

Integral Quadratic Separation
applied to polytopic systems

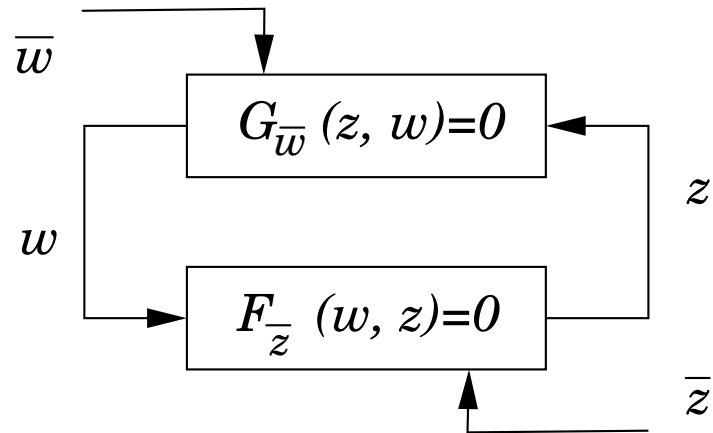
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Well-posedness & topological separation



Well-Posedness:

Bounded $(\bar{w}, \bar{z}) \Rightarrow$ unique bounded (w, z)

- [Safonov 80] $\exists \theta$ topological separator:

$$\mathcal{F}(\bar{z}) = \{(w, z) : F_{\bar{z}}(w, z) = 0\} \subset \{(w, z) : \theta(w, z) > -\phi_1(\|\bar{z}\|)\}$$

$$\mathcal{G}^I(\bar{w}) = \{(w, z) : G_{\bar{w}}(z, w) = 0\} \subset \{(w, z) : \theta(w, z) \leq \phi_2(\|\bar{w}\|)\}$$

Related results :

- Stability (θ Lyapunov certificate), Passivity (θ storage function), IQC ...
- Robust analysis of Linear uncertain systems [Iwasaki, Scherer]

■ Integral Quadratic Separation

- For the case of linear application with uncertain operator

$$\mathcal{E}z(t) = \mathcal{A}w(t) \quad , \quad w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ with \mathcal{E}_1 full column rank,

- Integral Quadratic Separator : $\exists \Theta$, matrix, solution of LMI

$$\begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp*} \Theta \begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp} > 0$$

and Integral Quadratic Constraint (IQC) $\forall \nabla \in \mathbb{W}$

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

- ▲ For some given \mathbb{W} , \exists LMI conditions for Θ solution to IQC.
- ▲ LMIs are conservative except in few special cases [Meinsma et al., 1997].

■ Integral Quadratic Separation Example: impulse-to-norm performance of

$$\begin{cases} E\dot{x} = Ax + Bv \\ g = Cx + Dv \end{cases}$$

● [ECC'09] Equivalent to well-posedness of

$$\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \varphi_0 g \\ g \end{pmatrix} = \begin{bmatrix} 0 & B \\ A & 0 \\ 0 & D \\ C & 0 \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

$$x(t) = \left[\mathcal{I} \begin{pmatrix} \varphi_0 x \\ \dot{x} \end{pmatrix} \right] (t) = x(0) + \int_0^t \dot{x}(\tau) d\tau$$

$$v = \nabla_{i2n} \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} : v = \alpha \varphi_0 \mathbf{1}_m, \quad |\alpha| \leq \frac{1}{\gamma} \left\| \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \right\|$$

■ Numerical tests for uncertain polytopic applications ?

$$\mathcal{E}(\xi)z(t) = \mathcal{A}(\xi)w(t) \quad , \quad w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E}(\xi) = \mathcal{E}_1(\xi)\mathcal{E}_2$ with $\mathcal{E}_1(\xi)$ full column rank and

$$\begin{bmatrix} \mathcal{E}_1(\xi) & -\mathcal{A}(\xi) \end{bmatrix} = \sum_{i=1}^{\bar{i}} \xi_i \begin{bmatrix} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{bmatrix}$$

is a modeling of parametric (constant) uncertainties constrained by

$$\xi \in \Xi = \left\{ \xi_i \geq 0 \quad , \quad \sum_{i=1}^{\bar{i}} \xi_i = 1 \right\} .$$

▲ Give LMI tests

▲ Control the numerical complexity / conservatism trade-off

■ General Slack Variables result

● If $\Theta^{[i]}$ satisfy the IQC conditions w.r.t. \mathbb{V} and $\exists H$ s.t. for all vertices

$$\Theta^{[i]} > H \begin{bmatrix} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{bmatrix} + \begin{bmatrix} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{bmatrix}^* H^*$$

well-posedness is satisfied for all ξ in the simplex Ξ .

- ▲ Large LMI conditions and large H matrix.
- ▲ H can be artificially increased by adding artificial rows/columns in \mathcal{E}_1 and \mathcal{A} .
- ▲ Unnecessary degrees of freedom ?
- ▲ On examples such tests encounter numerical problems.

■ ① Interpretation of slack variable via Finsler Lemma

● H is such that, for some τ , the following quantity is negligible

$$\begin{bmatrix} \mathcal{E}_1^*(\xi) \\ -\mathcal{A}^*(\xi) \end{bmatrix} \left(H^* - \tau \begin{bmatrix} \mathcal{E}_1(\xi) & -\mathcal{A}(\xi) \end{bmatrix} \right), \quad \forall \xi \in \Xi$$

▲ If $\begin{bmatrix} \mathcal{E}_1(\xi) & -\mathcal{A}(\xi) \end{bmatrix}$ have zero columns, one can choose the same for H .

▲ Reduces number of decision variables.

■ ② Factorization of uncertain rows

▲ If \mathcal{E}_1 is square without uncertainties.

● Algorithm for factorization as

$$\begin{bmatrix} \mathcal{E}_1^{-1} \mathcal{A}^{[i]} \\ \mathbf{1}_b \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathbf{0}_{b,\underline{a}} & \mathbf{1}_b \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} \mathcal{C}^{[i]} \\ \mathbf{1}_b \end{bmatrix}$$

where \mathcal{B}_2 gathers all the rows without uncertainties,

$\mathcal{C}^{[i]}$ gathers the uncertain rows (nb rows($\mathcal{C}^{[i]}$) = $\underline{a} \leq a =$ nb rows($\mathcal{A}^{[i]}$)),

and $\mathcal{B}_1^* \mathcal{B}_1 = \mathbf{1}_{\underline{a}}$, $\mathcal{B}_1^* \mathcal{B}_2 = 0$.

▲ Computation of the factorization: less than 1% of LMI test.

- If $\Theta^{[i]}$ satisfy the IQC conditions w.r.t. ∇ and $\exists H$ s.t. for all vertices

$$\mathcal{D}^* \Theta^{[i]} \mathcal{D} > \hat{H} \begin{bmatrix} \mathbf{1}_{\underline{a}} & -\mathcal{C}^{[i]} \end{bmatrix} + \begin{bmatrix} \mathbf{1}_{\underline{a}} \\ -\mathcal{C}^{[i]'} \end{bmatrix} \hat{H}^*$$

well-posedness is satisfied for all ξ in the simplex Ξ .

- ▲ No conservatism compared to general slack variable result.
- ▲ Size of LMIs reduced from $(a + b) \times (a + b)$ to $(\underline{a} + b) \times (\underline{a} + b)$
- ▲ Size of variable \hat{H} also reduced by factor $(a - \underline{a})$ compared to H
- ▲ Suppressed unnecessary degrees of freedom
- ▲ One can expect improved numerical properties

③ Case of unique separator

- If Θ satisfy the IQC conditions w.r.t. \mathbb{V} and $\exists H$ s.t. for all vertices

$$\Theta > H \begin{bmatrix} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{bmatrix} + \begin{bmatrix} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{bmatrix}^* H^*$$

well-posedness is satisfied for all ξ in the simplex Ξ .

- ▲ Is the slack variable H useful in that case ?

- ▲ If \mathcal{E}_1 is square and not affected by uncertainties: it is not.

- If Θ satisfy the IQC conditions w.r.t. \mathbb{V}

and if $\begin{bmatrix} 1_a & 0 \end{bmatrix} \Theta \begin{bmatrix} 1_a & 0 \end{bmatrix}^* \leq 0$ (which is the case for known sets \mathbb{V})
s.t. for all vertices

$$\begin{bmatrix} \mathcal{E}_1^{-1} \mathcal{A}^{[i]} \\ 1 \end{bmatrix}^* \Theta \begin{bmatrix} \mathcal{E}_1^{-1} \mathcal{A}^{[i]} \\ 1 \end{bmatrix} > 0$$

well-posedness is satisfied for all ξ in the simplex Ξ .

- ▲ This non conservative case \equiv usual “quadratic stability” framework.

Example of impulse-to-norm performance

- LMIs for the general slack variable result ($E = 1$ for simplicity)

$$\begin{aligned}
 & \begin{bmatrix} -P^{[i]} & 0 & 0 & 0 & | & 0 & 0 \\ 0 & 0 & 0 & 0 & | & -P^{[i]} & 0 \\ 0 & 0 & -\tau^{[i]} \mathbf{1} & 0 & | & 0 & 0 \\ 0 & 0 & 0 & -\tau^{[i]} \mathbf{1} & | & 0 & 0 \\ \hline 0 & -P^{[i]} & 0 & 0 & | & 0 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & Q^{[i]} \end{bmatrix} \\
 & > H \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & -B^{[i]} \\ 0 & 1 & 0 & 0 & | & -A^{[i]} & 0 \\ 0 & 0 & 1 & 0 & | & 0 & -D^{[i]} \\ 0 & 0 & 0 & 1 & | & -C^{[i]} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 & -B^{[i]} \\ 0 & 1 & 0 & 0 & | & -A^{[i]} & 0 \\ 0 & 0 & 1 & 0 & | & 0 & -D^{[i]} \\ 0 & 0 & 0 & 1 & | & -C^{[i]} & 0 \end{bmatrix}^* H^*
 \end{aligned}$$

$$P^{[i]} > 0, \quad \text{trace}(Q^{[i]}) \leq \tau^{[i]} \gamma^2$$

Example of impulse-to-norm performance

▲ If B and D are not affected by uncertainty

● Factorization of rows gives following non conservative LMIs

$$\begin{aligned}
 & \left[\begin{array}{cc|cc} 0 & 0 & -P^{[i]} & 0 \\ 0 & -\tau^{[i]} \mathbf{1} & 0 & 0 \\ \hline -P^{[i]} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B^* P^{[i]} B - \tau^{[i]} D^* D + Q^{[i]} \end{array} \right] \\
 & > \hat{H} \left[\begin{array}{cc|cc} 1 & 0 & -A^{[i]} & 0 \\ 0 & 1 & -C^{[i]} & 0 \end{array} \right] + \left[\begin{array}{cc|cc} 1 & 0 & -A^{[i]} & 0 \\ 0 & 1 & -C^{[i]} & 0 \end{array} \right]^* \hat{H}^* \\
 & P^{[i]} > 0, \quad \text{trace}(Q^{[i]}) \leq \tau^{[i]} \gamma^2
 \end{aligned}$$

- Noticing the zero columns, gives following non conservative LMIs

$$\left[\begin{array}{cc|c} 0 & 0 & -P^{[i]} \\ 0 & -\tau^{[i]} \mathbf{1} & 0 \\ \hline -P^{[i]} & 0 & 0 \end{array} \right] > \tilde{H} \left[\begin{array}{cc|c} 1 & 0 & -A^{[i]} \\ 0 & 1 & -C^{[i]} \end{array} \right] + \left[\begin{array}{cc|c} 1 & 0 & -A^{[i]} \\ 0 & 1 & -C^{[i]} \end{array} \right]^* \tilde{H}^*$$

$$Q^{[i]} > B^* P^{[i]} B + \tau^{[i]} D^* D$$

$$P^{[i]} > 0, \quad \text{trace}(Q^{[i]}) \leq \tau^{[i]} \gamma^2$$

- Removed slack variables when solving for a unique separator (“quadratic stability”)

$$\begin{aligned} A^{[i]*} P + P A^{[i]} + \tau C^{[i]*} C^{[i]} &< 0, & P &> 0, \\ Q &< B^{[i]*} P B^{[i]} + \tau D^{[i]*} D^{[i]}, & \text{trace}(Q) &< \tau \gamma^2. \end{aligned}$$

- ▲ Note that when $D = 0$,
impulse-to-norm is equivalent to H_2 norm performance.

■ Integral Quadratic Separation and Slack Variables

● SV's can be used in the general IQS framework

(includes issues such as performances, robustness, time-delay...
in descriptor form)

● Proposed methodology for coding efficiently the SV results

(removed unnecessary degrees of freedom and reduced size of LMIs)

■ Future work

▲ Results currently coded in a toolbox for robust analysis

Romuald (Matlab/YALMIP based tool - will be freely distributed)

(extension of RoMulOC www.laas.fr/OLOCEP/romuloc)

▲ Preliminary tests done on a satellite attitude control example

▲ More testing on real medium size applications to come

Thank you

