

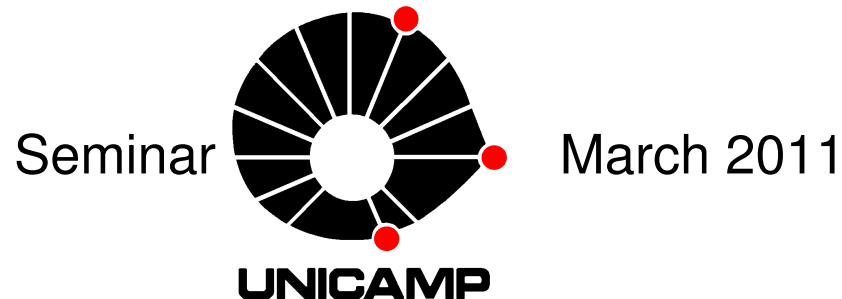
Integral Quadratic Separation Framework

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① Topological separation & related theory

- Well-posedness definition and main result
- Relations with Lyapunov theory
- The case of linear uncertain systems : quadratic separation
- The Lur'e problem
- Relations with IQC framework

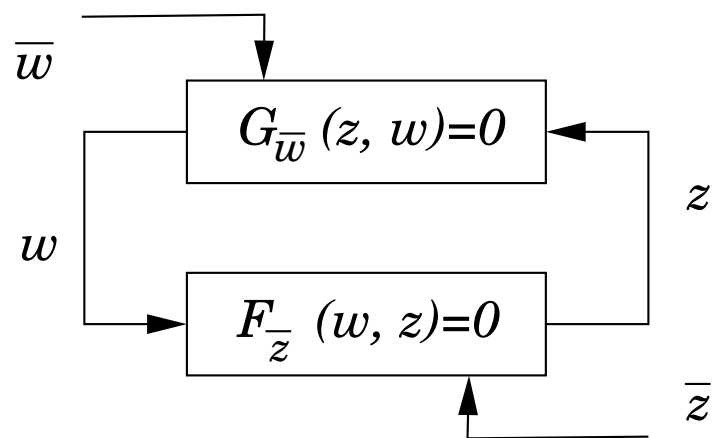
② Integral Quadratic Separation (IQS) for the descriptor case

③ Performances in the IQS framework

④ System augmentation : a way towards conservatism reduction

⑤ The Romuald toolbox

Well-posedness



Well-Posedness:

Bounded $(\bar{w}, \bar{z}) \Rightarrow$ unique bounded (w, z)

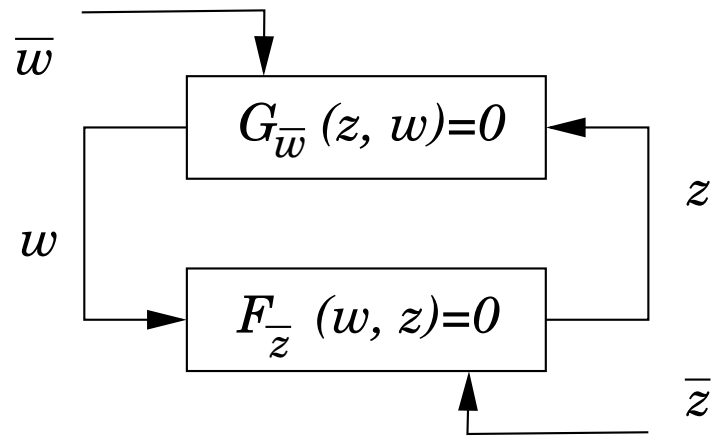
- In case $\underbrace{z = Aw + \bar{z}}_F$ and $\underbrace{w = \Delta z + \bar{w}}_G$ are linear applications

Well-posedness : $(1 - A\Delta)$ non-singular

- ▲ What if $\Delta = \triangle \in \triangle$ is uncertain ?
- ▲ If $A = T(j\omega)$ is an LTI system ?
- ▲ If G is non-linear ?

...

Well-posedness & topological separation



Well-Posedness:

Bounded (\bar{w}, \bar{z})

$$\Rightarrow \exists!(w, z) , \exists \gamma : \begin{vmatrix} w \\ z \end{vmatrix} \leq \gamma \begin{vmatrix} \bar{w} \\ \bar{z} \end{vmatrix}$$

● [Safonov 80] $\exists \theta$ topological separator:

$$\mathcal{G}^I(\bar{w}) = \{(w, z) : G_{\bar{w}}(z, w) = 0\} \subset \{(w, z) : \theta(w, z) \leq \phi_2(\|\bar{w}\|)\}$$

$$\mathcal{F}(\bar{z}) = \{(w, z) : F_{\bar{z}}(w, z) = 0\} \subset \{(w, z) : \theta(w, z) > -\phi_1(\|\bar{z}\|)\}$$

● ϕ_1 and ϕ_2 are positive functions. When $\bar{w} = 0, \bar{z} = 0$ separation reads as

$$\mathcal{G}^I(0) = \{(w, z) : G_0(z, w) = 0\} \subset \{(w, z) : \theta(w, z) \leq 0\}$$

$$\mathcal{F}(0) = \{(w, z) : F_0(w, z) = 0\} \subset \{(w, z) : \theta(w, z) > 0\}$$

- For dynamic systems $\dot{x} = f(x)$, topological separation \equiv Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^F, \quad \overbrace{w(t) = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^G$$

- ▲ \bar{w} : contains information on initial conditions ($x(0) = 0$ by convention)
- Well-posedness \Rightarrow for zero initial conditions and zero perturbations :
 $w = z = 0$ (equilibrium point).
- Well-posedness (global stability)
 \Rightarrow whatever bounded perturbations the state remains close to equilibrium

1 Topological separation & related theory

■ For dynamic systems $\dot{x} = f(x)$, topological separation \equiv Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^F, \quad \overbrace{w(t) = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^G$$

● Assume a Lyapunov function $V(0) = 0$, $V(x) > 0$, $\dot{V}(x) < 0$

▲ Topological separation w.r.t. $\mathcal{G}^I(0)$ is obtained with

$$\theta(w = x, z = \dot{x}) = \int_0^\infty -\frac{\partial V}{\partial x}(x(\tau))\dot{x}(\tau)d\tau = \lim_{t \rightarrow \infty} -V(x(t)) < 0$$

▲ Topological separation w.r.t. $\mathcal{F}(0)$ does hold as well

$$\theta(w, z = f(w)) = \int_0^\infty -\dot{V}(w(\tau))d\tau > 0$$

- For linear systems : quadratic Lyapunov function, *i.e. quadratic separator*

$$\overbrace{z(t) = Aw(t) + \bar{z}(t)}^F, \quad \overbrace{w(t) = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^G$$

- A possible separator based on quadratic Lyapunov function $V(x) = x^T P x$

$$\theta(w, z) = \int_0^\infty \begin{pmatrix} z^T(\tau) & w^T(\tau) \end{pmatrix} \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix} d\tau$$

- ▲ Quadratic separation w.r.t. $\mathcal{G}^I(0)$:

$$\lim_{t \rightarrow \infty} -x^T(t) P x(t) \leq 0, \quad \text{i.e. } P > 0$$

- ▲ Quadratic separation w.r.t. $\mathcal{F}(0)$ guaranteed if

$$\forall t > 0, \quad -2w^T(t) P A w(t) > 0, \quad \text{i.e. } A^T P + P A < 0$$

■ Topological separation : alternative to Lyapunov theory

▲ Needs to manipulate systems in a new form

● Suited for systems described as feedback connected blocs

Any linear system with rational dependence w.r.t. parameters writes as such

$$\dot{x} = (A + B_{\Delta} \Delta (1 - D_{\Delta} \Delta)^{-1} C_{\Delta}) x \quad \xleftrightarrow{\text{LFT}} \quad \begin{cases} \dot{x} = Ax + B_{\Delta} w_{\Delta} \\ z_{\Delta} = C_{\Delta} x + D_{\Delta} w_{\Delta} \\ w_{\Delta} = \Delta z_{\Delta} \end{cases}$$

▲ Finding a topological separator is *a priori*

as complicated as finding a Lyapunov function

● Allows to deal with several features simultaneously in a unified way

■ Quadratic separation [Iwasaki & Hara 1998]

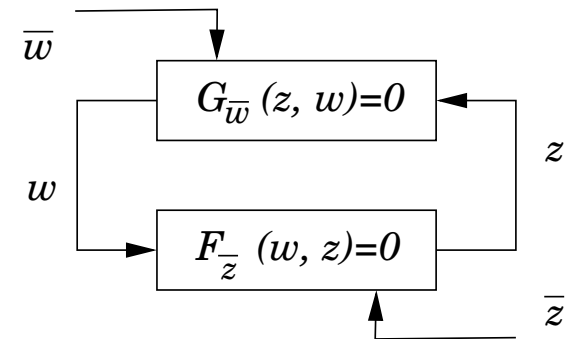
- If $F(w) = Aw$ is a linear transformation and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbb{\Delta}$ convex set it is necessary and sufficient to look for a quadratic separator

$$\theta(z, w) = \int_0^\infty \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

- If $F(w) = A(\omega)w$ is a linear parameter dependent transformation and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbb{\Delta}$ convex set necessary and sufficient to look for a parameter-dependent quadratic separator

$$\theta(z, w) = \int_0^\infty \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta(\omega) \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

■ A well-known example : the Lur'e problem

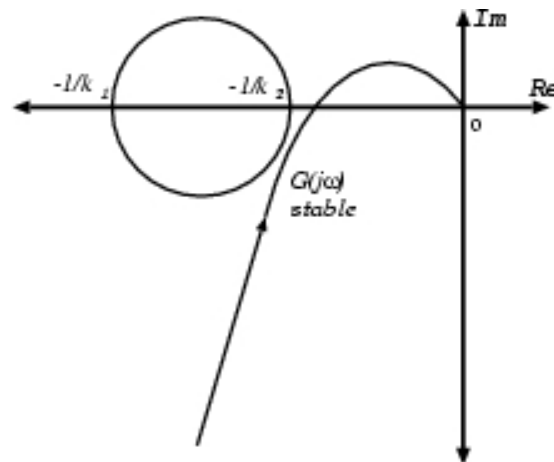


▲ $F = T(j\omega)$ is a transfer function

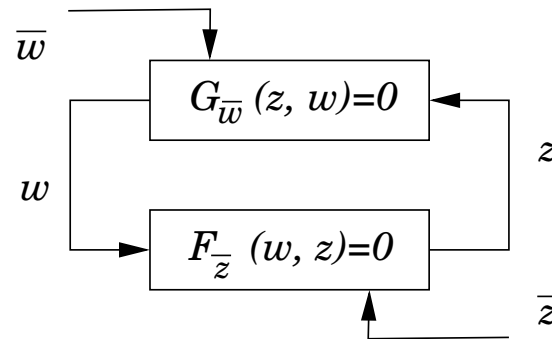
▲ $G(z)/z \in [-k_1, -k_2]$ is a sector-bounded gain

(i.e. the inverse graph of G is in $[-1/k_1, -1/k_2]$)

● Circle criterion : exists a quadratic separator (circle) for all ω



Another example : parameter-dependent Lyapunov function



▲ $F = A(\delta)$ parameter-dependent LTI state-space model

▲ $G = \mathcal{I}$ is an integrator

● Necessary and sufficient to have

$$\Theta(\delta) = \begin{bmatrix} 0 & -P(\delta) \\ -P(\delta) & 0 \end{bmatrix}$$

■ Direct relation with the IQC framework

▲ $F = T(j\omega)$ is a transfer matrix

▲ $G = \Delta$ is an operator known to satisfy an Integral Quadratic Constraint (IQC)

$$\int_{-\infty}^{+\infty} \begin{bmatrix} 1 & \Delta^*(j\omega) \end{bmatrix} \Pi(\omega) \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} d\omega \leq 0$$

● Stability of the closed-loop is guaranteed if for all ω

$$\begin{bmatrix} T^*(j\omega) & 1 \end{bmatrix} \Pi(\omega) \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} > 0$$

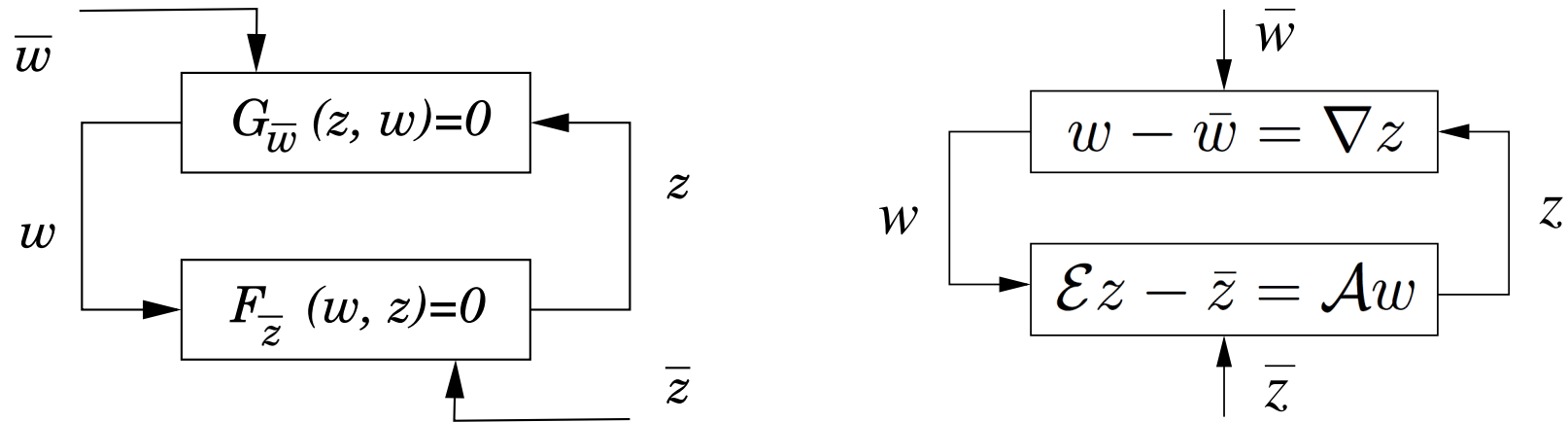
▲ Knowing Δ how to choose $\Pi = \Theta$? (i.e. the quadratic separator)

Plenty of results in μ -analysis and IQC theory

D-scalings, DG-scalings etc. but still, conservative

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2 IQS for the descriptor case



- Linear implicit application in feedback loop with an uncertain operator

$$\underbrace{\mathcal{E}z(t) = \mathcal{A}w(t)}_F, \quad \underbrace{w(t) = [\nabla z](t)}_G \quad \nabla \in \mathbb{W}$$

- ∇ is bloc-diagonal contains scalar, full-bloc, LTI and LTV uncertainties
and other operators such as integrators, delays...
- Result also extend to time-varying linear applications $\mathcal{E}(t), \mathcal{A}(t)$
and to polytopic linear applications $\left[\begin{array}{c} \mathcal{E}(\xi) \\ \mathcal{A}(\xi) \end{array} \right] \in CO \left\{ \left[\begin{array}{c} \mathcal{E}^{[i]} \\ \mathcal{A}^{[i]} \end{array} \right] \right\}$.

2 IQS for the descriptor case

■ Integral Quadratic Separation [Automatica'08, CDC'08]

- For the case of linear application with uncertain operator

$$\mathcal{E}z(t) = \mathcal{A}w(t) \quad , \quad w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ with \mathcal{E}_1 full column rank,

- Integral Quadratic Separator (IQS) : $\exists \Theta$, matrix, solution of LMI

$$\begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp*} \Theta \begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp} > 0$$

and Integral Quadratic Constraint (IQC) $\forall \nabla \in \mathbb{W}$

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

2 IQS for the descriptor case

- For given ∇ , there exist (conservative) LMI conditions for Θ solution to IQC

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

- ▲ Θ is build out of IQS for elementary blocs of ∇
- ▲ Improved DG -scalings, full-bloc S-procedure, vertex separators...
- ▲ Building Θ and related LMIs is tedious but can be automatized
www.laas.fr/OLOCEP/romuloc/
- ▲ It is conservative except in few special cases [Meinsma et al., 1997].

■ Robust analysis in IQS framework:

- 1- Write the robust analysis problem as a well-posedness problem

$$\mathcal{E}z = \mathcal{A}w, \quad w = \nabla z = \begin{bmatrix} \nabla_1 & & 0 \\ & \ddots & \\ 0 & & \nabla_j \end{bmatrix} z$$

- 2- Build Integral Quadratic Separators for each elementary bloc ∇_j
- 3- Apply the IQS results to get (conservative) LMIs

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- Induced L_2 norm (H_∞ in the LTI case)

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

- ▲ Prove that system is asymptotically stable
- ▲ and $\|g\| < \gamma\|v\|$ for zero initial conditions $x(0) = 0$
(strict upper bound on the L_2 gain attenuation)

- Equivalent to well-posedness with respect to

Integrator with zero initial conditions $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau) d\tau$

and signals such that $\|v\| \leq \frac{1}{\gamma} \|g\|$

3 Performance analysis in quadratic separation framework

■ Induced L_2 norm

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

▲ Define ∇_{n2n} the fictitious non-causal uncertain operator such that

$$v = \nabla_{n2n}g \quad \text{iff} \quad \|v\| \leq \frac{1}{\gamma} \|g\|$$

● Induced L_2 norm problem is equivalent to well-posedness of

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

3 Performance analysis in quadratic separation framework

■ Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_A \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

● Elementary IQS for bloc \mathcal{I}_1 is

$$\Theta_{\mathcal{I}_1} = \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} : P > 0$$

Indeed (recall $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau) d\tau$ and $x(0) = 0$)

$$\left\langle \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \middle| \Theta_{\mathcal{I}_1} \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \right\rangle_T = -x^*(T) P x(T) \leq 0$$

③ Performance analysis in quadratic separation framework

■ Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_A \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

● Elementary IQS for bloc ∇_{n2n} is (small gain theorem)

$$\Theta_{\nabla_{n2n}} = \begin{bmatrix} -\tau \mathbf{1} & 0 \\ 0 & \tau \gamma^2 \mathbf{1} \end{bmatrix} : \tau > 0$$

Indeed (recall $v = \nabla_{n2n} g$ iff $\|v\| \leq \frac{1}{\gamma} \|g\|$)

$$\left\langle \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \middle| \Theta_{\nabla_{n2n}} \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \right\rangle = \tau (-\|g\|^2 + \gamma^2 \|v\|^2) \leq 0$$

- Apply IQS and get (for non-descriptor case $E = 1$)

$$P > 0, \quad \tau > 0$$
$$\begin{bmatrix} A^*P + PA + \tau C^*C & PB + \tau C^*D \\ B^*P + \tau D^*C & -\tau\gamma^2\mathbf{1} + \tau D^*D \end{bmatrix} < 0$$

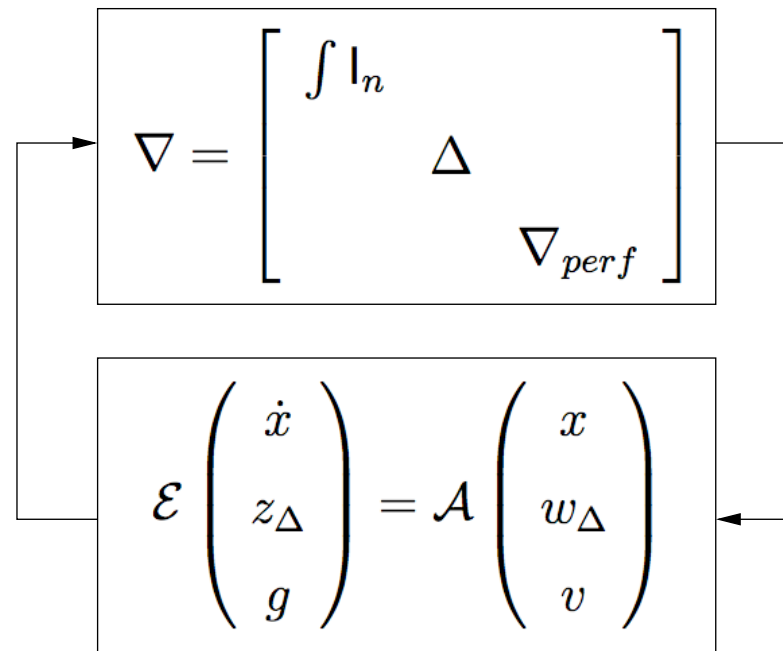
which is the classical H_∞ result.

- No difficulty to generate LMIs for descriptor case
- No difficulty to handle systems with uncertainties, time-delays...

③ Performance analysis in quadratic separation framework

■ Generic robust performance analysis problem:

● Well-posedness of



▲ $\int 1_n$ integrator

▲ Δ matrix of uncertainties

▲ ∇_{perf} operator related to performances

(induced L_2 , H_∞ , H_2 , impulse-to-norm, norm-to-peak, impulse-to-peak)

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4 System augmentation and conservatism reduction

■ Towards less-conservative conditions: System augmentation

▲ Example of stability of uncertain system with parametric uncertainty ($\dot{\delta} = 0$)

$$\dot{x} = (A + \delta B_{\Delta}(1 - \delta D_{\Delta})^{-1}C_{\Delta})x$$

▲ Corresponds to well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_{\Delta} \end{pmatrix} = \begin{bmatrix} A & B_{\Delta} \\ C_{\Delta} & D_{\Delta} \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix}, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 \mathbf{1}_n & 0 \\ 0 & \delta \mathbf{1}_m \end{bmatrix}$$

▲ [Meinsma] rule indicates results may be conservative

4 System augmentation and conservatism reduction

▲ Well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{pmatrix} x \\ w_\Delta \end{pmatrix}, \quad \nabla = \begin{bmatrix} \mathcal{I}1_n & 0 \\ 0 & \delta 1_m \end{bmatrix}$$

● adding the fact that $\dot{w}_\Delta = \delta \dot{z}_\Delta$, is also equivalent to well-posedness of

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_\Delta & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{pmatrix} \dot{w}_\Delta \\ \dot{x} \\ z_\Delta \\ \dot{z}_\Delta \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & A & B_\Delta & 0 \\ 0 & C_\Delta & D_\Delta & 0 \\ 0 & 0 & 0 & D_\Delta \\ 1 & 0 & -1 & 0 \end{array} \right] \begin{pmatrix} w_\Delta \\ x \\ w_\Delta \\ \dot{w}_\Delta \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}1_{m+n} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$$

4 System augmentation and conservatism reduction

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_{\Delta} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{pmatrix} \dot{w}_{\Delta} \\ \dot{x} \\ z_{\Delta} \\ \dot{z}_{\Delta} \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & A & B_{\Delta} & 0 \\ 0 & C_{\Delta} & D_{\Delta} & 0 \\ 0 & 0 & 0 & D_{\Delta} \\ 1 & 0 & -1 & 0 \end{array} \right] \begin{pmatrix} w_{\Delta} \\ x \\ w_{\Delta} \\ \dot{w}_{\Delta} \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}1_{m+n} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$$

- ▲ It is descriptor model.
- More decisions variables in the separator (increased dimensions of ∇)
- Bigger LMI conditions ($m + n$ rows)

4 System augmentation and conservatism reduction

- Lyapunov function is with respect to the augmented state
(vector involved in the integrator operator)

$$\begin{pmatrix} w_{\Delta}^* & x^* \end{pmatrix} P \begin{pmatrix} w_{\Delta} \\ x \end{pmatrix}$$

- ▲ Recalling that



$$w_{\Delta} = \delta(1 - \delta D_{\Delta})^{-1} C_{\Delta} x$$

the result corresponds to looking for a parameter dependent Lyapunov function

$$x^* \begin{bmatrix} \delta(1 - \delta D_{\Delta})^{-1} C_{\Delta} \\ 1 \end{bmatrix}^* P \begin{bmatrix} \delta(1 - \delta D_{\Delta})^{-1} C_{\Delta} \\ 1 \end{bmatrix} x$$

- Proves to be less conservative than for LMIs obtained on original system.

4 System augmentation and conservatism reduction

- Towards less-conservative conditions: System augmentation
- Adding more equations for higher derivatives of the state:
less conservative LMI conditions
- Same technique works for time varying uncertainties
(if known bounds on derivatives)
- Has been applied successfully to time-delay systems [Gouaisbaut]:
gives sequences of LMI conditions with decreasing conservatism
- ▲ Related to SOS representations of positive polynomials [Sato 2009]:
conservatism decreases as the order of the representation is augmented
- No need to manipulate by hand LMIs (Schur complements etc.), polynomials...
- ▲ Does conservatism vanishes? Exactly? Asymptotically? 
- ▲ Is it possible to cope with non-linearities in the same way? 

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5 The Romuald toolbox

■ Freely distributed software to test the theoretical results

● Existing software : RoMulOC

`www.laas.fr/OLOCEP/romuloc`

▲ Contains some of the analysis results plus some state-feedback features

● Currently developed software : Romuald

▲ Dedicated to analysis of descriptor systems

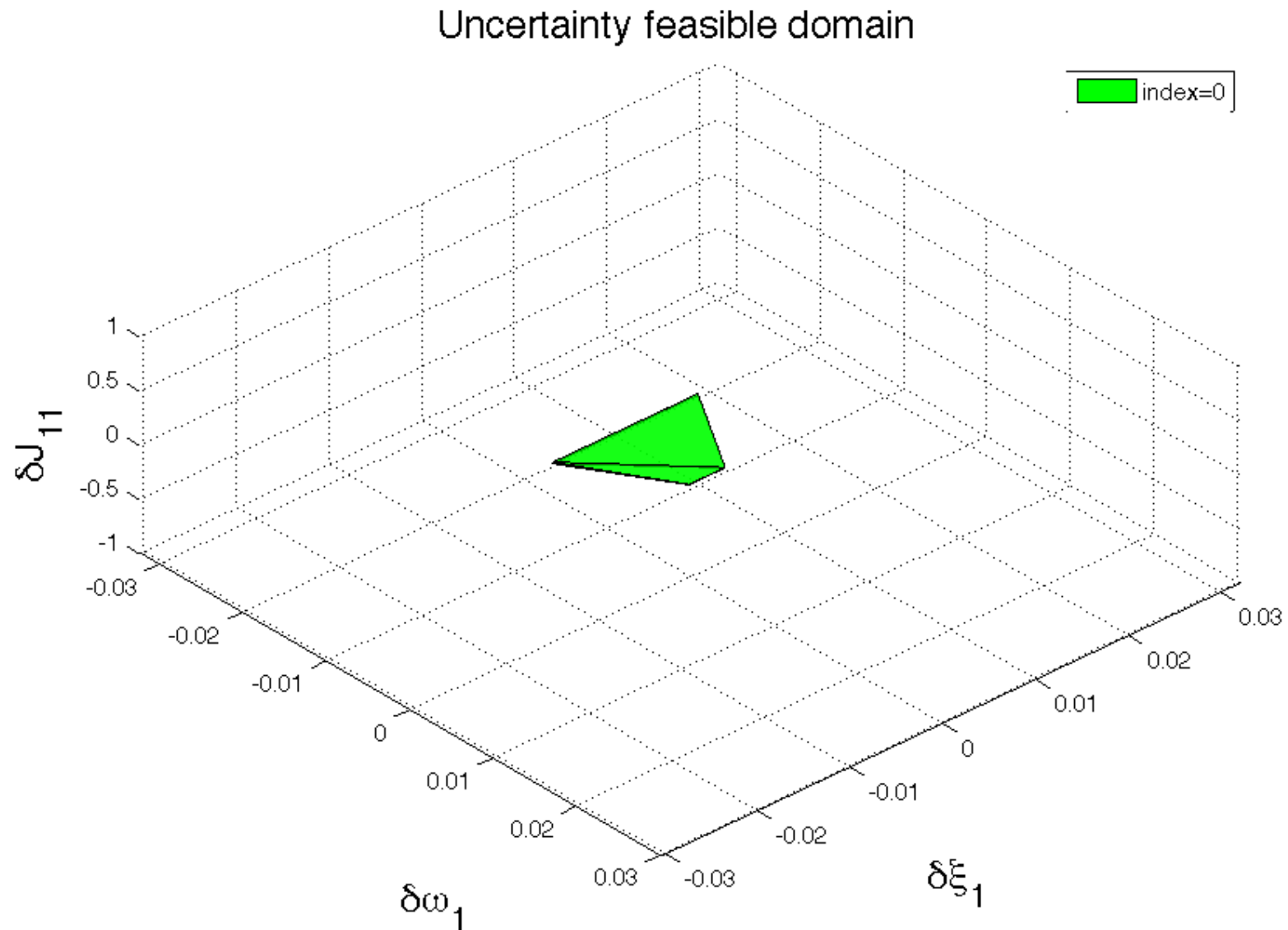
▲ Fully coded using the quadratic separation theory

▲ Allows systematic system augmentation

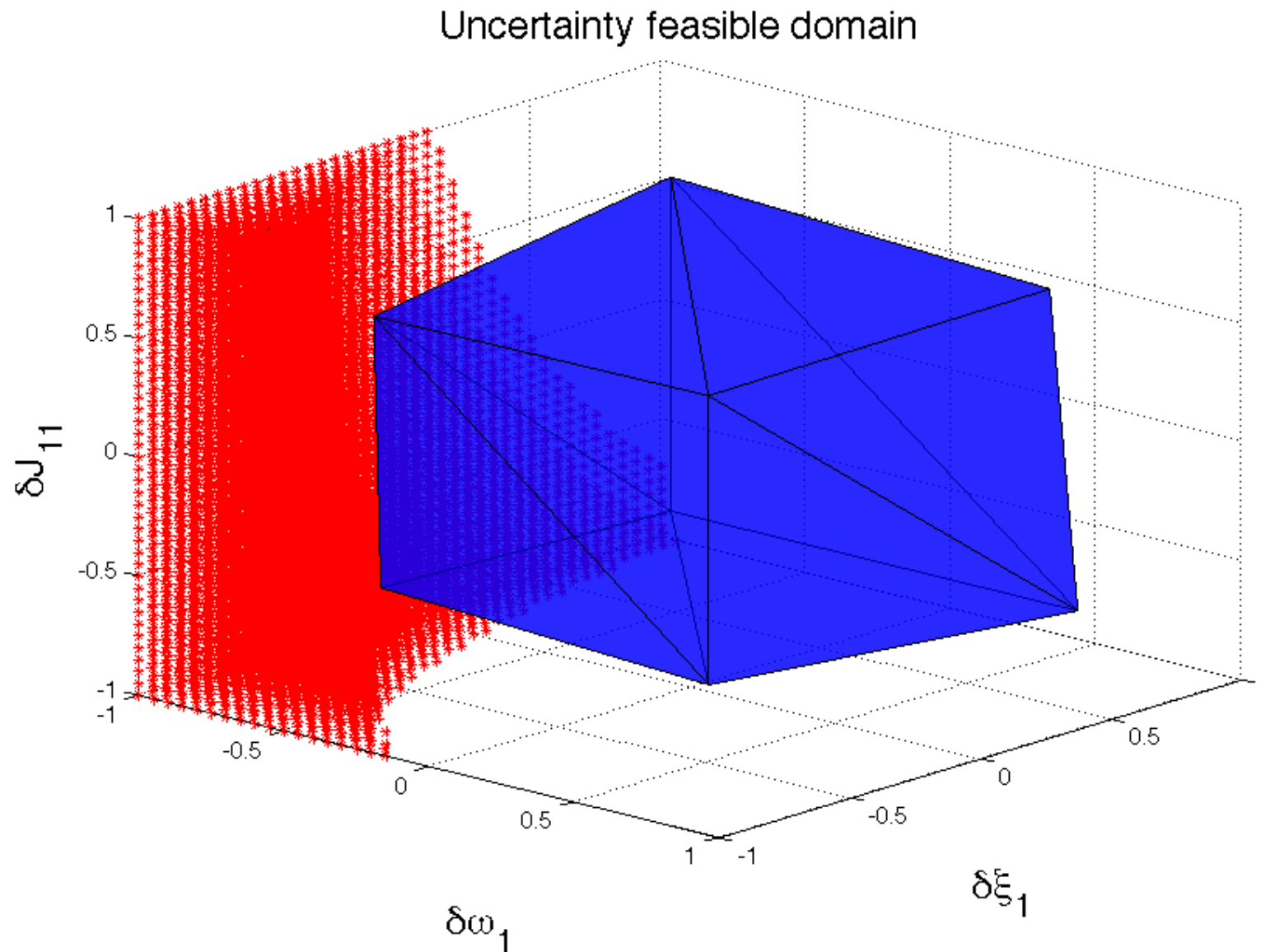
▲ First preliminary tests currently done for satellite and plane applications

```
>> quiz = ctrpb( OrderOfAugmentation ) + h2 (usys);  
>> result = solvesdp( quiz )
```

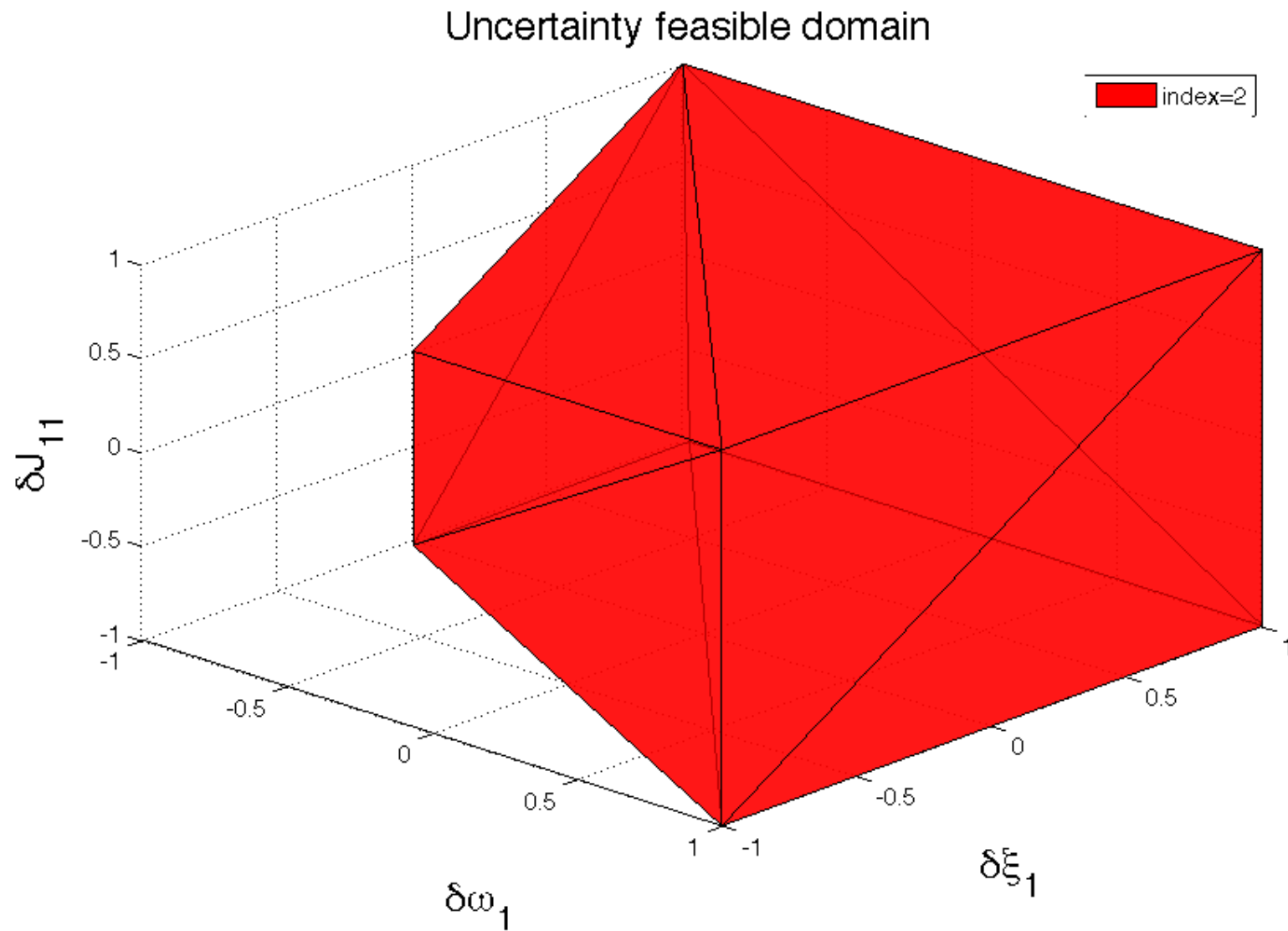
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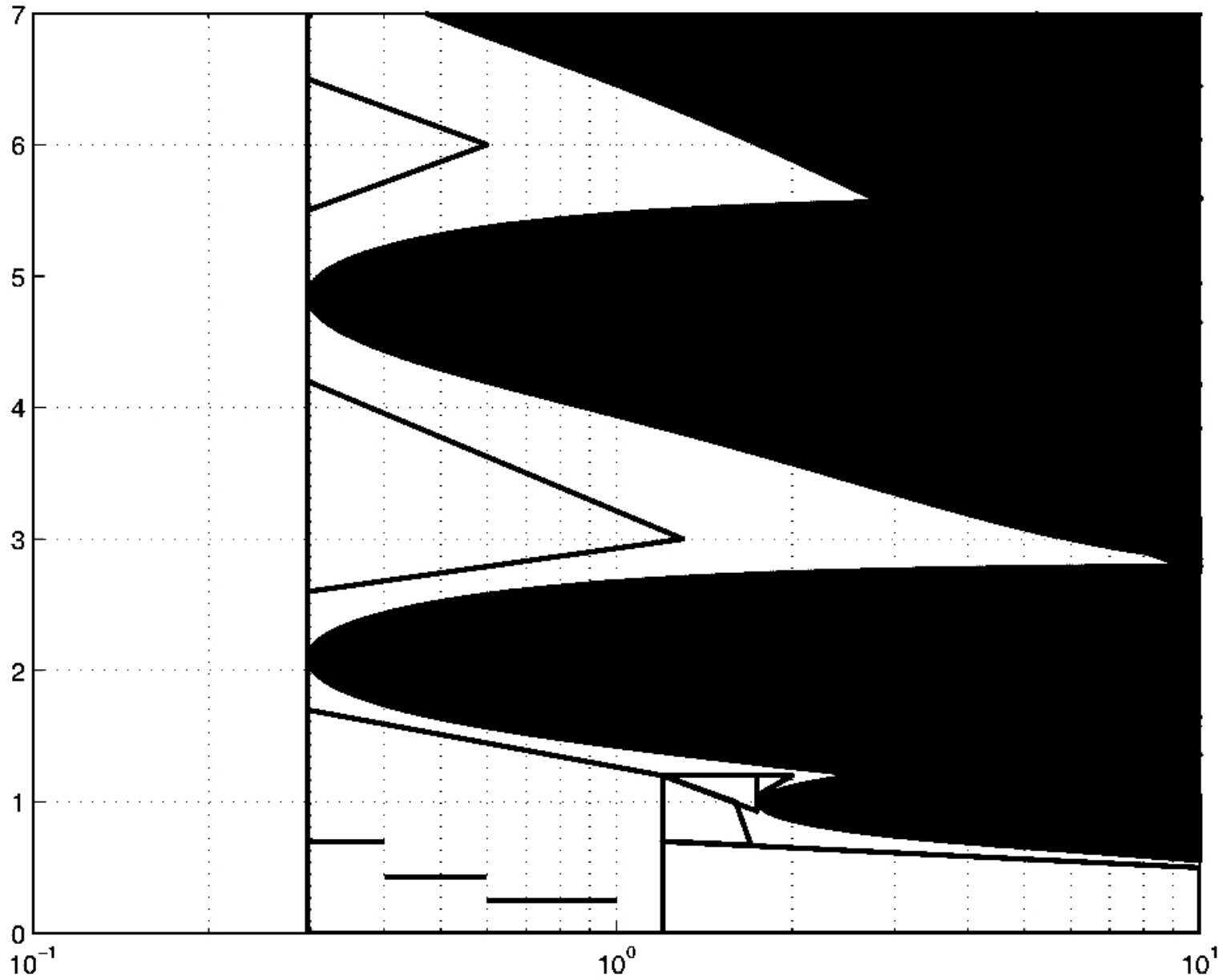
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Conclusions

