Structured adaptive control,
or how to solve LMIs with Simulink

Alexandru - Razvan LUZI
Dimitri PEAUCELLE

IEIIT-CNR Torino, october 2012
**Introduction**

- **Direct adaptive control:**
  Adaptation of control gains done directly based on measurements.

- **Indirect adaptive control:**
  Estimator of model parameters + scheduled control gain

- Feedback-loop stabilizing gains, MRAC not considered

- Lyapunov based stability proofs, not gradient approximation ‘MIT rule’

- Framework initiated by V.A. Yakubovich in the late 1960’s

- Contributions: new adaptive control law with asymptotic structure + may solve LMIs
1. Passivity-based adaptive control
2. LMIs are strict-passifiable systems
3. Structured adaptive control
4. Numerical Example
Passivity-based adaptive control of LTI systems

Theorem

The following two conditions are equivalent:

1. There exists a static control $u(t) = Fy(t) + w(t)$ for the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad z(t) = y(t)$$

that makes the closed-loop strictly passive (with respect to $w/z$).

2. For all $\Gamma \succ 0$ the following adaptive control

$$u(t) = K(t)y(t) + w(t), \quad \dot{K}(t) = -y(t)y^T(t)\Gamma$$

makes the closed-loop globally strictly-passive.
Strict-passivity includes asymptotic stability of $x = 0$

Adaptive control converges to $K(\infty)$: strictly-passifying static gain

Theorem for square systems - extensions exist for non-square systems

Not all stabilizable systems are strictly-passifiable
- modified adaptive laws exist for stabilizable systems

Condition also reads in terms of matrix inequalities as

$$\exists Q \succ 0 : (A + BF C)^T Q + Q(A + BF C) \prec 0, \quad QB = C^T$$

It happens to be an LMI constraint!

$$\exists Q \succ 0 : A^T Q + QA + C^T(F^T + F)C \prec 0, \quad QB = C^T$$

Finding $F$ solution to the LMI is equivalent to simulating the system with the adaptive control law and taking $F = K(\infty)$.
All LMIs define strict-passifiable systems

Let us consider an example: LMIs for an upper bound on the $H_\infty$ norm

$$\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D \\
B^T P + D^T C & -\gamma^2 I + D^T D
\end{bmatrix} \prec 0, \quad P = P^T \succ 0.$$
LMIs are strict-passifiable systems

All LMIs define strict-passifiable systems

Let us consider an example: LMIs for an upper bound on the $H_\infty$ norm

\[
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D \\
B^T P + D^T C & -\gamma^2 I + D^T D
\end{bmatrix} \prec 0 , \quad P = P^T > 0.
\]

All LMI constraints can be gathered in one

\[
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D & 0 \\
B^T P + D^T C & -\gamma^2 I + D^T D & 0 \\
0 & 0 & -P
\end{bmatrix} \prec 0 , \quad P = P^T
\]
All LMI constraints can be gathered in one

\[
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D & 0 \\
B^T P + D^T C & -\gamma^2 I + D^T D & 0 \\
0 & 0 & -P
\end{bmatrix} \prec 0 , \quad P = P^T
\]
\( \Delta \) All LMI constraints can be gathered in one

\[
\begin{bmatrix}
A^T P + PA + C^T C & PB + C^T D & 0 \\
B^T P + D^T C & -\gamma^2 I + D^T D & 0 \\
0 & 0 & -P
\end{bmatrix} \prec 0, \quad P = P^T
\]

\( \Delta \) Can be decomposed in a sum with elementary matrix variables

\[
A + B_P^T \begin{bmatrix}
0 & P & 0 \\
P & 0 & 0 \\
0 & 0 & -P
\end{bmatrix} B_P + B_\gamma^T ( -\gamma^2 I ) B_\gamma \prec 0, \quad P = P^T
\]

\[
A = \begin{bmatrix}
C^T C & C^T D & 0 \\
D^T C & D^T D & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B_P = \begin{bmatrix}
A & B & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad B_\gamma = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix}
\]
LMIs are strict-passifiable systems

Can be decomposed in a sum with elementary matrix variables

\[ A + B_P^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} B_P + B_\gamma^T (-\gamma^2 1) B_\gamma \prec 0, \quad P = P^T \]
Can be decomposed in a sum with elementary matrix variables

\[
A + B_P^T \begin{bmatrix}
0 & P & 0 \\
0 & 0 & 0 \\
0 & 0 & -P
\end{bmatrix} B_P + B_\gamma^T (\gamma^2 I) B_\gamma < 0 , \quad P = P^T
\]

Or equivalently when gathering all variables in a block-diagonal matrix

\[
A + B^T F B < 0 , \quad B = \begin{bmatrix} B_P & B_\gamma \end{bmatrix}
\]

with the structural equality constraints

\[
F = \begin{bmatrix} F_P & 0 \\
0 & F_\gamma \end{bmatrix} , \quad F_P = \begin{bmatrix} 0 & P & 0 \\
P & 0 & 0 \\
0 & 0 & -P \end{bmatrix} , \quad P = P^T , \quad F_\gamma = -\gamma^2 I
\]
LMIs are strict-passifiable systems

\[ A + B_P^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} B_P + B_\gamma^T (-\gamma^2 I) B_\gamma \prec 0 \ , \ P = P^T \]

\[ \text{Or equivalently when gathering all variables in a block-diagonal matrix} \]

\[ A + B^T F B \prec 0 \ , \ B = \begin{bmatrix} B_P & B_\gamma \end{bmatrix} \]

with the structural equality constraints

\[ F = \begin{bmatrix} F_P & 0 \\ 0 & F_\gamma \end{bmatrix} \ , \ F_P = \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} \ , \ P = P^T \ , \ F_\gamma = -\gamma^2 I \]

\[ \text{The constraint } A + B^T F B \prec 0 \text{ holds iff} \]

\[ (A, B, C = B^T) \text{ is strictly-passifiable by } F. \]
LMIs are strict-passifiable systems

Can be decomposed in a sum with elementary matrix variables

\[
A + B^T_P \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} B_P + B^T_\gamma (-\gamma^2 I) B_\gamma < 0 \ , \ P = P^T
\]

Or equivalently when gathering all variables in a block-diagonal matrix

\[
A + B^T F B < 0 \ , \ B = \begin{bmatrix} B_P & B_\gamma \end{bmatrix}
\]

with the structural equality constraints

\[
F = \begin{bmatrix} F_P & 0 \\ 0 & F_\gamma \end{bmatrix} \ , \ F_P = \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & -P \end{bmatrix} \ , \ P = P^T \ , \ F_\gamma = -\gamma^2 I
\]

The constraint \( A + B^T F B < 0 \) holds iff

\((A, B, C = B^T)\) is strictly-passifiable by \( F \).

LMI converted to strict-passification problem, with equality constraints.
Procedure applies to any LMI: concludes with search of passifying

\[
F = \begin{bmatrix}
F_1 & 0 \\
& \ddots \\
0 & F_N
\end{bmatrix}
\]

for a (symmetric) system \((A, B, C = B^T)\) with additional structural equality constraints that can be compacted in

\[
\text{vec}(F_i) = S_i x_i \iff U_i \text{vec}(F_i) = 0
\]

where \(\text{vec}(F_i)\) is the vector composed of stacked columns of \(F_i\), \(x_i\) are vectors of independent scalar decision variables and \(U_i = S_i^{-\perp}\).
LMIs are strict-passifiable systems

- Procedure applies to any LMI: concludes with search of passifying

\[
F = \begin{bmatrix}
F_1 & 0 \\
\vdots & \ddots \\
0 & F_N
\end{bmatrix}
\]

for a (symmetric) system \((A, B, C = B^T)\) with additional structural equality constraints that can be compacted in

\[
\text{vec}(F_i) = S_i x_i \iff U_i \text{vec}(F_i) = 0
\]

where \(\text{vec}(F_i)\) is the vector composed of stacked columns of \(F_i\), \(x_i\) are vectors of independent scalar decision variables and \(U_i = S_i^\perp\).

- When starting from the canonical representation \(L_0 + \sum_j \hat{x}_j L_j \prec 0\), then the structural constraints are all of the type

\[
F_j = \begin{bmatrix}
\hat{x}_j 1_{r_{j1}} & 0 \\
0 & -\hat{x}_j 1_{r_{j2}}
\end{bmatrix}
\]

and \(r_{j1} + r_{j2}\) can be very large and \(U_j\) is huge.

- \(F\) and \(U_i\)s expected to be smaller when matrix representation.
Block-diagonal adaptive control with asymptotic structure

Theorem

The following two conditions are equivalent:

1. There exists a decentralized static control $u_i(t) = F_i y_i(t) + w_i(t)$ satisfying the structural constraints $U_i \text{vec}(F_i) = 0$ for the system

$$\dot{x}(t) = Ax(t) + \sum \mathbb{B}_i u_i(t) , \quad y_i(t) = C_i x(t) , \quad z(t) = y(t)$$

that makes the closed-loop strictly passive (with respect to $w/z$).

2. For all $\Gamma_i \succ 0, \alpha_i > 0$ the following adaptive control

$$u_i(t) = K_i(t) y_i(t) + w_i(t) , \quad \dot{K}_i(t) = -y_i(t) y_i^T(t) \Gamma_i - \alpha_i \cdot \text{mat} \left( U_i^T U_i \cdot \text{vec}(K_i(t)) \right) \Gamma_i$$

makes the closed-loop globally strictly-passive.

‘mat’ is the function such that $\text{mat}(\text{vec}(F)) = F$. 
Proof of $\mathbf{1} \implies \mathbf{2}$

$\mathbf{1}$ reads as

$$\exists F, \exists Q \succ 0 : \quad (A + BF C)^T Q + Q(A + BF C) < 0, \quad QB = C^T,$$

$$F = \text{diag} [ \cdots \ F_i \cdots ], \quad U_i \cdot \text{vec}(F_i) = 0$$  \hspace{1cm} (1)
Proof of $1 \Rightarrow 2$

$1$ reads as

$$\exists F, \exists Q \succ 0 \quad : \quad (A + BF C)^T Q + Q(A + BF C) < 0, \quad QB = CT, \quad F = \text{diag} \left[ \cdots F_i \cdots \right], \quad U_i \cdot \text{vec}(F_i) = 0$$

(1)

Let the Lyapunov function for the non-linear system (with adaptive law)

$$V(x, K) = \frac{1}{2} \left( x^T Q x + \sum_i \text{Tr} \left( (K_i - F_i) \Gamma^{-1} (K_i - F_i)^T \right) \right)$$
Proof of $\textcircled{1} \Rightarrow \textcircled{2}$

$\textcircled{1}$ reads as

$$\exists F, \exists Q \succ 0 : \quad (A + BF C)^T Q + Q(A + BF C) < 0, \quad QB = C^T, \\
F = \text{diag} [ \cdots F_i \cdots ], \quad U_i \cdot \text{vec}(F_i) = 0$$

Let the Lyapunov function for the non-linear system (with adaptive law)

$$V(x, K) = \frac{1}{2} \left( x^T Q x + \sum_i \text{Tr} \left( (K_i - F_i)\Gamma^{-1}(K_i - F_i)^T \right) \right)$$

After manipulations and using $QB = C^T$, $U_i \cdot \text{vec}(F_i) = 0$, we get:

$$\dot{V}(x, K) = x^T (A + BF C)^T Q x + w^T z - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i))$$
Proof of ① ⇒ ② (continued)

\[ \dot{V}(x, K) = x^T (A + BF\mathbb{C})^T Q x + w^T z - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)). \]
Structured adaptive control

Proof of \(1 \Rightarrow 2\) (continued)

\[
\dot{V}(x, K) = x^T (A + BF\mathcal{C})^T Q x + w^T z - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)).
\]

\(\blacktriangle\) First term is strictly negative due to (1), until \(x = 0\),

\(\blacktriangle\) Last term is strictly negative, until \(U_i \cdot \text{vec}(K_i) = 0\)

- If no perturbations \((w = 0)\) the system converges to the attractor

\[
\mathcal{A} = \{(x, K) : x = 0, U_i \cdot \text{vec}(K_i) = 0\}
\]

- On the attractor \(\dot{K}_i = 0\): the gains \(K_i(\infty)\) are constant.
Proof of $\textcircled{1} \Rightarrow \textcircled{2}$ (continued)

\[
\dot{V}(x, K) = x^T (A + BF) C^T Q x + w^T z - \sum_{i} \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)).
\]
Proof of $\mathbf{1} \Rightarrow \mathbf{2}$ (continued)

\[ \dot{V}(x, K) = x^T (A + B F C)^T Q x + w^T z - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)). \]

For initial conditions at equilibrium and nonzero perturbations

\[ 0 \leq V(x(t), K(t)) = \int_0^t \dot{V}(x, K) \, d\tau < \int_0^t w^T z \, d\tau \]

$\Rightarrow$ the system is strictly passive.
Proof of $2 \Rightarrow 1$

- The system with adaptive control is globally asymptotically stable, it converges to a asymptotically stable equilibrium: $F_i = K_i(\infty)$ are stabilizing gains.

- Same reasoning holds for passivity.
Summary

- All LMI problems are equivalent to static output-feedback strict-passification problems with structure constraints:
  - gain $F$ is block-diagonal
  - sub-blocks should satisfy $U_i \text{vec}(F_i) = 0$.

- If a structured strict-passification problem admits solutions, the block-diagonal adaptive law with asymptotic structure will converge to one of these.

- The LMIs can be solved by simulating the adaptive controlled systems.

- If the system converges $K_i(\infty)$ contain solutions of the LMIs.

- If does not converges the LMIs are infeasible.
Numerical example

Consider the transfer function:

\[ G(s) = \frac{s^2 + s + 1}{s^2 + s + 2} \]

Problem: compute the \( H_\infty \) norm (or at least an upper bound).

In Matlab: \( \text{norm}(G, \text{Inf}, 1\text{e}-4) = 1.3251 \)
Numerical example

Consider the transfer function:

\[
G(s) = \frac{s^2 + s + 1}{s^2 + s + 2}
\]

Problem: compute the \( H_\infty \) norm (or at least an upper bound).

\[\text{In Matlab: } \text{norm}(G, \text{Inf}, 1e^{-4}) = 1.3251\]

\[\text{LMI problem converted to adaptive passification} \]

\[
\dot{K}_i = -y_i y_i^T \Gamma_i - \alpha_i \cdot \text{mat} \left( U_i^T U_i \cdot \text{vec}(K_i) \right) \Gamma_i , \quad y_1 \in \mathbb{R}^6, \quad y_2 \in \mathbb{R}
\]

with structural asymptotic constraints:

\[
F_1 = \begin{bmatrix}
0 & P & 0 \\
P^T & 0 & 0 \\
0 & 0 & -P
\end{bmatrix}, \quad P = P^T \in \mathbb{R}^{2 \times 2}, \quad F_2 = -\gamma^2 \mathbf{1} = -\gamma^2.
\]
Numerical Example

- Parameters for simulating the adaptive law (simulation in Simulink)
- Initial conditions $x = (1 \ldots 1)^T$ and $K_i = 0$
- $\Gamma_1 = 1000 \cdot 1$, $\Gamma_2 = 10$, $\alpha_1 = \alpha_2 = 1$
Parameters for simulating the adaptive law (simulation in Simulink)

- Initial conditions $x = (1 \ldots 1)^T$ and $K_i = 0$
- $\Gamma_1 = 1000 \cdot 1$, $\Gamma_2 = 10$, $\alpha_1 = \alpha_2 = 1$
- Convergence to zero of the ‘outputs’ $y_i$
Convergence to structured values of the adapted gains $K_i$:

$$K_1(\infty) = \begin{bmatrix} 0 & 0 & 4.6330 & 1.0671 & 0 & 0 \\ 0 & 0 & 1.0671 & 10.7960 & 0 & 0 \\ 4.6330 & 1.0671 & 0 & 0 & 0 & 0 \\ 1.0671 & 10.7960 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.6330 & -1.0671 \\ 0 & 0 & 0 & 0 & -1.0671 & -10.7960 \end{bmatrix}$$

$$K_2(\infty) = -7.1307$$
\section*{Numerical Example}

\textbf{Evolution of the (1 : 2, 3 : 4) elements of $K_1$ that converge to $P$}

\begin{itemize}
\item \textbf{Solution of the LMIs}
\end{itemize}

\begin{equation*}
P = \begin{bmatrix}
4.6330 & 1.0671 \\
1.0671 & 10.7960
\end{bmatrix}, \quad \gamma = 2.6703 \geq 1.3251 = \gamma_{opt}
\end{equation*}
Test for feasible / unfeasible cases

Only $K_1$ is adapted, $\gamma$ is slowly linearly modified

Unstable behavior when $\gamma < 1.3251 = \gamma_{opt}$.
Conclusions et perspectives

- LMI feasibility problems can be solved by simulating systems

- Need for a parser to convert LMIs to adaptive control problem

- Simulation time is large - what is the best implementation?

- Is simulation time polynomial w.r.t. size of problem?

- What about LMI optimization problems?

- Decreasing parameters until system becomes unstable?

- Minimizing gap with dual LMI problem (it works).

- Other?

- Solving time-varying LMI problems?