either one needs extensive simulations [9], or a two-steps strategy is developed, first by designing a linear controller for the non-saturating plant and then by modifying the controller to take the saturation into account [6, 7, 10].

The objective of this paper is to propose a systematic method to compute a saturating controller and a domain of local stability for discrete-time systems, while addressing the problem of reference tracking with additive disturbances on the state. No assumption is made on the open-loop stability. In this sense, the addressed problem is a problem of local stabilization. When some assumptions on the open-loop stability are considered, solutions have been proposed in a global and semi-global context [11, 12]. Here, the objective is expressed in terms of output tracking of a given reference level by bringing the steady-state error to zero. The reference level and disturbances under consideration are time-varying. The saturating controller containing nonlinear actions and the local ellipsoidal domain of stability are computed by using some relaxation schemes and a linear matrix inequalities (LMIs) formulation. Moreover, an ellipsoidal domain of admissible disturbances and reference is computed, for which the stability is ensured. This paper follows other work [13, 14] on continuous-time systems and proposes an extension of previous work [15], since the disturbances and the references are assumed to be time-varying.

2 List of symbols

The transpose of a vector \( y(t) \) is denoted by \( y(t)' \). The index \( e \) indicates that the variable is considered at the equilibrium. Matrix \( I_p \) denotes the identity matrix in \( \mathbb{R}^{p \times p} \). For any matrix \( M, M_{ij} \) (resp. \( M_{i,j} \)) denotes its \( i \)th row and \( j \)th column. Given any vector \( f \in \mathbb{R}^p \), the matrix \( \Delta f(M) \) denotes the diagonal matrix with components \( \Delta f_{ij} \), for \( i = 1, \ldots, p \). For any matrix \( M \in \mathbb{R}^{P \times P} \), \( M > 0 \) (resp. \( M \geq 0 \)) means that \( M \) is positive definite (resp. semi-definite). For two vectors \( x, y \in \mathbb{R}^m \), the notation \( x \preceq y \) is component-wise; this means that \( x_{ij} \preceq y_{ij} \), for \( i = 1, \ldots, m \). The index \( i \) denotes the convex hull. For any matrix \( N \in \mathbb{R}^{m \times n} \), with \( m \leq n \) and \( \text{rank}(N) = m \), \( N^N = N^T(NN^T)^{-1} \) denotes the right pseudo-inverse of \( N \), i.e. \( NN^N = I_m \).
saturation function $\text{sat}_o(v)$ is generically described by its components:

$$
\text{sat}_o(w(s)) = \begin{cases} 
    w(s) & \text{if } w(s) > w(0) \\
    w(0) & \text{if } -w(0) \leq w(s) \leq w(0) \\
    -w(0) & \text{if } w(s) < -w(0) 
\end{cases}
$$

(1)

3 Problem formulation

Consider the discrete-time system subject to actuator saturation described by

$$
x(k+1) = Ax(k) + B\text{sat}_o(u(k)) + d(k)
$$

$$
y(k) = Cx(k) + d\hat{w}(k)
$$

$$
e(k) = y(k) - r(k)
$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^l$ is the output vector, $d \in \mathbb{R}^r$ and $\hat{w} \in \mathbb{R}^s$ are vectors of disturbances, $r \in \mathbb{R}^l$ is the desired reference to follow and $e \in \mathbb{R}^l$ is the tracking error. The matrices $A$, $B$ and $C$ are constant real matrices of appropriate dimensions. We assume that the following hypotheses hold with respect to eqn 2.

Assumption 1: Matrices $B$ and $C$ are full rank ($\text{rank}(B) = m$ and $\text{rank}(C) = l$).

Assumption 2: For the system in eqn. 2, the number of inputs is greater than or equal to the number of outputs, i.e. $m \geq l$ [16].

Assumption 3: Pairs $(A, B)$ and $(C, \hat{A})$ are assumed to be controllable and observable. Furthermore,

$$
\text{rank} \left( \begin{bmatrix} I_n & -\hat{A} & -B \end{bmatrix} \right) = n + l
$$

The control objective is to achieve asymptotic output-reference tracking despite both actuator saturation and external disturbances. In other words, our aim consists of bringing the output $y(k)$ to the reference $r(k)$ at the equilibrium point for the closed-loop system with the anti-wind-up term $A(h)(\text{sat}_o(v(k)) - v(k))$. Thus, we introduce an additional state variable $q(k) \in \mathbb{R}^l$ with an anti-wind-up term [17]:

$$
q(k+1) = q(k) + e(k) + A(h)(\text{sat}_o(v(k)) - v(k))
$$

(3)

with $v \in \mathbb{R}^l$ an additional control input and $A(h) \in \mathbb{R}^{l \times l}$ a diagonal positive matrix. $A(h)$ is the anti-wind-up gain matrix. As for the continuous-time case [9], we introduce the error co-ordinates' representation using the new state vector

$$
z = \begin{bmatrix} e \\ x_2 \\ q \end{bmatrix} \in \mathbb{R}^{n+l}
$$

(4)

where $x_2 \in \mathbb{R}^{n+l}$ is defined by $x_2 = M_1x$, $M_1 \in \mathbb{R}^{(n-l) \times n}$, $M_1$ is chosen such that

$$
M_2 = \begin{bmatrix} \hat{C} \\ M_1 \end{bmatrix} \in \mathbb{R}^{n \times n}
$$

is nonsingular (this is always possible from Assumption 1).

The initial system in eqn. 2 can then be written as

$$
z(k+1) =Az(k) + B_2(\text{sat}_o(v(k)) - v(k)) + B_d(k)
$$

(5)

with

$$
A = \begin{bmatrix} M_2\hat{A}M_2^{-1} & 0 \\ E' & I_n \end{bmatrix} \in \mathbb{R}^{(n+l) \times (n+l)},
$$

$$
E = \begin{bmatrix} I_l \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+l) \times l},
$$

$$
B_1 = \begin{bmatrix} M_2B \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+l) \times m},
$$

$$
B_2 = \begin{bmatrix} 0 \\ A(h) \end{bmatrix} \in \mathbb{R}^{(n+l) \times l},
$$

$$
B_3 = \begin{bmatrix} M_2 \begin{bmatrix} I_n & -\hat{A} \end{bmatrix}E & -(I_n - \hat{A})E \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
d = \begin{bmatrix} [\hat{d} \hat{w} \hat{r} \hat{A}\hat{w} \hat{A}r]' \end{bmatrix} \in \mathbb{R}^{n+l}
$$

Remark 1. Note that in the system in eqns. 5 and 6 the new vector of perturbations $d(k)$ results from both the disturbances $\hat{d}(k), \hat{w}(k)$, the reference input $r(k)$ and the disturbance and reference rate $A(h) = \hat{w}(k+1) - \hat{w}(k)$ and $\Delta r(k) = r(k+1) - r(k)$.

Our control objective can be formulated into the following problem.

Problem 1. Compute two matrices $F_1 \in \mathbb{R}^{m \times (n+l)}$ and $F_2 \in \mathbb{R}^{r \times (n+l)}$, a set of initial conditions $X_0$ and a set $D_0$ of admissible disturbances $d$, such that $u(k) = F_1z(k)$ and $r(k) = F_2z(k)$ locally asymptotically stabilise the system in eqn. 5 for any initial condition in $X_0$ and any disturbance in $D_0$, i.e. asymptotically stabilise the closed-loop system:

$$
z(k+1) = Az(k) + B_2(\text{sat}_o(F_2z(k))) + B_d(k)
$$

In the disturbance-free case (i.e. $d(k) = 0$, $\hat{d}(k) = 0$, $\hat{w}(k) = 0$ and $r(k) = 0$, $\forall k$), stabilising feedback gains $F_1$ and $F_2$ being given, and the resulting nonlinear closed-loop system (eqn. 7) possesses a basin of attraction of the equilibrium point $z_k = 0$ [5, 18]. That is not the case when $d(k) \neq 0$, since it is not possible to strictly define one equilibrium point for the closed-loop system with the time-varying disturbance $d(k)$. At a given instant $k$ with $d(k) = d_s$, however, a corresponding equilibrium point $z(k+1) = z(k) = z_s$ could be computed, implying that associated to any disturbance $d_s \in D_0$, there exists a set of equilibrium points $Z_s$. In this sense, we could say that the closed-loop system (eqn. 7) exhibits a basin of attraction of this set of equilibrium points. The determination of this basin is a very hard (if not impossible) task, even when $d(k) = 0$ (no disturbance) [19–21].

An interesting way to overcome this difficulty is therefore to determine a set of admissible initial conditions $X_0$ from which the stability of the system in eqn. 7 is guaranteed [19]. Recall that our control objective consists
of satisfying the tracking condition \( e_k = 0 \), in particular or equivalently \( y(k) = r(k) \). Thus, the set of equilibrium points of interest can be defined as follows:

\[
Z_e = \left\{ z_e = \begin{bmatrix} 0 \\
x_{2e} \\
k_e \end{bmatrix}; z(k+1) = z(k) = z_e \right\} \tag{8}
\]

Remark 2. In the constant disturbance case, when \( r(k) = r \) and \( \tilde{w}(k) = \tilde{w} \) (i.e. \( \Delta \tilde{w}(k) = 0 \) and \( \Delta r(k) = 0 \)), \( Z_e \) reduces to one equilibrium point. Such a case has been treated in a continuous-time context elsewhere.

The resolution of Problem 1 consists of being able to characterise the sets \( X_0 \) and \( D_0 \) such that the following properties hold with respect to the system in eqn. 7:

1. When \( d(k) = 0 \), \( \forall k \) (disturbance-free case), \( z(k) \) asymptotically converges to the origin for any \( z(0) \in X_0 \)
2. When \( d(k) \neq 0 \), \( z(k) \) converges into \( Z_e \), as defined in eqn. 8, for any \( z(0) \in X_0 \) and \( d(k) \in D_0 \), \( k = 0, 1, \ldots \)

For pursuing such a solution, we use the S-procedure and the Lasalle invariance principle [19]. In other words, we search both sets \( X_0 \), \( D_0 \) and a continuous function \( V(z) \) satisfying

1. \( V(z(0)) \in X_0 \), \( \forall k \), \( z(k) \in X_0 \), \( \forall k = 0, 1, \ldots \), \( V(d(k)) \in D_0 \)
2. \( F: X_0 \to \Re \) is a function such that \( \Delta V(z(k)) = V(z(k+1)) - V(z(k)) \), \( z(k) \leq 0 \), \( \forall z(k) \in X_0 \), \( \forall d(k) \in D_0 \)
3. \( Z_e \subset X_0 \), with \( Z_e \) defined in eqn. 8.

(The links between \( Z_e, X_0, D_0 \) and \( V(z) \) are studied with our results.)

Remark 3: When saturations do not occur, i.e. when \( z(k) \in S(F_1, u_0) \cap S(F_2, v_0) \) defined by

\[
S(F_1, u_0) = \{ z \in \Re^{n+1}; -u_0 \leq F_1 z \leq u_0 \} \tag{9}
\]

\[
S(F_2, v_0) = \{ z \in \Re^{n+1}; -v_0 \leq F_2 z \leq v_0 \} \tag{10}
\]

the closed-loop system (eqn. 7) admits the following linear model

\[
z(k+1) = (A + B_1 F_1) z(k) + B_2 d(k) \tag{11}
\]

Note that we cannot conclude, without additional conditions, that any trajectory initiated in \( S(F_1, u_0) \cap S(F_2, v_0) \) is a trajectory of the system in eqn. 11, i.e. remains confined in \( S(F_1, u_0) \cap S(F_2, v_0) \).

Remark 4: When the open-loop system is stable, the global stabilization of the system in eqn. 7 can be studied, see, for example, cases A and B of the numerical example developed by Glattfelder and Schaufelberger [22]. In this case, the set \( X_0 \) in Problem 1 is equal to \( \Re^{n+1} \) (see elsewhere [23] for the continuous-time case). Throughout there is no assumption on the open-loop stability (it can be unstable). In this sense, the problem to be solved is a problem of local stabilization. This implies that we should limit the evolution of the state \( z \) so that the control law is able to stabilise the system despite saturations. Case C of the numerical example is a very well suited case of local stabilization [22].

Remark 5: In current work on anti-wind-up, it is often assumed that the dead-zone nonlinearity directly acts on \( u \) [17]. Here we have relaxed this constraint by introducing an additional control input \( v \). This brings an additional degree of freedom to the control that may be used to improve the performance of the closed-loop system (in terms of convergence rate, for example). However, this cannot be directly specified in the synthesis problem, but can be a posteriori analysed.

4 Preliminaries

4.1 Existence conditions of equilibrium set

Here we set different conditions to obtain, for the system in eqn. 7, an equilibrium set \( Z_e \) in the form of eqn. 8.

Lemma 1: Suppose that there exists an equilibrium point \( z_e \) for the system in eqn. 7. Then this equilibrium point belongs to \( Z_e \) defined in eqn. 8, provided that \( sat_{u_0}(F_1 z_e) = F_1 z_e \) (i.e. provided that \( z_e \in S(F_1, u_0) \) defined in eqn. 10, or, equivalently, if \( Z_e \subset S(F_2, v_0) \)).

Proof: Consider the system in eqn. 7 at the equilibrium \( z_e \). We then get

\[
z_e = A z_e + B_1 \text{sat}_{u_0}(F_1 z_e) + B_2 d_e
\]

\[
+ B_2 \text{sat}_{v_0}(F_2 z_e) - F_2 z_e + B_2 d_e \tag{12}
\]

From eqn. 6, this equality can be decomposed into the following:

\[
0 = -(I - A) \begin{bmatrix} e_x \\
x_{2e} \\
k_e \end{bmatrix} + B_1 \text{sat}_{u_0}(F_1 z_e) + B_2 d_e \tag{13}
\]

\[
0 = E \left[ \begin{bmatrix} e_x \\
x_{2e} \end{bmatrix} + A(h) (\text{sat}_{u_0}(F_2 z_e) - F_2 z_e) \right] \tag{14}
\]

eqn. 14 is equivalent to

\[
0 = e_x + A(h) (\text{sat}_{u_0}(F_2 z_e) - F_2 z_e) \tag{15}
\]

Hence, we obtain \( e_x = 0 \) in eqn. 15, provided that \( \text{sat}_{u_0}(F_2 z_e) - F_2 z_e = 0 \) (since by definition, \( A(h) \) is a positive definite diagonal matrix), which is equivalent to \( z_e \in S(F_2, v_0) \).}

From the equality in eqn. 13, we can determine a set of values for vectors \( x_{2e} \) and \( k_e \). Indeed, the saturation term \( \text{sat}_{u_0}(F_1 z_e) \) allows us to describe 3 regions of saturation in \( \Re^{n+1} \) [1]. Hence, according to this description for one value \( d_e \), eqn. 13 can lead to 3 regions

\[
\begin{bmatrix} 0 \\
x_{2e} \\
k_e \end{bmatrix} \tag{16}
\]

Each of these points could be locally studied in terms of existence (if the computed point does belong to the region of saturation considered for its calculation) and in terms of stability. However, to simplify our study, we restrict our attention to the case where

\[
\begin{bmatrix} 0 \\
x_{2e} \\
k_e \end{bmatrix} \tag{17}
\]

Therefore, we want to consider the set \( Z_e \) of equilibrium points

\[
z_e = \begin{bmatrix} 0 \\
x_{2e} \\
k_e \end{bmatrix} \tag{18}
\]

satisfying both

\[
|F_1 z_e| \leq u_0 \tag{19}
\]

\[
|F_2 z_e| \leq v_0 \tag{20}
\]

It is clear from eqn. 13 that eqns. 19 and 20 imply that the perturbation vector \( d_e \) and therefore \( d(k) \) have to satisfy Lemma 2.
Lemma 2: If \( A + B_1 F_1 \) is asymptotically stable and \( d(k) \) satisfies, \( \forall k = 0, 1, \ldots \) (there always exists such a matrix \( F_1 \) because, from Assumption 3, pair \((A, B_1)\) is stabilisable)

\[
-u_0 \leq F_1 (I_{n+1} - (A + B_1 F_1))^{-1} B_2 d(k) \leq u_0 \tag{21}
\]

\[
v_0 \leq F_2 (I_{n+1} - (A + B_1 F_1))^{-1} B_3 d(k) \leq v_0 \tag{22}
\]

then the following properties hold:

1. the set of equilibrium points \( Z_e \) is given by

\[
Z_e = \{ z_e = (I_{n+1} - (A + B_1 F_1))^{-1} B_2 d_e, \forall d_e \} \tag{23}
\]

satisfying eqns. 21 and 22

2. \( Z_e \subseteq S(F_1, u_0) \cap S(F_2, v_0) \)

3. \( Z_e \) is the unique equilibrium set for the system in eqn. 7.

Proof: First if \((A + B_1 F_1)\) is asymptotically stable, then matrix \( I_{n+1} - (A + B_1 F_1) \) is nonsingular. Next if \( d(k) \) satisfies eqns. 21 and 22, then by imposing \( d(k) = d_e \) at an instant \( k \) such that \( z(k+1) = z(k) = z_e \), we obtain,

\[
(I_{n+1} - (A + B_1 F_1)) z_e + B_2 d_e = 0
\]

which verifies eqns. 19 and 20. Properties 1 and 2 follow. Since the linear system in eqn. 11 is only valid in \( S(F_1, u_0) \cap S(F_2, v_0) \), property 3 follows.

Lemma 3: The set \( Z_e \) as defined in eqns. 8 and 23 is included in \( S(F_1, u_0) \), \( \forall k \), and therefore also that \( d_e \in S(F_1, u_0) \) to compute the set \( Z_e \) given in eqn. 23. We could also consider that \( d(k) \) does not belong temporarily in \( S(F_1, u_0) \) and may re-enter it and settle down \( z(k) \) to \( z_e \) (this last case is not investigated here).

Lemmas 1, 2 and 3 prove that the output tracking objective cannot be carried out for any disturbance \( d \), i.e., for any disturbance \( d \) and \( z(k) \), any reference input \( Y \) and any variations \( d_G \) and \( d_r \).

Remark 6: In the case \( m = l \), since we get

\[
\text{rank}\left(\begin{bmatrix} I_n - \tilde{A} & -\tilde{B}_1 \\ -E' & 0 \end{bmatrix}\right) = n + m,
\]

it suffices to replace in eqn. 24

\[
\begin{bmatrix} I_n - \tilde{A} & -\tilde{B}_1 \\ -E' & 0 \end{bmatrix}
\]

by

\[
\begin{bmatrix} I_n - \tilde{A} & -\tilde{B}_1 \\ -E' & 0 \end{bmatrix}^{-1}
\]

Remark 7: For simplicity, we have considered that \( d(k) \in S(F_1, u_0) \), \( \forall k \), and therefore also that \( d_e \in S(F_1, u_0) \) to compute the set \( Z_e \) given in eqn. 23. We could also consider that \( d(k) \) does not belong temporarily in \( S(F_1, u_0) \) and may re-enter it and settle down \( z(k) \) to \( z_e \) (this last case is not investigated here).

Lemmas 1, 2 and 3 prove that the output tracking objective cannot be carried out for any disturbance \( d \), i.e., for any disturbance \( d \) and \( z(k) \), any reference input \( Y \) and any variations \( d_G \) and \( d_r \).

4.2 Polytopic model

Our results are based on a polytopic model representation of the saturated system [24–26]. Thus, with respect to the system in eqn. 7 the two saturation terms can be written as

\[
\text{sat}_{u_0}(F_1 z(k)) = \Delta(a_1(z(k))) F_1 z(k)
\]

\[
\text{sat}_{v_0}(F_2 z(k)) = \Delta(a_2(z(k))) F_2 z(k)
\]

\[
\Delta(a_1(z(k))) \in \mathbb{R}^{m \times m} \quad \text{and} \quad \Delta(a_2(z(k))) \in \mathbb{R}^{l \times l}
\]

are diagonal matrices, whose diagonal elements \( a_{1g}(z(k)), g = 1, \ldots, m \) and \( a_{2g}(z(k)), g = 1, \ldots, l \), are defined by

\[
\begin{align*}
\Delta_{u_0}(z(k)) &= \begin{cases} u_{0(0)} & \text{if } F_{10}(z(k)) > u_{0(0)} \\
1 & \text{if } -u_{0(0)} \leq F_{10}(z(k)) \leq u_{0(0)} \\
-u_{0(0)} & \text{if } F_{10}(z(k)) < -u_{0(0)}
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\Delta_{v_0}(z(k)) &= \begin{cases} v_{0(0)} & \text{if } F_{20}(z(k)) > v_{0(0)} \\
1 & \text{if } -v_{0(0)} \leq F_{20}(z(k)) \leq v_{0(0)} \\
v_{0(0)} & \text{if } F_{20}(z(k)) < -v_{0(0)}
\end{cases} \\
\end{align*}
\]

Thus we get

\[
[F_{11} F_{12}] z_e = F_1 z_e
\]

\[
= \begin{bmatrix} I_n & \tilde{A} & -\tilde{B}_1 \\ -E' & 0 & B_3 d_e \end{bmatrix} (30)
\]

Hence the result of the lemma follows.

Remark 7: For the case of a constant reference (i.e., \( r(k) = r \), \( \forall k \)), the only way to solve our output tracking objective for any reference \( r \) is to verify \( (I_n - \tilde{A}) E = 0 \). Such a condition requires some structural properties between matrices \( A \) and \( C \).
and satisfy \(0 < \alpha_{1}(z(k)) \leq 1\) and \(0 < \alpha_{2}(z(k)) \leq 1\). By this way, the system in eqn. 7 can be written in the equivalent form:

\[
z(k + 1) = (A + B_{1}A(z(k)))F_{1}z(k) + B_{2}(A(z(k)))I_{1}F_{2}z(k) + B_{3}d(k)
\]

Note that the coefficient \(\alpha_{1}(z(k))\) (resp. \(\alpha_{2}(z(k))\)) can be viewed as an indicator of the saturation degree of the \(i\)th entry of \(u(k)\) (resp. \(v(k)\)). Thus, the smaller the value of \(\alpha_{1}(z(k))\) (resp. \(\alpha_{2}(z(k))\)), the farther the state vector \(z\) is from the region of linearity \(S(F_{1}, u_{i})\) (resp. \(S(F_{2}, v_{i})\)).

Since we are interested in the problem of local stabilization we have to limit \(z\), and therefore to give a lower bound for both \(\alpha_{1}(z(k))\) and \(\alpha_{2}(z(k))\). If we consider any compact set \(\Omega_{0} \subset \mathbb{R}^{n+1}\), then for \(z(k) \in \Omega_{0}\) we may define a lower bound for the positive vectors \(\alpha_{1}(z(k))\) and \(\alpha_{2}(z(k))\) as follows:

\[
a_{1\text{min}}(z) = \min_{z(k) \in \Omega_{0}} \{ \alpha_{1}(z) \}, i = 1, \ldots, m
\]

\[
a_{2\text{min}}(z) = \min_{z(k) \in \Omega_{0}} \{ \alpha_{2}(z) \}, g = 1, \ldots, l
\]

**Remark 6:** For example, by considering \(\Omega_{0} = \{ z \in \mathbb{R}^{n+1}; \ z \leq \zeta, P = P > 0 \}\) it follows that

\[
a_{1\text{min}}(z) = \min \left( 1, \frac{\mu_{1}(z)}{\sqrt{F_{10}P_{0}\Gamma P_{10}}} \right), i = 1, \ldots, m,
\]

and

\[
a_{2\text{min}}(z) = \min \left( 1, \frac{\gamma_{0}(z)}{\sqrt{F_{20}P_{0}\Gamma P_{20}}} \right), g = 1, \ldots, l
\]

Thus, \(\forall z(k) \in \Omega_{0}\), the scalars \(\alpha_{1}(z(k))\) and \(\alpha_{2}(z(k))\) satisfy \(0 < \alpha_{1}(z(k)) \leq 1\), \(\forall i = 1, \ldots, m\) and \(0 < \alpha_{2}(z(k)) \leq 1\), \(\forall g = 1, \ldots, l\). Hence, we can define 2\(m\) matrices \(A_{j}, j = 1, 2, \ldots, 2m\) and 2\(l\) matrices \(B_{q}^{*}\), \(q = 1, 2, \ldots, 2l\):

\[
A_{j} = A + B_{1}A(z_{j})F_{1}
\]

\[
B_{q}^{*} = B_{2}(A(z_{q}) - I_{1})F_{2}
\]

\(\Delta(z_{j})\) (resp. \(\Delta(z_{q})\)) is a diagonal matrix, whose positive diagonal elements \(\Delta(z_{j})_{i,0} = \tau_{i}(z_{j})\) (resp. \(\Delta(z_{q})_{i,0} = \tilde{\tau}_{q}(z_{q})\)) take the values 1 or \(\alpha_{1}(z_{j})\), \(i = 1, \ldots, m\) (resp. 1 or \(\alpha_{2}(z_{q})\), \(g = 1, \ldots, l\)). Hence, if \(z(k) \in \Omega_{0}\), then \(z(k + 1)\) can be determined by the following polytopic system:

\[
z(k + 1) = \sum_{j = 1}^{2m} \lambda_{j}A_{j}z(k) + \sum_{q = 1}^{2l} \lambda_{q}B_{q}z(k) + B_{3}d(k)
\]

**Remark 10:** To solve Problem 1, we need to determine matrices \(F_{1}, F_{2}\), set \(\Omega_{0}\) and vectors \(\alpha_{1\text{min}}, \alpha_{2\text{min}}\).

**5 Results**

To solve Problem 1, we choose ellipsoidal sets \(\Omega_{0}\) and \(\mathcal{D}_{0}\), derived from symmetric positive definite matrices \(P\) and \(S\), respectively:

\[
\Omega_{0} = \mathcal{E}(P, \zeta) = \{ z \in \mathbb{R}^{n+1}; zPz \leq \zeta, P = P > 0, \zeta > 0 \}
\]

\[
\mathcal{D}_{0} = \mathcal{E}(S, \sigma) = \{ d \in \mathbb{R}^{n+1}; dSd \leq \sigma, S = S > 0, \sigma > 0 \}
\]

Proposition 1 can therefore be stated.

**Proposition 1:** If there exist matrices \(F_{1}, F_{2}, P = P > 0, S = S > 0\), \(\zeta > 0, \sigma > 0\), vectors \(\alpha_{1\text{min}}\) and \(\alpha_{2\text{min}}\), positive scalars \(\zeta > 0, \sigma > 0\), and \(\omega > 0\), satisfying

\[
\begin{cases}
-\mu P & * & * & * \\
0 & -\omega S & * & * \\
P(A + B_{1}A(z_{j})F_{1} + B_{2}(A(z_{q}) - I_{1})F_{2}) & * & * & * \\
-\xi_{j}^{2}(I_{n+1} - A - B_{1}F_{1}) & * & * & * \\
z_{j \text{min}}(z_{j})F_{1} & * & * & * \\
z_{q \text{min}}(z_{q}) & * & * & * \\
F_{1} & * & * & * \\
\xi_{q}^{2}(z_{q}) & * & * & * \\
\end{cases}
\]

\[
\begin{array}{ll}
0 & (46) \\
\end{array}
\]

\(*\) is the substitute for blocks ensuring matrix symmetry.

\[
0 < \alpha_{1\text{min}} \leq 1, \forall i = 1, \ldots, m
\]

\[
0 < \alpha_{2\text{min}} \leq 1, \forall g = 1, \ldots, l
\]

\[
\begin{cases}
\mu P & a_{1\text{min}}z_{j \text{max}}^{2} & * & * & * \\
0 & -\omega S & * & * & * \\
P(A + B_{1}A(z_{j})F_{1} + B_{2}(A(z_{q}) - I_{1})F_{2}) & * & * & * & * \\
-\xi_{j}^{2}(I_{n+1} - A - B_{1}F_{1}) & * & * & * & * \\
z_{j \text{min}}(z_{j})F_{1} & * & * & * & * \\
z_{q \text{min}}(z_{q}) & * & * & * & * \\
F_{1} & * & * & * & * \\
\xi_{q}^{2}(z_{q}) & * & * & * & * \\
\end{cases}
\]

\[
\begin{array}{ll}
0 & (49) \\
\end{array}
\]

\[
\begin{cases}
\mu P & a_{1\text{min}}z_{j \text{max}}^{2} & * & * & * \\
0 & -\omega S & * & * & * \\
P(A + B_{1}A(z_{j})F_{1} + B_{2}(A(z_{q}) - I_{1})F_{2}) & * & * & * & * \\
-\xi_{j}^{2}(I_{n+1} - A - B_{1}F_{1}) & * & * & * & * \\
z_{j \text{min}}(z_{j})F_{1} & * & * & * & * \\
z_{q \text{min}}(z_{q}) & * & * & * & * \\
F_{1} & * & * & * & * \\
\xi_{q}^{2}(z_{q}) & * & * & * & * \\
\end{cases}
\]

\[
\begin{array}{ll}
0 & (50) \\
\end{array}
\]

with \(F\) defined in eqn. 24.

The state feedback gains \(F_{1}\) and \(F_{2}\) are derived from the set of admissible initial conditions \(\mathcal{E}(P, \zeta)\) and the set of admissible disturbances \(\mathcal{E}(S, \sigma)\), then solve Problem 1.
Proof: Note that, with $A_i$ and $B_i$ defined in eqns. 39 and 40, with $F_2 = Z(I_{n-1} - (A + B,F_1))$ and using the Schur's complement, eqn. 46 equivalently reads

$$
\begin{bmatrix}
(A_j^r + B_j^r)P(A_j + B_j) - \mu P & B_j^rP(A_j + B_j) \\
0 & \langle A_j^r + B_j^r \rangle P B_j - \omega S \\
\end{bmatrix}
\leq 0 \quad (53)
$$

From eqns. 39 and 40, $\phi_j^r$ satisfies eqns. 47 and 48, and inclusion relations (eqns. 49 and 50), then the polytopic model (eqn. 41) can represent the saturated systems in eqn. 7. Moreover, by considering the quadratic function $V(z(k + 1))$ along the trajectories of the system in eqn. 41, it follows that

$$
V(z(k + 1)) = [z(k)'M(j,q)z(k) + d(k)'B_j^rP(A_j + B_j) + B_j^rP(j)]
$$

From eqns. 39 and 40, $M(j,q) = \sum_1^{2^n} \sum_{k=1}^{2^n} A_j^r + \sum_1^{2^n} \sum_{k=1}^{2^n} B_j^r$. First, we have to prove that $\forall (k + 1) \leq \xi^r, \forall (k + 1) \leq \xi^r$, and $\forall d(k)Sd(k) \leq \sigma - 1$, respectively. By using the $S$-procedure [27], it is possible to show that it is equivalent to seek $\mu > 0$ and $\sigma > 0$, such that

$$
V(z(k + 1)) - \frac{1}{\xi^r} - \mu \left( z(k)'P_z(k) - \frac{1}{\xi^r} \right) - \omega \left( d(k)'Sd(k) - \frac{1}{\sigma} \right) \leq 0 \quad (55)
$$

Thanks to convexity properties, if eqn. 46 is verified, then eqn. 55 holds. Furthermore, the satisfaction of eqn. 46 implies the asymptotic stability of matrix $A + B,F_1$. Moreover, if eqns. 51 and 52 are verified, then the inclusion relation $\mathcal{E}(S, \sigma) \subseteq \mathcal{S}(F_1, I_k) \cap \mathcal{S}(F_2, I_k)$, or, equivalently, $\mathcal{E}(S, \sigma) \subseteq \mathcal{S}(F_1, I_k) \cap \mathcal{S}(F_2, I_k)$ is satisfied, and therefore $z_k \in Z_2 \subseteq \mathcal{S}$, and $Z_2 = \infty$ is included in the set $\mathcal{S}(F_1, I_k) \cap \mathcal{S}(F_2, I_k)$. Hence, provided that there exist matrices $F_1, Z, P$ and $S$, vectors $\alpha_{2min}$ and $\alpha_{2min}$, positive scalars $\xi, \sigma$, and $\omega$ satisfying eqns. 46-52, Problem 1 is solved. □

Remark 11: A special case of the anti-wind-up term is given by $\Delta(h) = 0$. The control then reduces to a simple integrating action, and eqn. 3 reads

$$
g(k + 1) = g(k) + c(k)
$$

Therefore, the closed-loop system (eqn. 7) becomes

$$
z(k + 1) = Az(k) + B, sat, (F_2 z(k) + B, d(k))
$$

This case is also equivalent to set $F_2 = 0$. Thus, Proposition 1 applies by considering eqn. 46, in which we set $Z_2 = 0$, and eqns. 47, 49 and 51.

The direct application of Proposition 1 for solving Problem 1 is difficult due to some nonlinearities in the variables. Hence, we get the following:

(a) Eqn. 46 is nonlinear in all the variables because it depends on the products $PB, \Delta(g_2)F_1, PB, \Delta(g_2) - 1, Z$.

(b) Eqns. 47 and 48 are linear in $\alpha_{min}$ and $\alpha_{2min}$.

(c) Eqns. 49 and 50 are nonlinear in $\alpha_{min}$, $F_1$, and $\alpha_{2min}$, $Z$, $F_1$, respectively, to products $\Delta_{min}F_1$ and $\Delta_{2min}$, $Z_{min}(A_2+B_1)$. These relations are linear in $P$ and $\xi$.

(d) Eqn. 51 is linear in $S$ and $\sigma$.

(e) Eqn. 52 is linear in $S, Z$ and $\sigma$.

Remark 12: The problem formulation and the use of semi-definite programming do not allow us to solve the complete synthesis problem (as defined by Problem 1) in a single step (it is not possible to compute $F_1$ and $F_2$ (or $Z$) together). This is, however, possible for continuous-time systems [13]. We emphasise that our problem formulation entails different sources of conservatism including the representation of the saturated system by a polytopic model, the search of a unique Lyapunov matrix $P$ shared by each vertex of the matrix polytope and even the use of the $S$-procedure. Moreover, some global optimisation methods for solving the nonlinear problem introduced in Proposition 1 could be used [28], but their worst-case complexities and the required computational effort may make the solution untractable. Even if we do not attain the global optimum, we will be satisfied with $a(a)$ (approximative) feasible solution, i.e. a sub-optimal solution.

Hence, relations of Proposition 1 become linear as soon as some variables are fixed. In this sense, the relaxation schemes below can be considered.

(i) Relax 1: fix $F_1, Z$ (or equivalently $F_2$), $P, S, \xi, \sigma$ and search $\alpha_{min}, \alpha_{2min}, \omega$, which solve eqns. 46-50.

(ii) Relax 2: fix $F_1, Z$ (or equivalently $F_2$), $\alpha_{min}, \alpha_{2min}, \omega$, and search $P, S, \xi, \sigma$ which solve eqns. 46, 49-52.

(iii) Relax 3: fix $P, Z, \alpha_{min}, \alpha_{2min}, \omega$, and search $F_1, S, \xi, \sigma$ which solve eqns. 46, 49-52.

(iv) Relax 4: fix $P, F_1, \alpha_{min}, \alpha_{2min}, \omega$, and search $Z, S, \xi, \sigma$ which solve eqns. 46, 49-52.

Furthermore, Proposition 1 provides a sufficient condition to derive gains $F_1$ and $F_2$ and sets $\mathcal{E}(P, \xi)$ and $\mathcal{E}(S, \sigma)$. It is then interesting to orient the solutions in order to obtain ellipsoids $\mathcal{E}(P, \xi)$ and $\mathcal{E}(S, \sigma)$ as large as possible. First, recall that the size of $\mathcal{E}(P, \xi)$ (resp. $\mathcal{E}(S, \sigma)$) is related both to $P$ (resp. $S$) and $\xi$ (resp. $\sigma$). In that sense, we can consider the optimisation problems described below:

(a) Optim 1: $\min \sum_{k}^{n} \mathcal{E}(P, \xi)$

(b) Optim 2: $\min \log(\det(\mathcal{E})) + \log(\det(\mathcal{S}))$

(c) Optim 3: $\min \xi + \sigma + \alpha_{min}P + \alpha_{2min}$

We can therefore combine these optimisation problems with the previous relaxation schemes.

Remark 13: The optimisation criterion Optim 1, associated with Relax 1 allows us to decrease $\alpha_{min}$ and $\alpha_{2min}$, which corresponds (according to the definition of $\alpha_{min}$ and $\alpha_{2min}$) to an increase in the tolerance to the saturation of the closed-loop system.

Remark 14: Note that the optimisation of both $\mathcal{E}(P, \xi)$ and $\mathcal{E}(S, \sigma)$ leads to some trade-off between these two sets, since one increases while the other decreases. Depending on the control objective, the optimisation criterion may be weighted to emphasise the initial admissible states set $\mathcal{E}(P,$
of \( \xi \) or the set \( \mathcal{E}(S, \sigma) \) involving both output tracking references and admissible disturbances.

**Remark 15:** Two maximal admissible tracking references may be derived for \( \mathcal{E}(P, \xi) \) and \( \mathcal{E}(S, \sigma) \). The first one corresponds to the maximal initial value \( [-r_1^0 \ 0 \ 0]^T \) belonging to \( \mathcal{E}(P, \xi) \), i.e. the maximal distance between the initial output \( y \) at the origin and the reference signal \( r \). The second one corresponds to the maximal reference, when no disturbance occurs, \( [0 \ 0 \ 0 \ \Delta r]^T \) belonging to \( \mathcal{E}(S, \sigma) \). Finally, \( r_{\text{max}} = \min(r_1, r_2) \). Note, however, that these maximal values \( r_1 \) and \( r_2 \) do not correspond to the maximal value \( r \) which may be attained by exploring the whole surface of \( \mathcal{E}(P, \xi) \) and \( \mathcal{E}(S, \sigma) \), respectively. Note also that, more important than \( r_2 \), \( \Delta r \) limits the gradient of reference steps (see example).

Based on the above comments, we propose the following procedure for solving our control design problem.

### 6 Algorithmic procedure

#### 6.1 Initialisation

Compute initial feasible matrices \( F_1 \), \( F_2 \) (and by extension \( Z \)) and \( R \). They can be the solution, for example, to the following Riccati equation:

\[
A'(P^{-1} + B_1 R^{-1} B_1')^{-1} A - P + Q = 0
\]

with given \( R = R' > 0 \) and \( Q = Q' > 0 \), and defined by

\[
F_1 = -R^{-1} B_1' (P^{-1} + B_1 R^{-1} B_1')^{-1} A \quad (58)
\]

\[
F_2 = \Delta(\rho) B_1' PA \quad (59)
\]

\[
Z = F_2 (I_{n+1} - (A + B_1 F_1))^{-1} \quad (60)
\]

where \( \Delta(\rho) \in \mathbb{R}^{[n+1] \times \mu} \) is a diagonal matrix of positive elements.

Set \( \alpha_{\text{min}}(i) = 1 \), \( \forall i = 1, \ldots, m \), and \( \alpha_{\text{max}}(i) = 1 \), \( \forall \sigma = 1, \ldots, l \).

Set \( 1 > \mu > \lambda_{\text{max}}((A + B_1 F_1) P (A + B_1 F_1)^{-1}) \) and \( \omega = 1 - \mu \) (the lower bound on \( \mu \) is derived from a definite positive condition on the upper left \( 2 \times 2 \) block of inequality (eqn. 46)). Compute \( S \), \( \xi \) and \( \sigma \) which solve eqns. 46, 49–52 and minimise \( \text{Optim} \ 4 \).

#### 6.2 Iterative procedure

1. Solve Relax 1 with the optimisation criterion \( \text{Optim} \ 4 \).
2. Solve Relax 2 with the optimisation criterion \( \text{Optim} \ 4 \).
3. Solve Relax 3 with the optimisation criterion \( \text{Optim} \ 4 \).
4. Solve Relax 4 with the optimisation criterion \( \text{Optim} \ 4 \).
5. If some conditions on the size of sets \( \mathcal{E}(P, \xi) \) and \( \mathcal{E}(S, \sigma) \) are fulfilled, then stop. Otherwise, return to Step 1.

**Remark 16:** Note that the initialisation step (choice of admissible solutions \( P, F_1, F_2 \), choice of \( \mu \), criterion) and the optimisation criteria of Relax 1, 2, 3 and 4 have a strong influence on the solution, both in terms of size of sets \( \mathcal{E}(P, \xi) \) and \( \mathcal{E}(S, \sigma) \) and in terms of closed-loop spectrum. They have to be chosen according to the control objective.

### 7 Example

Consider the following example, borrowed from Chella-bona et al. [29] in its discrete-time form. The purpose consists of determining a controller for the pitch-axis longitudinal dynamics model of the F-16 fighter aircraft.

Thus, the system in eqn. 2 is described by the following data:

\[
\begin{bmatrix}
A & 0.1025 & 0.2080 \\
B & 1.1167 & 4.1522 \\
C & 0.0951 & 1.0716
\end{bmatrix}
\begin{bmatrix}
0.0879 & 0.0097 \\
1.8038 & 0.2140 \\
0.0992 & 0.0326
\end{bmatrix}
\]

Furthermore, vector \( \bar{u}(k) \) of the system in eqn. 2 is replaced by \( \bar{B} \bar{d}(k) \). In \( s(k) = [x_1(k), x_2(k), x_3(k)]' \), \( x_1 \) is the pitch angle, \( x_2 \) is the pitch rate and \( x_3 \) is the angle of attack. In \( u(k) = [u_1(k), u_2(k)]', \) \( u_1 \) is the elevator deflection and \( u_2 \) is the flap deflection. In the following we consider

\[
u_0 = \begin{bmatrix}
60 \\
60
\end{bmatrix}, \quad \nu_1 = 60, \quad \Delta(h) = 1 \quad \text{and} \quad M_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

which gives \( M_2 = I_1 \).

**Remark 17:** According to the structure of the example (\( \bar{B} \bar{d}(k) \) despite \( \bar{d}(k) \)), matrix \( B \) is modified as follows (with respect to eqn. 6):

\[
B_3 = \begin{bmatrix}
M_2 \bar{B} & (I_n - \bar{A})E & -(I_n - \bar{A})E & E & -E
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\in \mathbb{R}^{[n+1] \times (n+1)}
\]

and \( S \in \mathbb{R}^{[n+1] \times (n+1)} \).

Several optimisations have been done to evaluate the performance of the approach and the influence of some parameters. They are initialised as described in the algorithmic procedure, with \( \mu = 0.9940 \) and \( \Delta(\rho) = 1 \).

#### 7.1 Optimisation procedure

Independent of the optimisation criterion, the main idea is to optimise the volume of both \( \mathcal{E}(S, \sigma) \) and \( \mathcal{E}(P, \xi) \), directly related to \( \sqrt{\det(S^{-1}/\sigma)} \) and \( \sqrt{\det(P^{-1}/\xi)} \), respectively. The main problem of the iterative procedure proposed in Section 6.2 is that it does not converge towards a global optimum but to successive local ones. This is illustrated in Fig. 1, which shows the plot of the iteration evolution of

\[
C_{\text{opt}} = \left( \sqrt[\omega]{\det(P^{-1}/\xi)} \sqrt[\omega]{\det(S^{-1}/\sigma)} \right)^{\frac{1}{\omega}}
\]

with optimisation criterion: \( \min(\text{trace}(S) + \sigma) \).

Moreover, the optimal solution is also strongly related to the optimisation criterion. Different cases are summarised in Table 1, where the optimal solution is associated to the best \( C_{\text{opt}} \) obtained on 50 iterations.

![Fig. 1](image-url)
Remark 18: The idea to take as a criterion $\text{trace}(S) + \sigma$, i.e. to optimise the size of $E(S, \sigma)$ preferably to $E(P, \xi)$, comes from the solution obtained when the criterion is $\min \xi + \sigma$. Indeed, when the optimisation criterion tends to optimise both domains, it is to the detriment of $E(S, \sigma)$, which is much smaller than $E(P, \xi)$. Moreover, increasing $E(S, \sigma)$ induces a larger desired reference $r$, and then a larger domain of admissible state. Hence, even if the criterion only refers to $E(S, \sigma)$, $E(P, \xi)$ cannot be strongly decreased.

7.2 Anti-wind-up term
Comparisons are presented between the solution given with free $F_2$ and a discrete version of the simple integrator $(F_2 = [0 \ 0 \ 0 \ 0])$. In practice, there is no significant difference between the solutions, neither when comparing the volumes of ellipsoidal sets nor when comparing the closed-loop spectrum. In fact, it has been proven that in the continuous-time case [13], the Lyapunov function decreasing rate with free $F_2$ was faster than with the simple integrator (or with the intelligent integrator $F_2 = [0 \ 0 \ 0 \ 1]$ as defined by Krikelis and Barkas [9]). It has not been possible to prove the same in the discrete-time case, and it depends on the chosen example.

7.3 Time responses
Comparisons of the time responses are presented in Fig. 2, to emphasise the effect of saturation on the closed-loop response. The feedback gains are those obtained at the 25th iteration with optimisation criterion on $\text{trace}(S) + \sigma$:

$$F_1 = \begin{bmatrix} 3.11 & 0.73 & 2.56 & 0.35 \\ 2.15 & 0.76 & 1.75 & 0.96 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.49 & 0.15 & 0.93 & 0.50 \end{bmatrix}$$

The simulation conditions were the following:

$$z(0) = [10 \ -25 \ 15 \ -50]$$

$e = [d_1 \ d_2]$ with $d_1$ and $d_2$ random numbers belonging to the intervals $[-1, 1]$, $[-2.5, 2.5]$, respectively.

$r$ is a square signal of amplitude $\pm 22$, period 100, associated with a rate limiter of rate $\Delta r$.

Different cases have been considered: $\Delta r = 1.4$ such that $d(k) \in E(S, \sigma)$, $\forall k$; $\Delta r = 5$ such that $d(k) \notin E(S, \sigma)$; $\Delta r = 5.4$ such that $d(k) \notin E(S, \sigma)$ and the saturated closed-loop system becomes unstable.

The simulation example shows some conservatism of the approach since in the case $\Delta r = 5$, $d(k) \notin E(S, \sigma)$, the system remains asymptotically stable. It also shows, however, the necessity to determine some admissible tracking reference domain before implementing the saturated state feedback since (as shown by the example) the step reference signal cannot be applied ($\Delta r \geq 5.4$ leads to instability).

8 Conclusions
We have addressed the problem of local stabilization and reference tracking of linear discrete-time systems subject to input saturation and disturbances. By using some relaxation schemes and an LMI formulation, we have proposed an algorithm allowing the simultaneous computation of a saturating controller, a local domain of stability and a domain of admissible reference and disturbances. By using iterative procedures induced by relaxation schemes and different optimisation criteria, we can emphasise either the stability or the performance, in terms of sizes of either the stability domain or the reference and disturbances domains. This trade-off has been illustrated by a simple example.
We have considered time-varying disturbances and references without knowledge of their dynamics. Thus, in the future, an interesting way could be to consider time-varying disturbances and reference, with a knowledge or an estimation of their dynamics, and eventually with a tolerance concerning the tracking performance (admissible varying disturbances and reference, with a knowledge or delay or tracking error). Another way could be the synthesis of output feedback type controller instead of a state tolerances concerning the tracking performance (admissible varying disturbances and reference, with a knowledge or delay or tracking error). We have to compute a dynamic output controller for the initial system in eqn. 2 or for the augmented one in eqn. 5.

9 References
15. YUAN, F., and PETIT, C.: 'Output-reference tracking problem for discrete-time systems with input saturations and constant references'. Accepted for 8th IEEE Mediterranean Conf. on Control & Automation, July 2000, Patras, Greece
24. YUAN, F., and BERNSTEIN, D.S.: 'Output-reference tracking problem for discrete-time systems with input saturations and constant references'. Accepted for 8th IEEE Mediterranean Conf. on Control & Automation, July 2000, Patras, Greece