Evaluating regions of attraction of LTI systems with saturation in IQS framework

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Saturated control of a linear system

\[ \dot{x} = Ax + Bu , \quad u = \text{sat}(Ky) , \quad y = Cx \]

Assume \( K \) designed for the linear system (no saturation)

System with saturation: Stability is (in general) only local

Problem: find (largest possible) set of \( x(0) \) such that \( x(\infty) = 0 \)

Goal of this presentation: formalize the problem in the IQS framework

Can "system augmentation" relaxations provide less conservative results?
Well-posedness of a feedback loop

Uniqueness and boundedness of internal signals for all bounded disturbances

\[ \exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2, \quad \begin{vmatrix} w - w_0 \end{vmatrix} \leq \gamma \begin{vmatrix} \bar{w} \end{vmatrix}, \quad G(z_0, w_0) = 0 \]
\[ F(\bar{w}, z) = \bar{z} \]

\[ \forall (\bar{w}, \bar{z}) \in L_2 \times L_2, \quad \begin{vmatrix} z - z_0 \end{vmatrix} \leq \gamma \begin{vmatrix} \bar{z} \end{vmatrix}, \quad F(w_0, z_0) = 0 \]

iff exists a topological separator \( \theta \)

Negative on the inverse graph of one component

Positive definite on the graph of the other component of the loop

\[ \mathcal{G}^I(\bar{w}) = \{(w, z) : G(z, w) = \bar{w}\} \subset \{(w, z) : \theta(w, z) \leq \phi_2(||\bar{w}||)\} \]
\[ \mathcal{F}(\bar{z}) = \{(w, z) : F(w, z) = \bar{z}\} \subset \{(w, z) : \theta(w, z) > -\phi_1(||\bar{z}||)\} \]

Issues: How to choose \( \theta \)? How to test the separation inequalities?
Example: the small gain theorem

Well-posedness of a feedback loop

In case of causal $G(z, w) : w = \Delta z$, $\Delta \in \mathcal{RH}_{\infty}^{m \times l}$ and stable proper LTI $F(w, z) : z = H(s)w$

Necessary and sufficient (lossless) choice of separator

$$\theta(w, z) = \|w\|^2 - \gamma^2\|z\|^2$$

Separation inequalities:

$$\theta(w, z) = \|w\|^2 - \gamma^2\|z\|^2 \leq 0, \forall w = \Delta z \iff \|\Delta\|_{\infty}^2 \leq \gamma^2$$

$$\theta(w, z) = \|w\|^2 - \gamma^2\|z\|^2 > 0, \forall z = H(s)w \iff \|H\|_{\infty}^2 < \frac{1}{\gamma^2}$$
Integral Quadratic Separation (IQS)

- Choice of an Integral Quadratic Separator

\[ \theta(w, z) = \left\langle \begin{pmatrix} z \\ w \end{pmatrix} \right| \Theta \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle = \int_0^\infty \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \Theta(t) \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \, dt \]

- Identical choice to IQC framework [Megretski, Rantzer, Jönsson]

\[ \theta(w, z) = \int_{-\infty}^{+\infty} \begin{pmatrix} z^T(j\omega) & w^T(j\omega) \end{pmatrix} \Pi(j\omega) \begin{pmatrix} z(j\omega) \\ w(j\omega) \end{pmatrix} \, d\omega \]

- \( \Pi \) is called a multiplier. \( \theta(w, z) \leq 0 \) is called an IQC.

- Conservatism reduction in IQC framework: \( \omega \)-dependent multipliers:

\[ \Pi(j\omega) = \begin{bmatrix} 1 & \Psi_1(j\omega)^* & \cdots & \Psi_r(j\omega)^* \end{bmatrix} \hat{\Pi} \begin{bmatrix} 1 \\ \Psi_1(j\omega) \\ \vdots \\ \Psi_r(j\omega) \end{bmatrix} \]
Main IQS result (both for $\omega$ or $t$ or $k$ dependent signals)

IQS is **necessary and sufficient** under assumptions (proof based on [Iwasaki 2001])

- One component is a linear application, can be descriptor form $F(w,z) = Aw - Ez$
  
- can be time-varying $A(t)w(t) - E(t)z(t)$ or frequency dep. $\hat{A}(\omega)\hat{w}(\omega) - \hat{E}(\omega)\hat{z}(\omega)$

- $A(t)$, $E(t)$ are bounded and $E(t) = E_1(t)E_2$ where $E_1(t)$ is full column rank

- The other component can be defined in a set

  $$G(z,w) = \nabla(z) - w, \quad \nabla \in \mathbb{W}$$

- $\mathbb{W}$ must have a linear-like property

  $$\forall(z_1,z_2), \forall \nabla \in \mathbb{W}, \exists \tilde{\nabla} \in \mathbb{W} : \nabla(z_1) - \nabla(z_2) = \tilde{\nabla}(z_1 - z_2)$$

- $\mathbb{W}$ need not to be causal

- The matrix $\Theta$ must satisfy an IQC over $\mathbb{W}$ + an LMI involving ($E$, $A$)
Global stability of a non-linear system $\dot{x} = f(x, t)$

$w \overset{G(z, w) = \bar{w}}{\rightarrow} \bar{z} \overset{z = \dot{x}, w = x}{\rightarrow} \int_{0}^{t} z(\tau)d\tau - w(t)$

$F(w, z, t) = f(w, t) - z(t)$

$\bar{w}$ plays the role of the initial conditions, $\bar{z}$ are external disturbances

Well-posedness: for all bounded initial conditions and all bounded disturbances, the state remains bounded around the equilibrium $\equiv$ global stability

For linear systems $\dot{x}(t) = A(t)x(t)$, $\nabla = s^{-1}1$

$IQS: \theta(w, z) = \int_{0}^{\infty} \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \begin{pmatrix} 0 & -P(t) \\ -P(t) & -\dot{P}(t) \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$

$\Delta \theta(w, z) \leq 0$ for all $G(z, w) = 0$ iff $P(t) \geq 0$

$\Delta \theta(w, z) > 0$ for all $F(w, z) = 0$ iff $A^T(t)P(t) + P(t)A(t) + \dot{P}(t) < 0$
Global stability of a system with a dead-zone

- $G_1(\dot{x}, x) = \int_0^t \dot{x}(\tau) d\tau - x(t)$,
- $G_2(g, v) = dz(g(t)) - v(t)$,
- $F_1(x, v, \dot{x}, t) = f_1(x, v, t) - \dot{x}(t)$,
- $F_2(x, v, g, t) = f_2(x, v, t) - g(t)$

IQS applies for linear $f_1, f_2$

- Dead-zone embedded in a sector uncertainty $\mathcal{W}_\infty = \{\nabla_\infty : 0 \leq \nabla_\infty(g) \leq g\}$

$$\mathcal{G}_2^I = \{(v, g) : G_2(g, v) = 0\} \subset \{(v, g) : v = \nabla_\infty(g), \nabla_\infty \in \mathcal{W}_\infty\}$$

This is the only source of conservatism

- LMI conditions obtained for the IQS defined by

$$\Theta = \begin{bmatrix} 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -p_1 \\ -P & 0 & 0 & 0 \\ 0 & -p_1 & 0 & 2p_1 \end{bmatrix}$$

where $P > 0$, $p_1 > 0$. 

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Examples - Topological Separation and Lyapunov
Launcher model

- Launcher in ballistic phase: attitude control
  - neglected atmospheric friction, sloshing modes, ext. perturbation, axes coupling: $I\ddot{\theta} = T$
  - Saturated actuator: $T = \text{sat}_T(u) = u - \bar{T}d(\frac{1}{T}u)$
  - PD control $u = -K_P\theta - K_D\dot{\theta}$

- Global stability LMI test fails
  - Sector uncertainty includes $\nabla_\infty = 1$ for which the system is $I\ddot{\theta} = 0$ (unstable)

- LMI test succeeds (whatever $\bar{g} < \infty$) if dead-zone is restricted to belong to

\[ \nabla_{\bar{g}} = \{ \nabla_{\bar{g}} : 0 \leq \nabla_{\bar{g}}(g) \leq \frac{1-\bar{g}}{\bar{g}} g \} \]

- Useful if one can prove for constrained $x(0)$ that $|g(\theta)| \leq \bar{g}$ holds $\forall \theta \geq 0$.

- How can one prove local properties in IQS framework?
Initial conditions dependent IQS

- Well-posedness of a feedback loop

\[ F(w, z) = z \]
\[ G(z, w) = w \]

- Uniqueness and boundedness of internal signals for all bounded disturbances

\[ \exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2, \quad \begin{bmatrix} \| w - w_0 \| \\ \| z - z_0 \| \end{bmatrix} \leq \gamma \begin{bmatrix} \| \bar{w} \| \\ \| \bar{z} \| \end{bmatrix}, \quad \begin{cases} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{cases} \]

▲ How to introduce initial conditions \( x(0) \) and “final” conditions \( g(\theta) \) in IQS framework?

- Square-root of the Dirac operator: linear operator such that

\[
\begin{align*}
\langle \varphi_{\theta_1} x | M \varphi_{\theta_2} x \rangle &= \int_0^\infty \varphi_{\theta_1} x^T(t) M \varphi_{\theta_2} x(t) dt = x^T(\theta) M x(\theta) \\
\langle \varphi_{\theta_1} x | M \varphi_{\theta_2} x \rangle &= 0 \quad \text{if} \quad \theta_1 \neq \theta_2
\end{align*}
\]

- Such operator is also used for PDE to describe states on the boundary
System with initial and final conditions writes as

\[
\begin{bmatrix}
\varphi_0 x \\
\mathcal{T}_\theta \dot{x} \\
\mathcal{T}_\theta g \\
\varphi_\theta g
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 1 \\
A & 0 & B & 0 \\
C & 0 & 0 & 0 \\
0 & C & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathcal{T}_\theta x \\
\varphi_\theta x \\
\mathcal{T}_\theta v \\
\varphi_0 x
\end{bmatrix}
\]

\(\mathcal{T}_\theta x\) is the truncated signal such that \(\mathcal{T}_\theta x(t) = x(t)\) for \(t \leq \theta\) and \(= 0\) for \(t > \theta\).

The integration operator is redefined as a mapping

\[
\begin{bmatrix}
\mathcal{T}_\theta x \\
\varphi_\theta x
\end{bmatrix} = \mathcal{I}
\begin{bmatrix}
\varphi_0 x \\
\mathcal{T}_\theta \dot{x}
\end{bmatrix}
\]
Initial conditions dependent IQS

\[
\begin{pmatrix}
\varphi_0 x \\
\mathcal{T}_\theta \dot{x} \\
\mathcal{T}_\theta g \\
\varphi_\theta g
\end{pmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 1 \\
A & 0 & B & 0 \\
C & 0 & 0 & 0 \\
0 & C & 0 & 0
\end{bmatrix}
\begin{pmatrix}
\mathcal{T}_\theta x \\
\varphi_\theta x \\
\mathcal{T}_\theta v \\
\varphi_0 x
\end{pmatrix}
\]

- Restricted sector constraint assumed to hold up to \( t = \theta \):

\[
\mathcal{T}_\theta v = \nabla \bar{g} \mathcal{T}_\theta g
\]
Goal is to prove the restricted sector condition holds strictly at time $\theta$ (whatever $\theta$)

\[ \text{i.e. find sets } 1 \geq x^T(0)Qx(0) = \langle \varphi_0x|Q\varphi_0x \rangle \text{ s.t. } |g(\theta)| = \|\varphi_\theta g\| < \bar{g} \]

\[ \text{ reformulated as well posedness problem where } \varphi_0x = \nabla_{ci}\varphi_\theta g \text{ defined by } \]

\[ w_{ci} = \nabla_{ci}z_{zi} : \bar{g}^2 < w_{ci}|Qw_{ci} \leq \|z_{ci}\|^2 \]
Initial conditions dependent IQS

\[
\begin{bmatrix}
\varphi_0 x \\
\mathcal{T}_\theta \dot{x} \\
\mathcal{T}_\theta g \\
\varphi_\theta g
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 1 \\
A & 0 & B & 0 \\
C & 0 & 0 & 0 \\
0 & C & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathcal{T}_\theta x \\
\varphi_\theta x \\
\mathcal{T}_\theta v \\
\varphi_0 x
\end{bmatrix}
\]

Problem defined in this way is a well-posedness problem with \( \nabla \) composed of 3 blocs

\[
\nabla = \begin{bmatrix}
\mathcal{I} & \nabla \bar{g} \\
\nabla_{ci}
\end{bmatrix}
\]

IQS framework applies and gives conservative LMI conditions

Equivalent to LaSalle invariance principle with \( V(x) = x^T Q x \) (ellipsoidal sets of IC)
How to reduce conservatism?

- Needed a description of the dead-zone better than sector uncertainty
- Needed to have dead-zone dependent sets of initial conditions

Both features derived via descriptor modeling of system augmented with $\dot{v}$ and $\dot{g}$

$$v = dz(g) : \begin{cases} 
  \text{if } g > 1 & v = g - 1 \quad \dot{v} = \dot{g} \\
  \text{if } |g| \geq 1 & v = 0 \quad \dot{v} = 0 \\
  \text{if } g < -1 & v = g + 1 \quad \dot{v} = \dot{g}
\end{cases}$$

- For IQS, link between $\dot{v}$ and $\dot{g}$ is embedded in $\dot{v} = \nabla\{0,1\} \dot{g}$, with $\nabla\{0,1\} \in \{0, 1\}$.
- Also needed to specify that $v$ is the integral of $\dot{v}$ (thus descriptor form)
Problem defined in this way is a well-posedness problem with $\nabla$ composed of 5 blocs

- IQS framework applies and gives less conservative LMI conditions
- Equivalent to LaSalle invariance principle with

$$V(x) = \begin{pmatrix} x \\ v \end{pmatrix}^T Q_a \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x \\ dz(Cx) \end{pmatrix}^T Q_a \begin{pmatrix} x \\ dz(Cx) \end{pmatrix}$$
Application to the launcher model

- LMIs tested on the launcher example

- Sets of initial conditions for which \( |g(\theta)| \leq 8 \) is guaranteed
- Improvement thanks to piecewise quadratic sets of initial conditions
Conclusions

- IQS framework can handle local stability issues
  - Provides LMI tests - conservative
  - System augmentation + descriptor modeling = reduction of conservatism

Prospectives
- Improved construction of the IQS \(\equiv\) “generalized sector conditions”
- Further system augmentation with higher derivatives (?)
- Simultaneous handling of saturation, uncertainties, delays...
- Hybrid systems?