General polynomial parameter-dependent Lyapunov functions for polytopic uncertain systems

Dimitri PEAUCELLE† & Yoshio EBIHARA‡
& Denis ARZELIER† & Tomomichi HAGIWARA‡

† LAAS-CNRS - Toulouse, FRANCE
‡ Dpt. Electrical Engineering - Kyoto Univ., JAPAN
Robust stability in the Lyapunov context

Let an uncertain LTI system: \( \dot{x} = A(\zeta)x \)

where \( \zeta \) is a notation that gathers all constant unknown bounded parameters.

Stability is equivalent to the existence of a parameter-dependent Lyapunov function (PDLF): \( V_\zeta(x) = x^T P(\zeta)x \)

such that for all admissible uncertainties the LMIs hold

\[
P(\zeta) > 0, \quad A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0
\]  

(1)

Considered case

→ Affine polytopic systems \( A(\zeta) = \sum_{i=1}^{N} \zeta_i A_i : \zeta_i \geq 0, \sum_{i=1}^{N} \zeta_i = 1 \)

→ Polynomial PDLF (PPDLF) \( P(\zeta) = \sum \alpha_j(\zeta) P_j : \alpha_j(\zeta) = \zeta_1^{j_1} \zeta_2^{j_2} \cdots \zeta_N^{j_N} \)

→ (1) is then a PPD-LMI.
Outline

① Overview of existing techniques from the literature
   "Sum-Of-Squares" / "Positive coefficients" / "Small-gain theorem"
   → Large LMI problems
   → Few results on convergence to exact robustness analysis tests
   → Complex mathematical formulations

② Proposed approach : "dilated LMIs"
   → Same drawbacks
   → Interpretations in terms of "redundant system modeling"

③ Numerical example - robust $H_2$ guaranteed cost computation
Overview of techniques from the literature

Solving PPD-LMIs such as \( A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0 \) ?

"Sum-Of-Squares" approach [Chesi et al] - [Lasserre], [Parrilo]

Express the PPD-LMI as a quadratic form of nomomials

\[-(A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta)) = (\alpha(\zeta) \otimes 1)^TQ(P)(\alpha(\zeta) \otimes 1)\]

is positive if SOS which is LMI problem : \( Q(P) + U(\tilde{P}) > 0 \)

⚠ No proof of necessity
⚠ Numerical construction of \( Q(P) \) and \( U(\tilde{P}) \) is complex
⚠ Large LMIs with large number of variables
⚠ Restrict to homogeneous forms to reduce the dimensions
Overview of techniques from the literature

Solving PPD-LMIs such as $A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0$?

"Positive coefficients" approach [Scherer], [Peres et al] - [Pólya]

As all parameters are positive $\zeta_i \geq 0$,

$$\sum_{i=1}^{N} \zeta_i^d (A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta)) = \sum \alpha_j(\zeta)T_j(P)$$

is negative if all coefficient matrices are negative : $T_j(P) < 0$

- Proof of necessity for $d$ large enough ($P(\zeta)$ of fixed degree)
- Numerical construction of $T_j(P)$ is complex
- Large LMIs but no additional variables

"Positive coefficients" approach [Scherer], [Peres et al] - [Pólya]
Overview of techniques from the literature

Solving PPD-LMIs such as $A^T(\zeta) P(\zeta) + P(\zeta) A(\zeta) < 0$?

"Small-gain theorem" approach [Bliman] - [Scherer], [Iwasaki]

\[ A(\zeta) = A_0 + \sum_{k=1}^{m} z_k \tilde{A}_k : |z_k| \leq 1 \]

\[ A^T(\zeta) P(\zeta) + P(\zeta) A(\zeta) = (z_1^{\{r\}} \otimes 1)^T R_1(P, z_2, \ldots, m)(z_1^{\{r\}} \otimes 1) \]

it is negative for all $|z_1| \leq 1$ if there exists $Q_1(z_2, \ldots, m) > 0$ such that

\[ M_1^T R_1(P, z_2, \ldots, m) M_1 < N_1^T \begin{bmatrix} Q_1(z_2, \ldots, m) & 0 \\ 0 & -Q_1(z_2, \ldots, m) \end{bmatrix} N_1 \]

Choose $Q_1(z_2, \ldots, m)$ polynomial and go on recursively with $z_2, \ldots, z_m$.

\[ \text{Numerical construction of the LMIs is complex} \]
\[ \text{Large LMIs and very large number of additional variables} \]
\[ \text{Proof of convergence to exact robustness test as degree of polynomials grow} \]
\[ \text{Extends to LFT modelling} \]
Some characteristics

- Numerical construction of the LMIs is complex
- Large LMIs and large number of additional variables
- No proof of convergence to exact robustness test

Alternative method

- Interpretation in terms of “redundant system modeling”
Proposed "dilated LMIs" approach

Central tool: "Finsler lemma" [Geromel 1998], [Peaucelle 2000]

Stability of $\dot{x} = A(\zeta)x$ is proved if

$$\dot{V}(x) = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}^T \begin{bmatrix} 0 & P(\zeta) \\ P(\zeta) & 0 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} < 0 : \begin{bmatrix} A(\zeta) & -1 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = 0$$

A sufficient condition for that is the existence of $G$ such that

$$\begin{bmatrix} 0 & P(\zeta) \\ P(\zeta) & 0 \end{bmatrix} + G \begin{bmatrix} A(\zeta) & -1 \end{bmatrix} + \begin{bmatrix} A(\zeta) & -1 \end{bmatrix}^T G^T < 0$$

→ If $P(\zeta)$ is affine (order 1 PPDLF) it suffices to test on vertices:

$$\zeta_i = 1 , \; \zeta_j \neq i = 0$$
Proposed "dilated LMIs" approach

Redundant modeling - CDC’05

Consider the system with 2 equations

\[ \dot{x} = A(\zeta)x, \quad A(\zeta)\dot{x} = A^2(\zeta)x \]

Applying the same methodology leads to:

\[
\begin{bmatrix}
0 & \Pi(\zeta) \\
\Pi(\zeta) & 0
\end{bmatrix}
+ G
\begin{bmatrix}
A(\zeta) & 0 & -1 & 0 \\
A(\zeta) & -1 & 0 & 0 \\
0 & 0 & A(\zeta) & -1
\end{bmatrix}
+ [\ast]^T < 0
\]

and one can prove that it corresponds to taking for \( \dot{x} = A(\zeta)x \) a PDLF

\[ P(\zeta) = \begin{bmatrix} 1 \\ A(\zeta) \end{bmatrix}^T \Pi(\zeta) \begin{bmatrix} 1 \\ A(\zeta) \end{bmatrix} \]

Special case of order 3 PPDLF.
Proposed "dilated LMIs" approach

Redundant modeling - ROCOND’06

Assume a given affine $M(\zeta)$ and the redundant equations

$$
\dot{x} = A(\zeta)x, \quad M(\zeta)\dot{x} = M(\zeta)A(\zeta)x
$$

Applying the same methodology leads to:

$$
\begin{bmatrix}
0 & \Pi(\zeta) \\
\Pi(\zeta) & 0
\end{bmatrix} + G
\begin{bmatrix}
A(\zeta) & 0 & -1 & 0 \\
M(\zeta) & -1 & 0 & 0 \\
0 & 0 & M(\zeta) & -1
\end{bmatrix} + [\ast]^T < 0
$$

and one can prove that it corresponds to taking for $\dot{x} = A(\zeta)x$ a PDLF

$$
P(\zeta) = \begin{bmatrix}
1 \\
M(\zeta)
\end{bmatrix}^T \Pi(\zeta) \begin{bmatrix}
1 \\
M(\zeta)
\end{bmatrix}
$$

Appropriate choices of $M(\zeta)$ improve the results.
Proposed "dilated LMIs" approach

Redundant modeling - MTNS’06

For a chosen set of monomial \( \alpha_j(\zeta) = \zeta_1^{j_1} \zeta_2^{j_2} \ldots \zeta_N^{j_N}, \ j \in \{k_1, \ldots, k_p\} \)
take the redundant equations \( \alpha_j(\zeta) \dot{x} = \alpha_j(\zeta) A(\zeta) x \)

Applying the same methodology leads to LMIs
for the robust analysis of \( \dot{x} = A(\zeta) x \) with a PPDLF

\[
P(\zeta) = \begin{bmatrix}
1 \\
\alpha_{k_1}(\zeta)1 \\
\vdots \\
\alpha_{k_p}(\zeta)1
\end{bmatrix}^T \Pi(\zeta) \begin{bmatrix}
1 \\
\alpha_{k_1}(\zeta)1 \\
\vdots \\
\alpha_{k_p}(\zeta)1
\end{bmatrix}
\]

where \( \Pi(\zeta) \) is affine with respect to \( \zeta \).
Numerical example

Results are extended to $H_2$ guaranteed cost

→ Allows on the numerical example to test the conservatism:

The smaller the guaranteed $H_2$ norm is, the smaller is the conservatism.

→ Academic example of order 3 ($x \in \mathbb{R}^3$) with 3 vertices ($N = 3$).

→ For $P(\zeta) = P$ ("quadratic stability") : $\gamma_2 = 18.15$ (6 vars in LMIs)

→ For $P(\zeta)$ of order 1 : $\gamma_2 = 8.31$ (52 vars in LMIs)

→ For $j \in \{(100)\}$ : $\gamma_2 = 4.83$ (217 vars in LMIs)

→ For $j \in \{(100), (200), (010), \}$ : $\gamma_2 = 3.73$ (499 vars in LMIs)

→ For $j \in \{(j_1j_2j_3) : \sum j_i \leq 2\}$ : $\gamma_2 = 2.67$ (2101 vars in LMIs)

→ Optimal value (expected by gridding) : $\gamma_2 = 1.32$

★ There is still work to be done:

Reduce computation burden, Reduce conservatism...

Compare numerically & theoretically the existing results.