General polynomial parameter-dependent Lyapunov functions for polytopic uncertain systems

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Robust stability in the Lyapunov context

Let an uncertain LTI system : $\dot{x} = A(\zeta)x$

where ζ is a notation that gathers all constant unknown bounded parameters.

Stability is equivalent to the existence of a

parameter-dependent Lyapunov function (PDLF) : $V_\zeta(x) = x^T P(\zeta) x$

such that for all admissible uncertainties the LMIs hold

$$P(\zeta) > 0 \ , \ A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0 \qquad (1)$$

Considered case

→ Affine polytopic systems $A(\zeta) = \sum_{i=1}^{N} \zeta_i A_i \ : \ \zeta_i \geq 0 \ , \ \sum_{i=1}^{N} \zeta_i = 1$

→ Polynomial PDLF (PPDLF) $P(\zeta) = \sum \alpha_j(\zeta) P_j \ : \ \alpha_j(\zeta) = \zeta_1^{j_1} \zeta_2^{j_2} \ldots \zeta_N^{j_N}$

→ (1) is then a PPD-LMI.

Outline

① Overview of existing techniques from the literature

”Sum-Of-Squares” / ”Positive coefficients” / ”Small-gain theorem”

→ Large LMI problems

→ Few results on convergence to exact robustness analysis tests

→ Complex mathematical formulations

② Proposed approach : ”dilated LMIs”

→ Same drawbacks

→ Interpretations in terms of ”redundant system modeling”

③ Numerical example - robust H_2 guaranteed cost computation

① Overview of techniques from the literature

Solving PPD-LMIs such as $A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0$?

"Sum-Of-Squares" approach [Chesi et al] - [Lasserre], [Parrilo]

Express the PPD-LMI as a quadratic form of nomomials

$$-(A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta)) = (\alpha(\zeta) \otimes 1)^T Q(P)(\alpha(\zeta) \otimes 1)$$

is positive if SOS which is LMI problem : $Q(P) + U(\tilde{P}) > 0$

↘ No proof of necessity

↘ Numerical construction of $Q(P)$ and $U(\tilde{P})$ is complex

↘ Large LMIs with large number of variables

↗ Restrict to homogeneous forms to reduce the dimensions

Solving PPD-LMIs such as $A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0$?

"Positive coefficients" approach [Scherer], [Peres et al] - [Pólya]

As all parameters are positive $\zeta_i \geq 0$,

$$(\sum_{i=1}^{N} \zeta_i)^d (A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta)) = \sum \alpha_j(\zeta) T_j(P)$$

is negative if all coefficient matrices are negative : $T_j(P) < 0$

↗ Proof of necessity for d large enough ($P(\zeta)$ of fixed degree)

↘ Numerical construction of $T_j(P)$ is complex

↗ Large LMIs but no additional variables

Solving PPD-LMIs such as $A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) < 0$?

"Small-gain theorem" approach [Bliman] - [Scherer], [Iwasaki]

↘ Assuming the sub-case $A(\zeta) = A_0 + \sum_{k=1}^{m} z_k \tilde{A}_k \; : |z_k| \leq 1$

$$A^T(\zeta)P(\zeta) + P(\zeta)A(\zeta) = (z_1^{\{r\}} \otimes 1)^T R_1(P, z_{2,\dots,m})(z_1^{\{r\}} \otimes 1)$$

it is negative for all $|z_1| \leq 1$ if there exists $Q_1(z_{2,\dots,m}) > 0$ such that

$$M_1^T R_1(P, z_{2,\dots,m}) M_1 < N_1^T \begin{bmatrix} Q_1(z_{2,\dots,m}) & 0 \\ 0 & -Q_1(z_{2,\dots,m}) \end{bmatrix} N_1$$

Choose $Q_1(z_{2,\dots,m})$ polynomial and go on recursively with $z_2, \dots, z_m$.

↘ Numerical construction of the LMIs is complex

↘ Large LMIs and very large number of additional variables

↗ Proof of convergence to exact robustness test as degree of polynomials grow

↗ Extends to LFT modelling

② Proposed "dilated LMIs" approach

Some characteristics

- ↘ Numerical construction of the LMIs is complex
- ↘ Large LMIs and large number of additional variables
- ↘ No proof of convergence to exact robustness test

- ↗ Alternative method
- ↗ Interpretation in terms of "redundant system modeling"

Central tool : "Finsler lemma" [Geromel 1998], [Peaucelle 2000]

Stability of $\dot{x} = A(\zeta)x$ is proved if

$$\dot{V}(x) = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}^T \begin{bmatrix} 0 & P(\zeta) \\ P(\zeta) & 0 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} < 0 \;:\; \begin{bmatrix} A(\zeta) & -1 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = 0$$

A sufficient condition for that is the existence of G such that

$$\begin{bmatrix} 0 & P(\zeta) \\ P(\zeta) & 0 \end{bmatrix} + G \begin{bmatrix} A(\zeta) & -1 \end{bmatrix} + \begin{bmatrix} A(\zeta) & -1 \end{bmatrix}^T G^T < 0$$

→ If $P(\zeta)$ is affine (order 1 PPDLF) it suffices to test on vertices :

$$\zeta_i = 1 \;,\; \zeta_{j\neq i} = 0$$

Redundant modeling - CDC'05

Consider the system with 2 equations

$$\dot{x} = A(\zeta)x \ , \ \ A(\zeta)\dot{x} = A^2(\zeta)x$$

Applying the same methodology leads to :

$$\begin{bmatrix} 0 & \Pi(\zeta) \\ \Pi(\zeta) & 0 \end{bmatrix} + G \left[\begin{array}{cc|cc} A(\zeta) & 0 & -1 & 0 \\ A(\zeta) & -1 & 0 & 0 \\ 0 & 0 & A(\zeta) & -1 \end{array}\right] + [*]^T < 0$$

and one can prove that it corresponds to taking for $\dot{x} = A(\zeta)x$ a PDLF

$$P(\zeta) = \begin{bmatrix} 1 \\ A(\zeta) \end{bmatrix}^T \Pi(\zeta) \begin{bmatrix} 1 \\ A(\zeta) \end{bmatrix}$$

Special case of order 3 PPDLF.

Redundant modeling - ROCOND'06

Assume a given affine $M(\zeta)$ and the redundant equations

$$\dot{x} = A(\zeta)x \ , \ \ M(\zeta)\dot{x} = M(\zeta)A(\zeta)x$$

Applying the same methodology leads to :

$$\begin{bmatrix} 0 & \Pi(\zeta) \\ \Pi(\zeta) & 0 \end{bmatrix} + G \left[\begin{array}{cc|cc} A(\zeta) & 0 & -1 & 0 \\ M(\zeta) & -1 & 0 & 0 \\ 0 & 0 & M(\zeta) & -1 \end{array}\right] + [*]^T < 0$$

and one can prove that it corresponds to taking for $\dot{x} = A(\zeta)x$ a PDLF

$$P(\zeta) = \begin{bmatrix} 1 \\ M(\zeta) \end{bmatrix}^T \Pi(\zeta) \begin{bmatrix} 1 \\ M(\zeta) \end{bmatrix}$$

Appropriate choices of $M(\zeta)$ improve the results.

Redundant modeling - MTNS'06

For a chosen set of monomial $\alpha_j(\zeta) = \zeta_1^{j_1}\zeta_2^{j_2}\dots\zeta_N^{j_N}$, $j \in \{k_1, \dots, k_p\}$

take the redundant equations $\alpha_j(\zeta)\dot{x} = \alpha_j(\zeta)A(\zeta)x$

Applying the same methodology leads to LMIs

for the robust analysis of $\dot{x} = A(\zeta)x$ with a PPDLF

$$P(\zeta) = \begin{bmatrix} 1 \\ \alpha_{k_1}(\zeta)1 \\ \vdots \\ \alpha_{k_p}(\zeta)1 \end{bmatrix}^T \Pi(\zeta) \begin{bmatrix} 1 \\ \alpha_{k_1}(\zeta)1 \\ \vdots \\ \alpha_{k_p}(\zeta)1 \end{bmatrix}$$

where $\Pi(\zeta)$ is affine with respect to ζ.

Results are extended to H_2 guaranteed cost

→ Allows on the numerical example to test the conservatism :

The smaller the guaranteed H_2 norm is, the smaller is the conservatism.

→ Academic example of order 3 ($x \in \mathrm{R}^3$) with 3 vertices ($N = 3$).

→ For $P(\zeta) = P$ ("quadratic stability") : $\gamma_2 = 18.15$ (6 vars in LMIs)

→ For $P(\zeta)$ of order 1 : $\gamma_2 = 8.31$ (52 vars in LMIs)

→ For $j \in \{(100)\}$: $\gamma_2 = 4.83$ (217 vars in LMIs)

→ For $j \in \{(100), (200), (010), \}$: $\gamma_2 = 3.73$ (499 vars in LMIs)

→ For $j \in \{(j_1 j_2 j_3)\ : \sum j_i \leq 2\}$: $\gamma_2 = 2.67$ (2101 vars in LMIs)

→ Optimal value (expected by gridding) : $\gamma_2 = 1.32$

✪ There is still work to be done :

Reduce computation burden, Reduce conservatism...

Compare numerically & theoretically the existing results.