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Ellipsoidal Output-Feedback Sets for Robust Multi-Performance Synthesis

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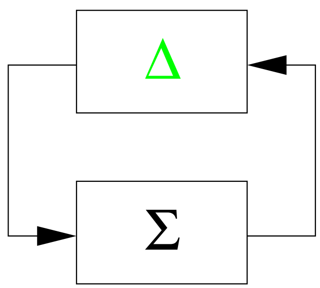
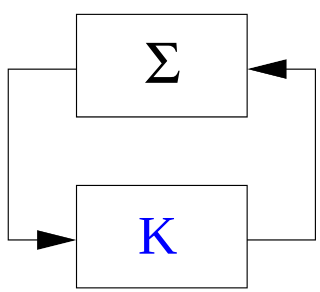
Motivation



$$\exists \Theta : \begin{cases} \begin{bmatrix} \mathbb{1} & \Delta' \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ \Delta \end{bmatrix} \leq \mathbb{0} & \forall \Delta \in \mathbb{\Delta} \\ \begin{bmatrix} \Sigma^*(j\omega) & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma(j\omega) \\ \mathbb{1} \end{bmatrix} > \mathbb{0} & \forall \omega \in \mathbb{R} \end{cases}$$

Topological separation for robust analysis ⇑ for control design ⇓



$$\exists \Theta : \begin{cases} \begin{bmatrix} \Sigma^*(j\omega) & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma(j\omega) \\ \mathbb{1} \end{bmatrix} > \mathbb{0} & \forall \omega \in \mathbb{R} \\ \begin{bmatrix} \mathbb{1} & K' \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} \leq \mathbb{0} & \text{non-empty set} \end{cases}$$

Overview

Overview

1 – Design of resilient SOF gains

- ➔ Matrix ellipsoids
- ➔ SOF design
- ➔ Resilience

1 - Design of resilient SOF gains

Definition of $\{X, Y, Z\}$-ellipsoids

$$\left\{ K \in \mathbb{R}^{m\times p} \;:\; \begin{bmatrix} \mathbb{1} & K' \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} \leq \mathbb{0} \;,\; Z > \mathbb{0} \right\}$$

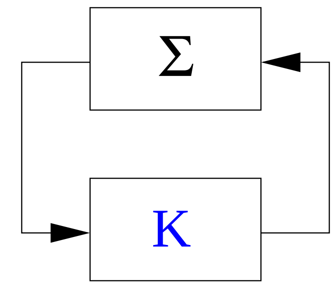
$$= \{\; K \in \mathbb{R}^{m\times p} \;:\; (K - K_o)'Z(K - K_o) \leq R \;,\; Z > \mathbb{0} \;\}$$

- **"Matrix ellipsoids"**
  - Centre $K_o = -Z^{-1}Y' \in \mathbb{R}^{m\times p}$
  - Radius $R = K_o{}'ZK_o - X \in \mathbb{R}^{p\times p}$
  - Geometry $Z \in \mathbb{R}^{m\times m}$ s.t. $Z > \mathbb{0}$, $\|Z\| = 1$
- **Closed convex sets**
- **Non empty iff** $\boxed{X \leq YZ^{-1}Y'}$

1 - Design of resilient SOF gains

SOF design

$\Sigma \star K$:



The system Σ : $\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \right.$ is stabilisable by static output feedback

iff

there exists a Lyapunov matrix $P > \mathbb{0}$ and an non-empty $\{X, Y, Z\}$-ellipsoid such that:

$$\begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix}' \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix} < \begin{bmatrix} C & D \\ \mathbb{0} & \mathbb{1} \end{bmatrix}' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} C & D \\ \mathbb{0} & \mathbb{1} \end{bmatrix}$$

$$X \leq YZ^{-1}Y'$$

1 - Design of resilient SOF gains

Resilience/Fragility

- Lyapunov function $V(x) = x'Px$ proves closed-loop stability for all control s.t. :

$$u(t) = K(t)y(t) \qquad \begin{bmatrix} \mathbb{1} & K'(t) \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ K(t) \end{bmatrix} \leq \mathbb{0}$$

- The centre K_o of the $\{X, Y, Z\}$-ellipsoid is resilient to dissipative uncertainty:

$$\Sigma\star(K_o + \Delta_K) \quad \text{stable if} \quad \Delta_K' Z \Delta_K \leq R$$

- Resilience constraints do not make more complex the SOF design problem
- Multiplicative uncertainty and norm-bounded uncertainty considered in IFAC'02.

2 – Multi-Performance Synthesis

- Robust H_∞ performance
- Robust H_2 performance
- Robust mixed H_2/H_∞ performance
- Multi-Performance

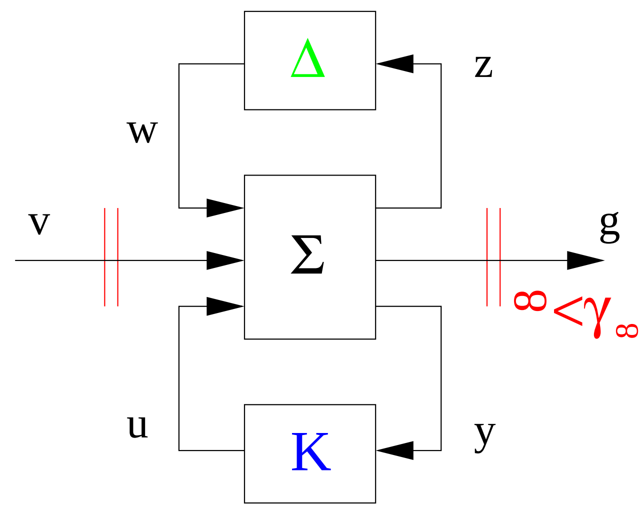
Multi-Performance Synthesis

Design for robust H_∞ performance

$\Delta \in \{X_{\text{lft}}, Y_{\text{lft}}, Z_{\text{lft}}\}$-ellipsoid

e.g. norm-bounded or positive real uncertainties

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + B_w w(t) + B_v v(t) + Bu(t) \\ z(t) &=& C_z x(t) + D_{zw} w(t) + D_{zv} v(t) + D_{zu} u(t) \\ g(t) &=& C_g x(t) + D_{gw} w(t) + D_{gv} v(t) + D_{gu} u(t) \\ y(t) &=& Cx(t) + D_{yw} w(t) + D_{yv} v(t) + Du(t) \end{array} \right.$$



$\Sigma \star \Delta = \Sigma(\Delta)$: open-loop uncertain system

Closed-loop robust H_∞ performance specification:

$$\forall \Delta \quad \|\Sigma(\Delta) \star K\|_\infty < \gamma_\infty$$

Multi-Performance Synthesis

Design for robust H_∞ performance

$$M_1' \begin{bmatrix} \mathbb{0} & P_\infty \\ P_\infty & \mathbb{0} \end{bmatrix} M_1 < \tau_{\text{lft}} M_2' \begin{bmatrix} X_{\text{lft}} & Y_{\text{lft}} \\ Y_{\text{lft}}' & Z_{\text{lft}} \end{bmatrix} M_2 + \tau_\infty M_3' \begin{bmatrix} -\mathbb{1} & \mathbb{0} \\ \mathbb{0} & \gamma_\infty^2 \mathbb{1} \end{bmatrix} M_3 + M_4' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} M_4$$

$$\tau_{\text{lft}} > 0 \qquad \tau_\infty > 0 \qquad Z > \mathbb{0} \qquad P_\infty > \mathbb{0} \qquad X \leq Y Z^{-1} Y'$$

$\Rightarrow$ $\{X, Y, Z\}$-ellipsoid is a set of stabilising gains such that :

$$||\Sigma(\Delta) \star K||_\infty < \gamma_\infty \quad \text{for all} \quad \Delta \in \{X_{\text{lft}}, Y_{\text{lft}}, Z_{\text{lft}}\}\text{-ellipsoid.}$$

$$M_1 = \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ A & B_w & B_v & B \end{bmatrix} \qquad M_2 = \begin{bmatrix} C_z & D_{zw} & D_{zv} & D_{zu} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} & \mathbb{0} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} C_g & D_{gw} & D_{gv} & D_{gu} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix} \qquad M_4 = \begin{bmatrix} C & D_{yw} & D_{yv} & D \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix}$$

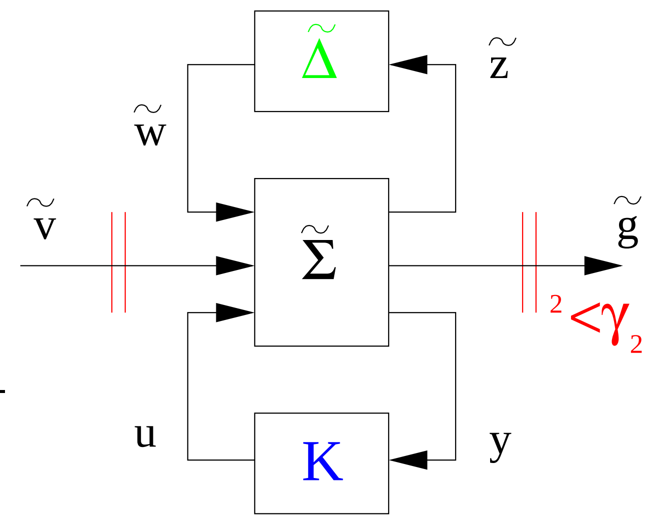
Multi-Performance Synthesis

Design for robust H_2 performance

$\tilde{\Delta} \in \{\tilde{X}_{\text{lft}}, \tilde{Y}_{\text{lft}}, \tilde{Z}_{\text{lft}}\}$-ellipsoid

e.g. norm-bounded or positive real uncertainties

$$\tilde{\Sigma} : \begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}_w\tilde{w}(t) + \tilde{B}_v\tilde{v}(t) + \tilde{B}u(t) \\ \tilde{z}(t) = \tilde{C}_z\tilde{x}(t) + \tilde{D}_{zw}\tilde{w}(t) + 0\,\tilde{v}(t) + \tilde{D}_{zu}u(t) \\ \tilde{g}(t) = \tilde{C}_g\tilde{x}(t) + \tilde{D}_{gw}\tilde{w}(t) + 0\,\tilde{v}(t) + \tilde{D}_{gu}u(t) \\ y(t) = \tilde{C}\tilde{x}(t) + \tilde{D}_{yw}\tilde{w}(t) + 0\,\tilde{v}(t) + \tilde{D}u(t) \end{cases}$$



$\tilde{\Sigma} \star \tilde{\Delta} = \tilde{\Sigma}(\tilde{\Delta})$: open-loop uncertain system

Closed-loop robust H_2 performance specification:

$$\forall \tilde{\Delta} \quad \|\tilde{\Sigma}(\tilde{\Delta}) \star K\|_2 < \gamma_2$$

Multi-Performance Synthesis

Design for robust H_2 performance

$$N_1' \begin{bmatrix} \mathbb{0} & P_2 \\ P_2 & \mathbb{0} \end{bmatrix} N_1 < \tilde{\tau}_{\text{lft}} N_2' \begin{bmatrix} \tilde{X}_{\text{lft}} & \tilde{Y}_{\text{lft}} \\ \tilde{Y}_{\text{lft}}' & \tilde{Z}_{\text{lft}} \end{bmatrix} N_2 - \tau_2 N_3' N_3 + N_4' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} N_4$$

$$\text{trace}(\tilde{B}_v' P_2 \tilde{B}_v) \leq \tau_2 \gamma_2^2 \qquad \tilde{\tau}_{\text{lft}} > 0 \qquad \tau_2 > 0 \qquad Z > \mathbb{0} \qquad P_2 > \mathbb{0} \qquad X \leq Y Z^{-1} Y'$$

$\Rightarrow$ $\{X, Y, Z\}$-ellipsoid is a set of stabilising gains such that :

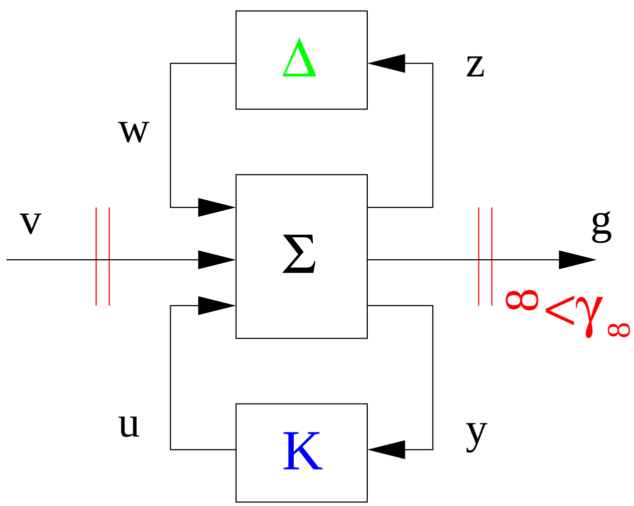
$$||\tilde{\Sigma}(\tilde{\Delta}) \star K||_2 < \gamma_2 \quad \text{for all} \quad \tilde{\Delta} \in \{\tilde{X}_{\text{lft}}, \tilde{Y}_{\text{lft}}, \tilde{Z}_{\text{lft}}\}\text{-ellipsoid.}$$

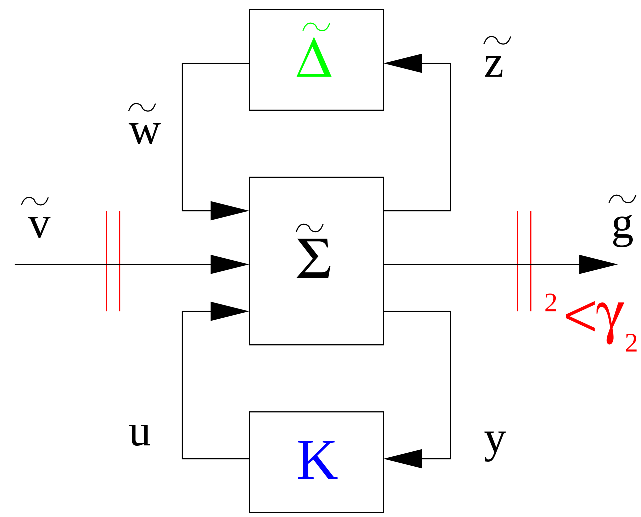
$$N_1 = \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \tilde{A} & \tilde{B}_w & \tilde{B} \end{bmatrix} \quad N_2 = \begin{bmatrix} \tilde{C}_z & \tilde{D}_{zw} & \tilde{D}_{zu} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix} \quad N_4 = \begin{bmatrix} \tilde{C} & \tilde{D}_{yw} & \tilde{D} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix}$$

$$N_3 = \begin{bmatrix} \tilde{C}_g & \tilde{D}_{gw} & \tilde{D}_{gu} \end{bmatrix}$$

Multi-Performance Synthesis

Design for robust mixed H_2/H_∞ performance





Unique control gain for performance levels defined for distinct models:

- ✪ Not the same disturbance input and controlled output signals
- ✪ Loop-shaping filters $\Rightarrow$ Models with different order
- ✪ Performances defined for distinct configurations of the system
- ✪ Different prescribed levels of uncertainty

Multi-Performance Synthesis

Design for robust mixed H_2/H_∞ performance

$$M_1' \begin{bmatrix} \mathbb{0} & P_\infty \\ P_\infty & \mathbb{0} \end{bmatrix} M_1 < \tau_{\mathrm{lft}} M_2' \begin{bmatrix} X_{\mathrm{lft}} & Y_{\mathrm{lft}} \\ Y_{\mathrm{lft}}' & Z_{\mathrm{lft}} \end{bmatrix} M_2 + \tau_\infty M_3' \begin{bmatrix} -\mathbb{1} & \mathbb{0} \\ \mathbb{0} & \gamma_\infty^2 \mathbb{1} \end{bmatrix} M_3 + M_4' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} M_4$$

$$\tau_{\mathrm{lft}} > 0 \qquad \tau_\infty > 0 \qquad Z > \mathbb{0} \qquad P_\infty > \mathbb{0}$$

$$N_1' \begin{bmatrix} \mathbb{0} & P_2 \\ P_2 & \mathbb{0} \end{bmatrix} N_1 < \tilde{\tau}_{\mathrm{lft}} N_2' \begin{bmatrix} \tilde{X}_{\mathrm{lft}} & \tilde{Y}_{\mathrm{lft}} \\ \tilde{Y}_{\mathrm{lft}}' & \tilde{Z}_{\mathrm{lft}} \end{bmatrix} N_2 - \tau_2 N_3' N_3 + N_4' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} N_4$$

$$\mathrm{trace}(\tilde{B}_v' P_2 \tilde{B}_v) \leq \tau_2 \gamma_2^2 \qquad \tilde{\tau}_{\mathrm{lft}} > 0 \qquad \tau_2 > 0 \qquad P_2 > \mathbb{0}$$

$$X \leq Y Z^{-1} Y'$$

- No additional conservatism: distinct Lyapunov functions for each performance
- No constraint on the systems models
- Entirely writes as LMIs + $X \leq Y Z^{-1} Y'$

Multi-Performance Synthesis

Design for multi-performance

Unique control gain for finite number of distinct model/performance specifications:

- ✪ H_∞ and/or robust H_∞
- ✪ H_2 and/or robust H_2
- ✪ Pole location and/or robust pole location (IFAC'02)
- ✪ Simultaneous stabilisation
- ✪ Overall resilience of the control

$\Rightarrow$ LMI constraints + $X \leq YZ^{-1}Y'$

Overview

3 – Algorithm and example

- Cone complementarity formulation
- Gradient type algorithm
- VTOL example

Algorithm and example

Cone complementarity formulation

$$L(Q,X,Y,Z) < 0 \qquad X \le YZ^{-1}Y'$$

$$\Updownarrow$$

$$0 = \min \ \mathrm{trace}(TS)$$

$$L(Q,X,Y,Z) < 0 \qquad S = \begin{bmatrix} \hat{X} & Y \\ Y' & Z \end{bmatrix} \ge 0 \qquad T = \begin{bmatrix} T_1 & T_2 \\ T_2' & T_3 \end{bmatrix} \ge 0$$

$$X \le \hat{X} \qquad T_1 \ge 1$$

Gradient type algorithm

$$(T_{k+1}, S_{k+1}) = \arg\min \ \mathrm{trace}(T_k S + T S_k)$$

Stopping criteria

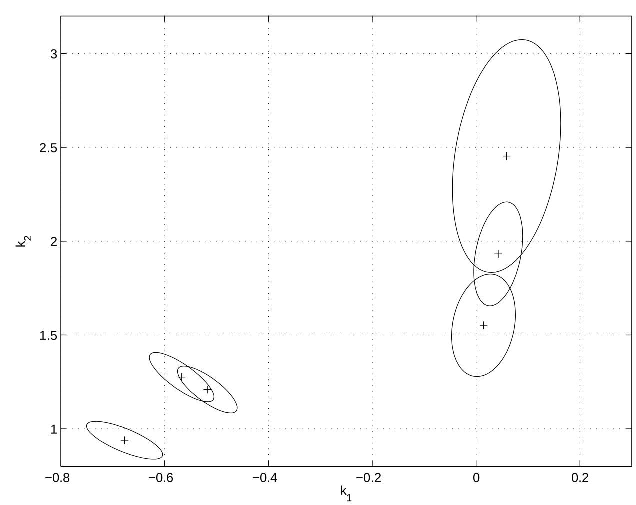
- At the optimum $TS = 0$ and $X \le \hat{X} = YZ^{-1}Y'$
- Stop as soon as $X \le YZ^{-1}Y'$

VTOL example Robust mixed H_2/H_∞ specifications:

$$||\Sigma(\Delta)\star K||_\infty < \gamma_\infty \qquad ||\tilde{\Sigma}(\Delta)\star K||_2 < \gamma_2 \qquad \forall \Delta \in \mathbb{\Delta}_\alpha$$

where $\mathbb{\Delta}_\alpha$ is norm-bounded uncertainty set - uncertainty level parametrised by α.

test	γ_∞	γ_2	α	iter	CPU	Tr(TS)	K_o'
(a)	0.5	0.3	3	4	8s	500	[0.014 1.55]
(b)	0.5	0.3	5	4	9s	700	[0.059 2.45]
(c)	0.5	0.3	7	4	9s	400	[0.043 1.93]
(d)	3	3	10	65	167s	0.006	[-0.68 0.94]
(e)	10	10	13	51	122s	0.02	[-0.52 1.21]
(f)	10	10	14	65	166s	0.01	[-0.57 1.27]
(g)	3	3	14	36	85s	4	fails



$$||\Sigma(\Delta)\star K_{o(f)}||_\infty < 0.61 \qquad ||\tilde{\Sigma}(\Delta)\star K_{o(f)}||_2 < 0.17 \qquad \forall \Delta \in \mathbb{\Delta}_{\alpha=14}$$

Conclusions - Prospectives

- Easy to manipulate formulation of general problems
- Needs to improve the algorithm
- Dynamic output feedback
- Robustness with respect to structured uncertainty $\rightarrow$ PDLF